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Stable difference scheme for a nonlocal boundary value heat conduction problem

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Abstract. In this paper, a new finite difference method to solve nonlocal boundary value problems for the heat equation is proposed. The most important feature of these problems is the non-self-adjointness. Because of the non-self-adjointness, major difficulties occur when applying analytical and numerical solution techniques. Moreover, problems with boundary conditions that do not possess strong regularity are less studied. The scope of the present paper is to justify possibility of building a stable difference scheme with weights for mentioned type of problems above.

Keywords. Heat equation, initial boundary problems, nonlocal boundary value problem, numerical method, difference scheme.

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1 Introduction

Currently, the attention of many scientists is attracted to mathematical physics problems with nonlocal (non-classical) additional conditions. The relevance of the study of these problems is presented in number of physical applications in the field of electrostatics, electrodynamics, the theory of elasticity, plasma physics. The study of numerical methods for solving problems with nonlocal additional conditions, which include finite-difference schemes, is not less important.

It should be noted the absence of any universal methods of research as differential problems with nonclassical conditions and also difference schemes approximating them. There are fundamental difficulties for the use of traditional methods, such as the potential method, the method of separation of variables, the maximum principle and the method of energy inequalities. This property of nonclassical problems is related to, first of all, huge freedom of choice and existence of variety of additional conditions. It makes sense to allocate some class of nonlocal problems of mathematical physics and corresponding difference schemes for research.

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The problems with boundary conditions that do not possess strong regularity are less studied. One of the first problems of this type, known as a problem of Samarskii-Ionkin, was investigated by N. I. Ionkin and A. V. Gulin [4], A. Yu. Mokin [5] and others. More specific classes of problems of difference methods were illustrated in the A. Ashyralyev papers [2, 3].

In this paper, the family of boundary value problems for a heat equation and finite difference schemes approximating these problems are considered. The peculiarity of the initial-boundary value problems is a special choice of the boundary conditions, which are not strengthened regular. The corresponding difference schemes do not have the property of self-adjointness.

2 Statement of the problem

In $\Omega = \{(x,t), \quad 0 < x < 1, \quad 0 < t < T\}$, we consider a problem of finding a solution $u(x,t)$ of the heat equation

$$u_t(x,t) - u_{xx}(x,t) = f(x,t), \tag{1}$$

satisfying the initial condition

$$u(x,0) = \phi(x), \quad 0 \le x \le 1, \tag{2}$$

and the boundary conditions of the general form

$$\begin{cases} a_1 u_x(0,t) + b_1 u_x(1,t) + a_0 u(0,t) + b_0 u(1,t) = 0, \\ c_1 u_x(0,t) + d_1 u_x(1,t) + c_0 u(0,t) + d_0 u(1,t) = 0. \end{cases} \tag{3}$$

The coefficients a_k, b_k, c_k, d_k, $(k = 0, 1)$ of the boundary conditions are real numbers, and $\phi(x)$, $f(x,t)$ are given functions.

Applying Fourier method for solving problem (1)-(3) leads to a spectral problem for a operator defined by differential expression

$$l(y) = -y''(x), \quad 0 < x < 1$$

and boundary conditions

$$\begin{cases} a_1 y'(0) + b_1 y'(1) + a_0 y(0) + b_0 y(1) = 0, \\ c_1 y'(0) + d_1 y'(1) + c_0 y(0) + d_0 y(1) = 0. \end{cases} \tag{4}$$

The boundary conditions (4) of this operator are called regular [5], if one of the following three conditions holds:

$$a_1 d_1 - b_1 c_1 \neq 0;$$

$$a_1 d_1 - b_1 c_1 = 0, \quad |a_1| + |b_1| > 0, \quad a_1 d_0 + b_1 c_0 \neq 0;$$

$$a_1 = b_1 = c_1 = d_1 = 0, \quad a_0 d_0 - b_0 c_0 \neq 0.$$

The regular boundary conditions are strongly regular in the first and third cases, and in the second case if the following condition holds:

$$a_1 c_0 + b_1 d_0 \neq \pm[a_1 d_0 + b_1 c_0].$$

In [6], the following lemma was proved.

