The Dirichlet problem for elliptic equation with several singular coefficients

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Abstract. Recently found all the fundamental solutions of a multidimensional singular elliptic equation are expressed in terms of the well-known Lauricella hypergeometric function in many variables. In this paper, we find a unique solution of the Dirichlet problem for an elliptic equation with several singular coefficients in explicit form. When finding a solution, we use decomposition formulas and some adjacent relations for the Lauricella hypergeometric function in many variables.

Keywords. Dirichlet problem, multidimensional elliptic equations, singular coefficients, decomposition formulas, Lauricella hypergeometric function.

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1 Introduction

It is known that the theory of boundary value problems for degenerate equations and equations with singular coefficients is one of the rapidly developing parts of the modern theory of partial differential equations, which is encountered in solving many important questions of an applied nature, for example, [3, 11]. A detailed bibliography and summary of studies of the basic boundary-value equations for degenerate equations of various types, in particular, for elliptic equations with singular coefficients, can be found in monographs [4, 14, 33, 34]. In addition, generalized axisymmetric potentials have been studied using various methods [2, 9, 10, 12, 13, 19, 22, 24, 35, 36]. Omitting a huge bibliography in which various local and non-local boundary-value problems for mixed-type equations containing elliptic equations with singular coefficients are studied, we note some papers which are close to the present work. In the work [15], fundamental solutions were constructed for the bi-axially symmetric Helmholtz equation, and in [29–31] the explicit solutions of the Dirichlet and Dirichlet-Neumann problems in one quarter of a circle was found.

Dirichlet and Dirichlet-Neumann problems for elliptic equation with one singular coefficient in some part of ball were investigated by Agostinelli [1] and

Olevskii [27]. Recently, Nazipov published a paper devoted to the investigation of the Tricomi problem in a mixed domain consisting of hemisphere and cone [25]. Fundamental solutions for the following three-dimensional elliptic equations with two and three singular coefficients

$$u_{xx} + u_{yy} + u_{zz} + \frac{2\alpha}{x}u_x + \frac{2\beta}{y}u_y = 0, \ 0 < 2\alpha, \ 2\beta < 1 \tag{1}$$

and

$$u_{xx} + u_{yy} + u_{zz} + \frac{2\alpha}{x}u_x + \frac{2\beta}{y}u_y + \frac{2\gamma}{z}u_z = 0, \ 0 < 2\alpha, \ 2\beta, 2\gamma < 1 \tag{2}$$

were constructed, respectively, in [21] and [16]. For equations (1) and (2), the Dirichlet, Neumann and Holmgren problems [20,26,32] were solved in some parts of the ball.

In this paper, we study the Dirichlet problem for the equation

$$H_\alpha^{(m,n)}(u) \equiv \sum_{i=1}^{m} u_{x_i x_i} + \sum_{k=1}^{n} \frac{2\alpha_k}{x_k} u_{x_k} = 0, \tag{3}$$

where $m \geq 2, 0 < n \leq m$; $\alpha = (\alpha_1, \cdots, \alpha)$ and α_k are constants with $0 < 2\alpha_k < 1$. Hereinafter in the present work, unless there are other reservations, the natural number k will vary from 1 to n, inclusive.

2 Preliminaries

Below we give some formulas for Euler gamma-function, Gauss hypergeometric function, multiple Lauricella hypergeometric function (that is, Lauricella hypergeometric function in several variables), which will be used in the next sections.

It is known that the Euler gamma-function $\Gamma(a)$ has properties [7, pp. 17-19, (2), (10), (15)]

$$\Gamma(a+m) = \Gamma(a)(a)_m; \quad \Gamma\left(a+\frac{1}{2}\right) = \frac{\sqrt{\pi}\Gamma(2a)}{2^{2a-1}\Gamma(a)}, \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{4}$$

Here $(a)_m$ is a Pochhammer symbol, for which an equality

$$(a)_{m+n} = (a)_m (a+m)_n \tag{5}$$

is true [7, p.67, (5)].

A function

$$F(a, b; c; x) \equiv F \begin{bmatrix} a, b; \\ c; \end{bmatrix} x := \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m m!} x^m, \ c \neq 0, -1, -2, \cdots; |x| < 1$$

is known as the Gauss hypergeometric function and an equality

$$F(a, b; c; 1) = \frac{\Gamma(c) \Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)} \quad [c \neq 0, -1, \cdots; \ Re(c - a - b) > 0] \quad (6)$$

holds [7, c.73, (14)]. Moreover, the following autotransformer formula [7, p.76, (22)]

$$F(a, b; c; x) = (1 - x)^{-b} F\left(c - a, b; c; \frac{x}{x - 1}\right) \tag{7}$$

is valid.

The Lauricella hypergeometric function in n variables has a form [23]

$$F_A^{(n)}(a, b_1, \cdots, b_n; c_1, \cdots, c_n; x_1, \cdots, x_n) \equiv F_A^{(n)} \begin{bmatrix} a, b_1, \cdots, b_n; \\ c_1, \cdots, c_n; \end{bmatrix} x_1, \cdots, x_n \end{bmatrix}$$

$$= \sum_{m_1, \cdots m_n = 0}^{\infty} \frac{(a)_{m_1 + \cdots + m_n} (b_1)_{m_1} \cdots (b_n)_{m_n}}{(c_1)_{m_1} \cdots (c_n)_{m_n}} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!}, \tag{8}$$

$$\left[c_k \neq 0, -1, -2, \cdots; \ k = \overline{1, n}; \ |x_1| + \cdots + |x_n| < 1 \right].$$

For a given multiple hypergeometric function, it is useful to fund a decomposition formula which would express the multivariable hypergeometric function in terms of products of several simpler hypergeometric functions involving fewer variables. Burchnall and Chaundy [5, 6] systematically presented a number of expansion and decomposition formulas for some double hypergeometric functions in series of simpler hypergeometric functions. For example, the Appell function

$$F_2(a, b_1, b_2; c_1, c_2; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} (b_1)_m (b_2)_n}{(c_1)_m (c_2)_n} \frac{x^m y^n}{m! \, n!},$$

$$[c_1, c_2 \neq 0, -1, -2, \cdots; \ |x| + |y| < 1]$$

has the expansion [5]

$$F_2(a, b_1, b_2; c_1, c_2; x, y)$$

$$= \sum_{i=0}^{\infty} \frac{(a)_i (b_1)_i (b_2)_i}{i! (c_1)_i (c_2)_i} x^i y^i F(a + i, b_1 + i; c_1 + i; x) F(a + i, b_2 + i; c_2 + i; y).$$

The Burchnall-Chaundy method, which is limited to functions of two variables, is based on the following mutually inverse symbolic operators [5]

$$\nabla\left(h\right) = \frac{\Gamma\left(h\right)\Gamma\left(\delta_1 + \delta_2 + h\right)}{\Gamma\left(\delta_1 + h\right)\Gamma\left(\delta_2 + h\right)}, \quad \Delta\left(h\right) = \frac{\Gamma\left(\delta_1 + h\right)\Gamma\left(\delta_2 + h\right)}{\Gamma\left(h\right)\Gamma\left(\delta_1 + \delta_2 + h\right)}, \quad (9)$$

where $\delta_1 = x\dfrac{\partial}{\partial x}$ and $\delta_2 = y\dfrac{\partial}{\partial y}$.

In order to generalize the operators $\nabla\left(h\right)$ and $\Delta\left(h\right)$, defined in (9), A. Hasanov and H. M. Srivastava [17, 18] introduced the operators

$$\tilde{\nabla}_{z_1;z_2,\cdots,z_n}\left(h\right) = \frac{\Gamma\left(h\right)\Gamma\left(\delta_1 + \cdots + \delta_n + h\right)}{\Gamma\left(\delta_1 + h\right)\Gamma\left(\delta_2 + \cdots + \delta_n + h\right)}, \quad (10)$$

$$\tilde{\Delta}_{z_1;z_2,\cdots,z_n}\left(h\right) = \frac{\Gamma\left(\delta_1 + h\right)\Gamma\left(\delta_2 + \cdots + \delta_n + h\right)}{\Gamma\left(h\right)\Gamma\left(\delta_1 + \cdots + \delta_n + h\right)}, \quad (11)$$

where $\delta_k = z_k\dfrac{\partial}{\partial z_k}$, with the help of which they managed to find decomposition formulas for a whole class of hypergeometric functions in several variables. For example, the hypergeometric Lauricella function $F_A^{(n)}$, defined by formula (8) has the decomposition formula [17]

$$F_A^{(n)}\left(a, b_1, \cdots, b_n; c_1, \cdots, c_n; z_1, \cdots, z_n\right)$$

$$= \sum_{m_2,\cdots,m_n=0}^{\infty} \frac{(a)_{m_2+\cdots+m_n}\,(b_1)_{m_2+\cdots+m_n}\,(b_2)_{m_2}\cdots(b_n)_{m_n}}{m_2!\cdots m_n!\,(c_1)_{m_2+\cdots+m_n}\,(c_2)_{m_2}\cdots(c_n)_{m_n}}z_1^{m_2+\cdots+m_n}z_2^{m_2}\cdots z_n^{m_n}$$

$$\times F\left(a + m_2 + \cdots + m_n, b_1 + m_2 + \cdots + m_n; c_1 + m_2 + \cdots + m_n; z_1\right)$$

$$\times F_A^{(n-1)}(a + m_2 + \cdots + m_n, b_2 + m_2, \cdots, b_n + m_n;$$

$$c_2 + m_2, \cdots, c_n + m_n; z_2, \cdots, z_n), n \in \mathbb{N}\backslash\{1\}. \quad (12)$$