Lemma 2.1. *If the boundary conditions are regular but not strongly regular, then they can always be reduced to the form*

$$\begin{cases} a_1 u_x(0,t) + b_1 u_x(1,t) + a_0 u(0,t) + b_0 u(1,t) = 0, \\ c_0 u(0,t) + d_0 u(1,t) = 0, \end{cases}$$

where $|a_1| + |b_1| > 0$, *of one of the following four types:*

$$\begin{aligned} a_1 + b_1 = 0, &\quad c_0 - d_0 \neq 0, \\ a_1 - b_1 = 0, &\quad c_0 + d_0 \neq 0, \\ c_0 - d_0 = 0, &\quad a_1 + b_1 \neq 0, \\ c_0 + d_0 = 0, &\quad a_1 - b_1 \neq 0. \end{aligned}$$

In this paper, we consider the boundary value problems of type II. The boundary value problems of type I is considered in [1]. We know that $a_1 - b_1 = 0$, and $|a_1| + |b_1| > 0$. Therefore, without loss of generality we can assume $a_1 = b_1 = 1$. Since $c_0 + d_0 \neq 0$, also without loss of generality we assume that $c_0 + d_0 = -1$. For convenience, we denote $c_0 = c$, so $d_0 = 1 - c$.

Therefore, the problem of type II can be formulated in the following form: *In* $\Omega = \{(x,t), 0 < x < 1, \quad 0 < t < T\}$, *find the solution of the heat equation (1), satisfying the initial condition (2), and the boundary conditions of type II*

$$\begin{cases} u_x(0,t) + u_x(1,t) + au(0,t) + bu(1,t) = 0, \\ cu(0,t) + (1-c)u(1,t) = 0. \end{cases} \tag{5}$$

Here, coefficients a, b, c of boundary conditions are arbitrary real numbers.

3 Reduction to the sequential solution of two problems

To solve this problem, we introduce the auxiliary functions

$$v(x,t) = \frac{[u(x,t) - u(1-x,t)]}{2}, \tag{6}$$

$$w(x,t) = u(x,t) - [1 - (1-2c)(2x-1)]v(x,t). \tag{7}$$

Note that the function $v(x,t)$ is odd on the interval $0 < x < 1$, and it is an odd part of the function $u(x,t)$, and function $w(x,t)$ is not an even part of the function $u(x,t)$, although it is even. This follows from the fact that $w(x,t)$ can be represented in the form:

$$w(x,t) = \frac{1}{2}[u(x,t) + u(1-x,t)] + (1-2c)(2x-1)v(x,t).$$

From (7) it is easy to see that if we find the functions $v(x,t)$ and $w(x,t)$, then the solution of original problem can be restored by the formula

$$u(x,t) = w(x,t) + [1 - (1-2c)(2x-1)]v(x,t). \tag{8}$$

In representation (8), the first term is even on $0 < x < 1$ and the second term at $1 - 2c \neq 0$ is neither even nor odd.

It is easy to make sure that the function $v(x,t)$ is a solution of the initial boundary problem

$$v_t(x,t) - v_{xx}(x,t) = f_0(x,t), \tag{9}$$

$$v(x,0) = \phi_0(x), \quad 0 \leq x \leq 1, \tag{10}$$

$$\begin{cases} v_x(0,t) + [a(1-c) - bc]v(0,t) = 0, \\ v_x(1,t) - [a(1-c) - bc]v(1,t) = 0, \end{cases} \quad 0 \leq t \leq T, \tag{11}$$

where

$$f_0(x,t) = \frac{1}{2}[f(x,t) - f(1-x,t)], \quad \phi_0 = \frac{1}{2}[\phi(x) - \phi(1-x)]. \tag{12}$$

First, we find the solution $v(x,t)$ of problem (9)-(11). Then, we obtain $w(x,t)$ as a solution of the following problem:

$$w_t(x,t) - w_{xx}(x,t) = f_1(x,t), \tag{13}$$

$$w(x,0) = \phi_1(x), \quad 0 \leq x \leq 1, \tag{14}$$

$$\begin{cases} w(0,t) = 0, \\ w(1,t) = 0, \end{cases} \quad 0 \leq t \leq T, \tag{15}$$

where

$$f_1(x,t) = f(x,t) - [1-(1-2c)(2x-1)]f_0(x,t) - 4(1-2c)v_x(x,t), \quad (16)$$

$$\phi_1(x) = \phi(x) - [1-(1-2c)(2x-1)]\phi_0(x). \quad (17)$$

From direct verification of (12) and (17), it is easy to see that if the initial data $\phi(x)$ of problem (1), (2), (5) satisfies the necessary (a classic and well-known) compatibility conditions, then the initial data $\phi_0(x)$ and $\phi_1(x)$ also satisfy the necessary compatibility conditions of their respective problems.