However, due to the recurrence of formula (12), additional difficulties may arise in the applications of this expansion. Further study of the properties of operators (10) and (11) showed that formula (12) can be reduced to a more convenient form.

Lemma 2.1. [8] *The following decomposition formula holds true at* $n \in \mathbb{N}\backslash\{1\}$

$$F_A^{(n)}\left(a, b_1, b_2, \cdots., b_n; c_1, c_2, \cdots., c_n; z_1, \cdots, z_n\right)$$

$$= \sum_{\substack{m_{i,j}=0 \\ (2 \le i \le j \le n)}}^{\infty} \frac{(a)_{A(n,n)}}{m_{ij}!}\prod_{k=1}^{n}[\frac{(b_k)_{B(k,n)}}{(c_k)_{B(k,n)}}z_k^{B(k,n)}F(a + A(k,n),$$

$$b_k + B(k,n); c_k + B(k,n); z_k)],\tag{13}$$

where

$$A(k,n) = \sum_{i=2}^{k+1}\sum_{j=i}^{n} m_{i,j},\ \ B(k,n) = \sum_{i=2}^{k} m_{i,k} + \sum_{i=k+1}^{n} m_{k+1,i}.\tag{14}$$

The formula (13) is proved by the method mathematical induction [8].

It should be noted here that the sum $\sum_{k=1}^{n} B(k,n)$ has the parity property, which plays an important role in the calculation of the some values of hypergeometric functions. In fact, by virtue of equality

$$\sum_{k=2}^{n}\sum_{i=2}^{k} m_{i,k} = \sum_{k=1}^{n-1}\sum_{i=k+1}^{n} m_{k+1,i}$$

we obtain

$$\sum_{k=1}^{n} B(k,n) = 2\sum_{k=2}^{n}\sum_{i=2}^{k} m_{i,k} = 2\sum_{k=1}^{n-1}\sum_{i=k+1}^{n} m_{k+1,i}.\tag{15}$$

In the present paper, R_m^{n+} denotes $1/2^n$ part of the Euclidean space R^m:

$$R_m^{n+} := \{(x_1,\cdots,x_m):\ x_1 > 0,\ \cdots,\ x_n > 0,\ 1 \leq n \leq m,\ m \geq 2\}.$$

All the fundamental solutions of equation (3) in the domain R_m^{n+} were found in [8], and we will use one of these solutions in the study of the problem:

$$q_n(\mathrm{x};\xi) = \gamma_n \prod_{i=1}^{n} \left[(x_i\xi_i)^{1-2\alpha_i}\right]$$

$$\times r^{-2\bar{\alpha}_n} F_A^{(n)}(\bar{\alpha}_n, 1-\alpha_1,\cdots,1-\alpha_n; 2-2\alpha_1,\cdots,2-2\alpha_n;\sigma),\tag{16}$$

where

$$\mathrm{x} := (x_1,\cdots,x_m), \xi := (\xi_1,\cdots,\xi_m), \sigma := (\sigma_1,\cdots,\sigma_n);$$

$$0 < 2\alpha_1,\cdots,2\alpha_n < 1;$$

$$\bar{\alpha}_n = \frac{m-2}{2} + n - \alpha_1 - \cdots - \alpha_n;\tag{17}$$

$$\gamma_n = 2^{2\bar{\alpha}_n - m}\frac{\Gamma(\bar{\alpha}_n)}{\pi^{m/2}}\prod_{j=1}^{n}\frac{\Gamma(1-\alpha_j)}{\Gamma(2-2\alpha_j)},\tag{18}$$

$$r^2 = \sum_{i=1}^{m} (x_i - \xi_i)^2, r_k^2 = (x_k + \xi_k)^2 + \sum_{i=1, i \neq k}^{m} (x_i - \xi_i)^2, \sigma_k = 1 - \frac{r_k^2}{r^2}.$$

It is easy to verify that the fundamental solution $q_n(\mathrm{x}; \xi)$ has the property

$$q_n(\mathrm{x}; \xi)|_{x_k=0} = 0.$$

3 Formulation of the problem and the uniqueness of the solution

Let $\Omega \subset \mathrm{R}_m^{n+}$ be a finite simple-connected domain bounded by planes $x_1 = 0$, $\cdots, x_n = 0$ and by smooth $m-$dimensional surface S. The intersection of this surface with plane $x_k = 0$ is denoted by χ_k. Designate as the domain S_k, a hyperplane $Ox_1 \cdots x_{k-1}x_{k+1} \cdots x_m$, bounded by $x_k = 0$ ($0 < x_l < a_l, -b_s < x_s < c_s$, $l = \overline{1,n}$, $l \neq k$; $n < s \leq m$) and by a curve χ_k. Here a_l, b_s and c_s are positive constants. We introduce the notation:

$$\tilde{x}_k := (x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_n, \cdots, x_m) \in S_k \subset \mathrm{R}_{m-1}^{(n-1)+} \subset \mathrm{R}^{m-1}.$$

Dirichlet problem. To find a function $u(\mathrm{x}) \in C(\bar{\Omega}) \cap C^2(\Omega)$, satisfying equation (3) in Ω and conditions

$$u|_{x_k=0} = \tau_k(\tilde{x}_k), \quad \tilde{x}_k \in \bar{S}_k, \tag{19}$$

$$u|_S = \varphi(\mathrm{x}), \quad \mathrm{x} \in \bar{S}, \tag{20}$$

where $\tau_k(\tilde{x}_k)$ and $\varphi(\mathrm{x})$ are given continuous functions fulfilling the following matching conditions:

$$\tau_i(0, \cdots, 0, x_{n+1}, \cdots, x_m) = \tau_j(0, \cdots, 0, x_{n+1}, \cdots, x_m), i \neq j, i, j = \overline{1,n},$$

$$\tau_k(\tilde{x}_k)|_{\chi_k} = \varphi(\mathrm{x})|_{\chi_k}. \tag{21}$$

One can readily check the validity of the following relation

$$x_1^{2\alpha_1} \cdots x_n^{2\alpha_n} \left[u H_\alpha^{(m,n)}(w) - w H_\alpha^{(m,n)}(u) \right]$$

$$= \sum_{i=1}^{m} \frac{\partial}{\partial x_i} \left[x_1^{2\alpha_1} \cdots x_n^{2\alpha_n} (uw_{x_i} - wu_{x_i}) \right], \quad n \leq m.$$

Let Ω_ε be a sub-domain of Ω at a distance $\varepsilon > 0$ from its boundary $\partial \Omega = \bigcup_{i=1}^{n} S_i \cup S$ and

$$\frac{\partial}{\partial \mathbf{n}} = \sum_{i=1}^{m} \cos(\mathbf{n}, x_i) \cdot \frac{\partial}{\partial x_i},$$

n is outer normal to $\partial\Omega$.

Integrate both sides of above given equality on the domain Ω_ε and use the classical formula of Gauss-Ostrogadsky:

$$\int_{\Omega_\varepsilon} x_1^{2\alpha_1} \cdots x_n^{2\alpha_n} \left[u H_\alpha^{(m,n)}(w) - w H_\alpha^{(m,n)}(u) \right] dx_1 \cdots dx_m$$

$$= \int_{\partial\Omega_\varepsilon} x_1^{2\alpha_1} \cdots x_n^{2\alpha_n} \sum_{i=1}^{m} (uw_{x_i} - wu_{x_i}) \cos(\mathbf{n}, x_i) \, d\vartheta. \tag{22}$$

Using the equality

$$x_1^{2\alpha_1} \cdots x_n^{2\alpha_n} \left[u H_\alpha^{(m,n)}(u) + \sum_{i=1}^{m} \left(\frac{\partial u}{\partial x_i} \right)^2 \right] = \sum_{i=1}^{m} \frac{\partial}{\partial x_i} \left(x_1^{2\alpha_1} \cdots x_n^{2\alpha_n} u \frac{\partial u}{\partial x_i} \right),$$

we obtain

$$\int_{\Omega_\varepsilon} x_1^{2\alpha_1} \cdots x_n^{2\alpha_n} u H_\alpha^{(m,n)}(u) \, dx_1 \cdots dx_m$$

$$+ \int_{\Omega_\varepsilon} x_1^{2\alpha_1} \cdots x_n^{2\alpha_n} \sum_{i=1}^{m} \left(\frac{\partial u}{\partial x_i} \right)^2 dx_1 \cdots dx_m$$

$$= \int_{\Omega_\varepsilon} \sum_{i=1}^{m} \frac{\partial}{\partial x_i} \left(x_1^{2\alpha_1} \cdots x_n^{2\alpha_n} u \frac{\partial u}{\partial x_i} \right) dx_1 \cdots dx_m.$$