Thus, the solution of the problem of type II (1), (2), (5) is reduced to the successive solution of two problems with homogeneous boundary conditions of Sturm type on the spatial variable:
(i) First, for $v(x,t)$ we need to obtain the solution of initial boundary problem (9) - (11) with homogeneous boundary conditions of Sturm type on the spatial variable,
(ii) Second, by the obtained value of $v(x,t)$, we need to solve the initial boundary problem (13) - (15) with homogeneous boundary conditions of the Dirichlet conditions on the spatial variable to get $w(x,t)$.

Therefore, the main results about existence, stability and convergence of numerical solution of the problem of type II (1), (2), (5) in the classical and generalized sense follow from the well-known classical results in the field of numerical methods for the solution of the heat equation with boundary conditions of Sturm type.

4 Study of problems by means of schemes with weights

In this section, we directly give the main result: the study of problems by means of schemes with weights. Schemes with weights for the heat equation were described in [6], where its error of approximation and the necessary conditions of stability were investigated. For completeness, these results are presented in more detail.

We introduce a grid $\delta_{h\tau} = \delta_h \times \delta_\tau$, where

$$\delta_h = \{x_i = ih, \quad i = 0,1,...,N, \quad Nh = 1\},$$

$$\delta_\tau = \{t_n = n\tau, \quad n = 0,1,...,K, \quad K\tau = T\}$$

and denote

$$y_i^n = y(x_i,t_n), \quad y_{t,i}^n = \frac{y_i^{n+1} - y_i^n}{\tau}, \quad y_{\overline{x}x,i}^n = \frac{y_{i+1}^n - 2y_i^n + y_{i-1}^n}{h^2}.$$

Differential problem (9) is replaced on the grid $\delta_{h\tau}$ by a difference problem

$$y_{t,i}^n = \sigma y_{\overline{x}x,i}^{n+1} + (1-\sigma)y_{\overline{x}x,i}^n + F_i^n, \quad i = 1,2,...,N, \quad n = 0,1,...,K-1, \quad (18)$$

where σ is a real number and F_i^n is a grid function, which replaces the function $f_0(x,t)$. We need to add difference form of initial and boundary conditions

$$\frac{y_1^n - y_0^n}{h} + [a(1-c) - bc]y_0^n = \frac{y_N^n - y_{N-1}^n}{h} - [a(1-c) - bc]y_N^n = 0, \quad (19)$$

where $n = 1, 2, ..., K-1$, and

$$y_i^0 = \phi_0(x_i), \quad i = 1, 2, ..., N. \quad (20)$$

The difference problem (18)-(20) is called a scheme with weights for the heat equation. The accuracy of the difference scheme is characterized by an error $z_i^n = y_i^n - v(x_i, t_n)$. For this error we have the following problem:

$$z_{t,i}^n = \sigma z_{\overline{x}x,i}^{n+1} + (1-\sigma) z_{\overline{x}x,i}^n + \psi_i^n \quad i = 1, 2, ..., N, \quad n = 0, 1, ..., K-1, \quad (21)$$

$$\frac{z_1^n - z_0^n}{h} + [a(1-c) - bc]z_0^n = \frac{z_N^n - z_{N-1}^n}{h} - [a(1-c) - bc]z_N^n = 0, z_i^0 = 0, \quad (22)$$

where $n = 1, 2, ..., K-1$ and $\psi_i^n = -v_{t,i}^n + \sigma v_{\overline{x}x,i}^{n+1} + (1-\sigma) u_{\overline{x}x,i}^n + F_i^n$ is the error of approximation schemes (18) - (20) in the solution of problem (1). It has been shown that from suitable choice of F_i^n we can take following relations

$$\begin{cases} \psi_i^n = O(\tau^2 + h^4), \text{ if } \sigma = \sigma_* = \frac{1}{2} - \frac{h^2}{12\tau}, \\ \psi_i^n = O(\tau^2 + h^2), \text{ if } \sigma = \frac{1}{2}, \\ \psi_i^n = O(\tau + h^2), \text{ if } \sigma \neq \sigma_*, \quad \sigma \neq \frac{1}{2}. \end{cases}$$