Applying again the formula of Gauss-Ostrogradsky to this equality and letting $\varepsilon \to 0$, we get

$$\int_{\Omega_\varepsilon} x_1^{2\alpha_1} \cdots x_n^{2\alpha_n} \sum_{i=1}^{m} \left(\frac{\partial u}{\partial x_i} \right)^2 dx_1 \cdots dx_m$$

$$= \sum_{k=1}^{n} \int_{S_k} x_1^{2\alpha_1} \cdots x_{k-1}^{2\alpha_{k-1}} x_{k+1}^{2\alpha_{k+1}} \cdots x_n^{2\alpha_n} \tau_k \nu_k dS_k$$

$$+ \int_{S} x_1^{2\alpha_1} \cdots x_n^{2\alpha_n} \varphi \frac{\partial u}{\partial \mathbf{n}} dS, \tag{23}$$

where

$$\nu_k(\tilde{x}_k) := \left(x_k^{2\alpha_k} \frac{\partial u}{\partial x_k} \right) \Big|_{x_k=0}.$$

To prove the uniqueness of the solution, as usual, we suppose that the problem has two v, w solutions. Denoting $u = v - w$ we have that satisfies homogeneous

Dirichlet problem ($\tau_k = 0, \quad \varphi = 0$). Further we have to prove that the homogeneous problem has only trivial solution. In this case from (23) one can easily get

$$\int_{\Omega_\varepsilon} x_1^{2\alpha_1} \cdots x_n^{2\alpha_n} \sum_{i=1}^{m} \left(\frac{\partial u}{\partial x_i} \right)^2 dx_1 \cdots dx_m = 0.$$

Hence, it follows that $u_{x_1} = \cdots = u_{x_m} = 0$, which implies that u is a constant function. Considering homogeneous conditions (19) and (20), we conclude that $u(\mathrm{x}) \equiv 0$ in $\overline{\Omega}$.

4   The existence of the solution

We prove the existence of the solution in a special case of the domain Ω in order to get the solution in an explicit form. Assume $R = a_k = b_k = c_k$ and let

$$\Omega = \left\{ \mathrm{x} : x_1^2 + \cdots + x_m^2 < R^2, \ x_1 > 0, \cdots, x_n > 0, \right.$$
$$\left. -R < x_{n+1} < R, \cdots, -R < x_m < R \right\}.$$

We find a solution of considered problem using method Green's functions [28]. Therefore, first we give a definition of Green's function for the formulated problem.

Definition 4.1. We call the function $G(\mathrm{x}; \xi)$ as Green's function of the Dirichlet problem, if it satisfies the following conditions:

  (i) this function is a regular solution of equation (3) in the domain Ω, expect at the point ξ, which is any fixed point of Ω;

 (ii) it satisfies boundary conditions

$$G(\mathrm{x}; \xi)|_{x_k=0} = 0, \quad G(\mathrm{x}; \xi)|_S = 0;$$

(iii) it can be represented as

$$G(\mathrm{x}; \xi) = q_n(\mathrm{x}; \xi) + q_n^*(\mathrm{x}; \xi), \tag{24}$$

where $q_n(\mathrm{x}; \xi)$ is the fundamental solution found earlier (see formula (16))

(iv) function

$$q_n^*(\mathrm{x}; \xi) = -\left(\frac{a}{R_0} \right)^{2\bar{\alpha}_n} q_n(\mathrm{x}; \bar{\xi})$$

is a regular solution of equation (3) in the domain Ω. Here

$$\bar{\xi} := (\bar{\xi}_1, \cdots, \bar{\xi}_m), \ \bar{\xi}_i = \frac{a^2}{R_0^2}\xi_i, \ R_0^2 = \xi_1^2 + \cdots + \xi_m^2.$$

Excise a small ball with its center at ξ and with radius $\rho > 0$ from the domain Ω. Designate the sphere of the excised ball as C_ρ and by Ω_ρ denote the remaining part of Ω.

Applying formula (22), we obtain

$$\int_{C_\rho} \mathbf{x}^{(2\alpha)} \left[u(\mathbf{x}) \frac{\partial G(\mathbf{x};\xi)}{\partial \mathbf{n}} - G(\mathbf{x};\xi) \frac{\partial u(\mathbf{x})}{\partial \mathbf{n}} \right] dC_\rho$$

$$= \sum_{k=1}^{n} \int_{S_k} G_k^* (x_1, \cdots, x_{k-1}, 0, x_{k+1}, \cdots, x_m; \xi) \, \tau_k(\tilde{x}_k) \, dS_k$$

$$+ \int_S \mathbf{x}^{(2\alpha)} \frac{\partial G(\mathbf{x};\xi)}{\partial \mathbf{n}} \varphi(S) dS, \tag{25}$$

where

$$\mathbf{x}^{(2\alpha)} := x_1^{2\alpha_1} \cdots x_n^{2\alpha_n}, \quad \tilde{x}_k^{(2\alpha)} := x_1^{2\alpha_1} \cdots x_{k-1}^{2\alpha_{k-1}} x_{k+1}^{2\alpha_{k+1}} \cdots x_n^{2\alpha_n}.$$

$$G_k^* (x_1, \cdots, x_{k-1}, 0, x_{k+1}, \cdots, x_m; \xi) = \tilde{x}_k^{(2\alpha)} \cdot \left(x_k^{2\alpha_k} \frac{\partial G(\mathbf{x};\xi)}{\partial x_k} \right)\Big|_{x_k=0},$$

$$(x_1, \cdots, x_{k-1}, 0, x_{k+1}, \cdots, x_m) \in S_k.$$

First, we consider an integral

$$\int_{C_\rho} \mathbf{x}^{(2\alpha)} u(\mathbf{x}) \frac{\partial G(\mathbf{x};\xi)}{\partial \mathbf{n}} dC_\rho.$$

Taking (24) into account we rewrite it as follows

$$\int_{C_\rho} \mathbf{x}^{(2\alpha)} u \frac{\partial G}{\partial \mathbf{n}} dC_\rho$$

$$\equiv \int_{C_\rho} \mathbf{x}^{(2\alpha)} u \frac{\partial q_n(\mathbf{x};\xi)}{\partial \mathbf{n}} dC_\rho + \int_{C_\rho} \mathbf{x}^{(2\alpha)} u \frac{\partial q_n^*(\mathbf{x};\xi)}{\partial \mathbf{n}} dC_\rho = I_1 + I_2.$$

Using the formula of differentiation

$$\frac{\partial}{\partial z_j} F_A^{(n)}(a, b_1, \cdots, b_n; c_1, \cdots, c_n; z_1, \cdots, z_n)$$

$$= \frac{a b_j}{c_j} F_A^{(n)}(a+1, b_1, \cdots, b_{j-1}, b_j+1, \tag{26}$$

$$b_{j+1}, \cdots, b_n; c_1, \cdots, c_{j-1}, c_j+1, c_{j+1}, \cdots, c_n; z_1, \cdots, z_n)$$

and the following adjacent relation

$$
\sum_{j=1}^{n} \frac{b_j}{c_j} z_j F_A^{(n)}(a+1, b_1, \cdots, b_{j-1}, b_j+1, b_{j+1}, \cdots, b_n;
$$

$$
c_1, \cdots, c_{j-1}, c_j+1, c_{j+1}, \cdots, c_n; z_1, \cdots, z_n)
$$

$$
= F_A^{(n)}(a+1, b_1, \cdots, b_n; c_1, \cdots, c_n; z_1, \cdots, z_n)
$$

$$
- F_A^{(n)}(a, b_1, \cdots, b_n; c_1, \cdots, c_n; z_1, \cdots, z_n),
$$

(27)

we calculate

$$
\frac{\partial q_n(\mathrm{x}; \xi)}{\partial \mathbf{n}} = \sum_{i=1}^{m} \frac{\partial q_n(\mathrm{x}; \xi)}{\partial x_i} \cdot \cos(\mathbf{n}, x_i).
$$

(28)

Below we get detailed evaluations for $\dfrac{\partial q_n(\mathrm{x}; \xi)}{\partial x_i}$, when $1 \le i \le n$. Indeed, using the formula of differentiation (26), we get

$$
\frac{\partial q_n(x, \xi)}{\partial x_k} = (1 - 2\alpha_k)\gamma_n x_k^{-2\alpha_k} \xi_k^{1-2\alpha_k}
$$

$$
\times \prod_{i=1, i \ne k}^{n} \left[(x_i \xi_i)^{1-2\alpha_i}\right] r^{-2\bar{\alpha}_n} F_A^{(n)} \left[\begin{array}{c} \bar{\alpha}_n, 1 - \alpha_1, \cdots, 1 - \alpha_n; \\ 2 - 2\alpha_1, \cdots, 2 - 2\alpha_n; \end{array} \sigma \right]
$$