We obtain estimates of the solution of the difference problem (18)-(20) through an initial data y_i^0 and a right-hand side of F_i^n, expressing the stability of the scheme with respect to the initial data and the right-hand side. The error estimates z_i^n through approximation error ψ_i^n, describing the convergence and accuracy of the scheme (18)-(20), will immediately follow from these estimates. It is known that the solution y_i^{n+1} of problem (18)-(20) can be written as

$$y_i^{n+1} = \sum_{k=1}^{N} \left[q_k c_k(t_n) + \frac{\tau}{1 + \sigma\tau\lambda_k} \widehat{F}_k(t_n) \right] \mu_k(x_i), \quad (23)$$

where $q_k = \frac{1-(1-\sigma)\tau\lambda_k}{1+\sigma\tau\lambda_k}$. Here, $c_k(t_n)$, $\widehat{F}_k(t_n)$ are Fourier coefficients of functions $y(x_i, t_n)$, $F(x_i, t_n)$, respectively; λ_k are eigenvalues and $\{\mu_k\}_{k=0}^N$ are an orthonormal basis of eigenfunctions of the operator A:

$$(Ay)_i = -y_{\overline{x}x,i}, \quad i = 1, 2, ..., N,$$

$$\frac{y_1^n - y_0^n}{h} + [a(1-c) - bc]y_0^n = \frac{y_N^n - y_{N-1}^n}{h} - [a(1-c) - bc]y_N^n = 0.$$

We require that $1 + \sigma\tau\lambda_k > 0, \quad k = 1, 2, ..., N.$

Denoting

$$\overline{y}_i^{n+1} = \sum_{k=1}^{N} q_k c_k(t_n)\mu_k(x_i),$$

$$\widetilde{y}_i^{n+1} = \sum_{k=1}^{N} \frac{\tau}{1 + \sigma\tau\lambda_k} \widehat{F}_k(t_n)\mu_k(x_i),$$

we find $y_i^{n+1} = \overline{y}_i^{n+1} + \widetilde{y}_i^{n+1}$.

We estimate norms of $\overline{y}^{n+1}$ and $\widetilde{y}^{n+1}$ separately. In view of the orthonormality of basis μ_k, we get

$$||\overline{y}^{n+1}||^2 = \sum_{i=1}^{N} \left(\overline{y}_i^{n+1}\right)^2 h = \sum_{k=1}^{N} q_k^2 \left(c_k(t_n)\right)^2$$

and hence

$$||\overline{y}^{n+1}|| \le \left(\sum_{i=1}^{N} (c_k(t_n))^2\right)^{\frac{1}{2}} \max_{1 \le k \le N} |q_k| = ||y^n|| \max_{1 \le k \le N} |q_k|.$$

It requires that the condition

$$|q_k| \le 1, \quad k = 1, ..., N \tag{24}$$

is held. It is easy to see that (24) is equivalent to

$$\sigma \ge \frac{1}{2} - \frac{1}{\tau\lambda_N}, \tag{25}$$

where λ_N is a largest eigenvalue of the operator A.

Note that from (25), the inequality

$$1 + \sigma\tau\lambda_k \ge \frac{\tau\lambda_k}{2} > 0$$

for any $k = 1, 2, ..., N$ follows, i.e., the inequality that we need.

So, if (25) holds, then the estimate

$$||\overline{y}^{n+1}|| \le ||y^n|| \tag{26}$$

is valid. Essentially, this estimate means that scheme (18)-(20) is stable with respect to the initial data. Indeed, if in equation (18) $F_i^n \equiv 0$, then $y_i^{n+1} = \overline{y}_i^{n+1}$ and from (26), we obtain

$$||y^{n+1}|| \leq ||y^n|| \leq ||y^{n-1}|| \leq ... \leq ||y^0||.$$

It means that scheme (18)-(20) is stable with respect to the initial data in the norm

$$||y|| = \left(\sum_{i=1}^{N} h y_i^2 \right)^{\frac{1}{2}}. \tag{27}$$

So, we reach to the following conclusion. If parameters (18)-(20) of the scheme are connected by inequality (25), then the scheme is stable with respect to the initial data and for any $y^0 \in H$, the estimate

$$||y^{n+1}|| \leq ||y^0||, \quad n = 0, 1, ..., K-1$$

is valid for solving problem (18) (for $F_i^n \equiv 0$). Here, the norm $||y||$ is determined in accordance with (27).