$$
-2\bar{\alpha}_n \gamma_n (x_k - \xi_k) \prod_{i=1}^{n} \left[(x_i \xi_i)^{1-2\alpha_i}\right] r^{-2\bar{\alpha}_n - 2} F_A^{(n)} \left[\begin{array}{c} \bar{\alpha}_n, 1 - \alpha_1, \cdots, 1 - \alpha_n; \\ 2 - 2\alpha_1, \cdots, 2 - 2\alpha_n; \end{array} \sigma \right]
$$

$$
-2\bar{\alpha}_n \gamma_n \xi_k \prod_{i=1}^{n} \left[(x_i \xi_i)^{1-2\alpha_i}\right] r^{-2\bar{\alpha}_n - 2}
$$

$$
\times F_A^{(n)} \left[\begin{array}{c} \bar{\alpha}_n + 1, 1 - \alpha_1, \cdots, 1 - \alpha_{k-1}, 2 - \alpha_k, 1 - \alpha_{k+1}, \cdots 1 - \alpha_n; \\ 2 - 2\alpha_1, \cdots, 2 - 2\alpha_{k-1}, 3 - 2\alpha_k, 2 - 2\alpha_{k+1}, \cdots 2 - 2\alpha_n; \end{array} \sigma \right]
$$

$$
-2\bar{\alpha}_n \gamma_n (x_k - \xi_k) \prod_{i=1}^{n} \left[(x_i \xi_i)^{1-2\alpha_i}\right] r^{-2\bar{\alpha}_n - 2} \sum_{i=1}^{n} \frac{1 - \alpha_i}{2 - 2\alpha_i} \sigma_i
$$

$$
\times F_A^{(n)} \left[\begin{array}{c} \bar{\alpha}_n + 1, 1 - \alpha_1, \cdots, 1 - \alpha_{i-1}, 2 - \alpha_i, 1 - \alpha_{i+1}, \cdots 1 - \alpha_n; \\ 2 - 2\alpha_1, \cdots, 2 - 2\alpha_{i-1}, 3 - 2\alpha_i, 2 - 2\alpha_{i+1}, \cdots 2 - 2\alpha_n; \end{array} \sigma \right].
$$

Considering adjacent relation (27) we obtain

$$\frac{\partial q_n(x,\xi)}{\partial x_k} = (1 - 2\alpha_k)\,\gamma_n x_k^{-2\alpha_k}\xi_k^{1-2\alpha_k}$$

$$\times \prod_{i=1,i\neq k}^{n} \left[(x_i\xi_i)^{1-2\alpha_i}\right] r^{-2\tilde{\alpha}_n} F_A^{(n)} \left[\begin{array}{c} \overline{\alpha}_n, 1 - \alpha_1, \cdots, 1 - \alpha_n; \\ 2 - 2\alpha_1, \cdots, 2 - 2\alpha_n; \end{array} \sigma\right]$$

$$-2\bar{\alpha}_n\gamma_n\xi_k\prod_{i=1}^{n}\left[(x_i\xi_i)^{1-2\alpha_i}\right]r^{-2\bar{\alpha}_n-2}$$

$$\times F_A^{(n)}\left[\begin{array}{c} \overline{\alpha}_n + 1, 1 - \alpha_1, \cdots, 1 - \alpha_{k-1}, 2 - \alpha_k, 1 - \alpha_{k+1}, \cdots 1 - \alpha_n; \\ 2 - 2\alpha_1, \cdots, 2 - 2\alpha_{k-1}, 3 - 2\alpha_k, 2 - 2\alpha_{k+1}, \cdots 2 - 2\alpha_n; \end{array} \sigma\right]$$

$$+2\bar{\alpha}_n\gamma_n\,(\xi_k - x_k)\prod_{i=1}^{n}\left[(x_i\xi_i)^{1-2\alpha_i}\right]r^{-2\bar{\alpha}_n-2}$$

$$\times F_A^{(n)}\left[\begin{array}{c} \bar{\alpha}_n + 1, 1 - \alpha_1, \cdots, 1 - \alpha_n; \\ 2 - 2\alpha_1, \cdots, 2 - 2\alpha_n; \end{array} \sigma\right]. \qquad (29)$$

Similarly we calculate $\dfrac{\partial q_n(\mathrm{x};\xi)}{\partial x_i}$, when $n + 1 \leq i \leq m$:

$$\frac{\partial q_n(x,\xi)}{\partial x_k} = 2\bar{\alpha}_n\gamma_n\,(\xi_k - x_k)\prod_{i=1}^{n}\left[(x_i\xi_i)^{1-2\alpha_i}\right]$$

$$\times r^{-2\bar{\alpha}_n-2}F_A^{(n)}\left[\begin{array}{c} \bar{\alpha}_n + 1, 1 - \alpha_1, \cdots, 1 - \alpha_n; \\ 2 - 2\alpha_1, \cdots, 2 - 2\alpha_n; \end{array} \sigma\right]. \qquad (30)$$