Now, we show the stability on the right-hand part and the convergence of the first problem. To evaluate the function $\widetilde{y}^{n+1}$, we strengthen the condition (18) and it requires that the inequality holds

$$\sigma \geq \frac{1}{2} - \frac{(1-\varepsilon)}{\tau \lambda_N} \tag{28}$$

with constant $\varepsilon \in (0,1)$. Then $\sigma \geq \frac{1}{2} - \frac{(1-\varepsilon)}{\tau\lambda_k}$ and for any $k = 1, 2, ..., N$, we obtain

$$1 + \sigma\tau\lambda_k \geq \frac{\tau\lambda_k}{2} + 1 - \frac{(1-\varepsilon)\lambda_k}{\lambda_N} > 1 - \frac{(1-\varepsilon)\lambda_N}{\lambda_N} = \varepsilon > 0,$$

i.e., $1 + \sigma\tau\lambda_k \geq \varepsilon > 0$. From the expansion

$$||\widetilde{y}^{n+1}||^2 = \sum_{k=1}^{N} \frac{\tau^2}{(1+\sigma\tau\lambda_k)^2} \left(\widehat{F}_k(t_n) \right)^2 \leq \frac{\tau^2}{\varepsilon^2} \sum_{k=1}^{N} \left(\widehat{F}_k(t_n) \right)^2,$$

we obtain

$$||\widetilde{y}^{n+1}|| \leq \frac{\tau}{\varepsilon} ||F^n||. \tag{29}$$

If $\sigma \geq 0$, then condition (28) becomes unnecessary, since $1 + \sigma\tau\lambda_k \geq 1$ and estimate (29) holds with $\varepsilon = 1$. From the triangle inequality

$$||y^{n+1}|| \leq ||\overline{y}^{n+1}|| + ||\widetilde{y}^{n+1}||$$

and estimates (26), (29) we obtain that the inequality

$$||y^{n+1}|| \leq ||y^n|| + \frac{\tau}{\varepsilon}||F^n|| \tag{30}$$

is valid for $n = 0, 1, ..., K-1$. Summing of (30) with respect to n leads to an estimate

$$||y^{n+1}|| \leq ||y^0|| + \tau \sum_{j=0}^{n} \tau ||F^j|| \tag{31}$$

and it means that problem (18) - (20) is stable with respect to the initial data and the right-hand part. From (31), taking into account the condition $\tau n \leq T$, we obtain

$$||y^{n+1}|| \leq ||y^0|| + \frac{T}{\varepsilon} \max_{0 \leq j \leq n} ||F^j||. \tag{32}$$

So, if condition (28) holds with $\varepsilon \in (0,1)$ then scheme (18)-(20) is stable with respect to the initial data and provides the validity of the estimate. If $\sigma \neq 0$ and condition (25) holds, then (32) is valid with $\varepsilon = 1$.

The convergence of scheme (18)-(20) follows from (32) and the approximation requirements. For problem (21), the estimate of (32) takes the form

$$||z^{n+1}|| \leq \frac{T}{\varepsilon} \max_{0 \leq j \leq n} ||\psi^j||.$$

To solve the problem (13)-(15) the difference scheme and the estimate of its convergence are constructed similarly. In both cases the weight σ is the same. We do not stop here on these details. We give the final result: $||z^{n+1}||$ has the same order of magnitude as the error of approximation. In particular, for

$$\sigma = \sigma_* = \frac{1}{2} - \frac{h^2}{12\tau},$$

$$F_i^n = f\left(x_i, t_{n+\frac{1}{2}}\right) + \frac{h^2}{12} f''\left(x_i, t_{n+\frac{1}{2}}\right) + O\left(\tau^2 + h^4\right),$$

we have $||\psi^j|| = O\left(\tau^2 + h^4\right)$, the stability condition (28) holds with $\varepsilon = \frac{2}{3}$. Therefore, $||z^{n+1}|| = O\left(\tau^2 + h^4\right)$, that is, the scheme has the second order accuracy with respect to τ and the fourth order accuracy with respect to h. If $\sigma = \frac{1}{2}$ and $F_i^n = f\left(x_i, t_{n+\frac{1}{2}} + O\left(\tau^2 + h^4\right)\right)$, then the stability condition holds for all τ and h and $||z^{n+1}|| = O\left(\tau^2 + h^2\right)$. For other values of σ we have $\left||z^{n+1}\right|| = O\left(\tau + h^2\right)$, if (28) holds with $\varepsilon \in (0,1)$ or if $\sigma \neq 0$ and (25) holds.

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