Taking (28), (29) and (30) into account we calculate

$$\frac{\partial q_n(x,\xi)}{\partial \mathbf{n}} = -\bar{\alpha}_n\gamma_n\prod_{i=1}^{n}\left[(x_i\xi_i)^{1-2\alpha_i}\right]r^{-2\bar{\alpha}_n}$$

$$\times F_A^{(n)}\left[\begin{array}{c} \bar{\alpha}_n + 1, 1 - \alpha_1, \cdots, 1 - \alpha_n; \\ 2 - 2\alpha_1, \cdots, 2 - 2\alpha_n; \end{array} \sigma\right]\frac{\partial}{\partial \mathbf{n}}\left[\ln r^2\right]$$

$$-2\bar{\alpha}_n\gamma_n\prod_{i=1}^{n}\left[(x_i\xi_i)^{1-2\alpha_i}\right]r^{-2\bar{\alpha}_n-2}\cos(\mathbf{n};x_i)$$

$$\times \sum_{i=1}^{n}\xi_i F_A^{(n)}\left[\begin{array}{c} \bar{\alpha}_n + 1, 1 - \alpha_1, \cdots, 1 - \alpha_{i-1}, 2 - \alpha_i, 1 - \alpha_{i+1}, \cdots 1 - \alpha_n; \\ 2 - 2\alpha_1, \cdots, 2 - 2\alpha_{i-1}, 3 - 2\alpha_i, 2 - 2\alpha_{i+1}, \cdots 2 - 2\alpha_n; \end{array} \sigma\right]$$

$$+\gamma_n \prod_{i=1}^{n} \left[(x_i \xi_i)^{1-2\alpha_i} \right] r^{-2\bar{\alpha}_n}$$

$$\times F_A^{(n)} \left[\begin{array}{c} \bar{\alpha}_n, 1-\alpha_1, \cdots, 1-\alpha_n; \\ 2-2\alpha_1, \cdots, 2-2\alpha_n; \end{array} \sigma \right] \cdot \sum_{i=1}^{n} \frac{1-2\alpha_i}{x_i} \cos(\mathbf{n}; x_i).$$

Now consider the integral

$$\int_{C_\rho} \mathbf{x}^{(2\alpha)} u \frac{\partial q_n(\mathbf{x}; \xi)}{\partial \mathbf{n}} dC_\rho = I_{11} + I_{12} + I_{13},$$

where

$$I_{11} = -\bar{\alpha}_n \gamma_n \prod_{i=1}^{n} \left[\xi_i^{1-2\alpha_i} \right] \int_{C_\rho} u \cdot \prod_{i=1}^{n} [x_i] r^{-2\bar{\alpha}_n}$$

$$\times F_A^{(n)} \left[\begin{array}{c} \bar{\alpha}_n + 1, 1-\alpha_1, \cdots, 1-\alpha_n; \\ 2-2\alpha_1, \cdots, 2-2\alpha_n; \end{array} \sigma \right] \frac{\partial}{\partial \mathbf{n}} \left[\ln r^2 \right] dC_\rho,$$

$$I_{12} = -2\bar{\alpha}_n \gamma_n \prod_{i=1}^{n} \left[\xi_i^{1-2\alpha_i} \right] \int_{C_\rho} u \cdot \prod_{i=1}^{n} [x_i] r^{-2\bar{\alpha}_n - 2} \sum_{i=1}^{n} \xi_i \cdot \cos(\mathbf{n}; x_i)$$

$$\times F_A^{(n)} \left[\begin{array}{c} \bar{\alpha}_n + 1, 1-\alpha_1, \cdots, 1-\alpha_{i-1}, 2-\alpha_i, 1-\alpha_{i+1}, \cdots 1-\alpha_n; \\ 2-2\alpha_1, \cdots, 2-2\alpha_{i-1}, 3-2\alpha_i, 2-2\alpha_{i+1}, \cdots 2-2\alpha_n; \end{array} \right] dC_\rho,$$

$$I_{13} = \gamma_n \prod_{i=1}^{n} \left[\xi_i^{1-2\alpha_i} \right] \int_{C_\rho} u \cdot \prod_{i=1}^{n} [x_i] r^{-2\bar{\alpha}_n}$$

$$\times F_A^{(n)} \left[\begin{array}{c} \bar{\alpha}_n, 1-\alpha_1, \cdots, 1-\alpha_n; \\ 2-2\alpha_1, \cdots, 2-2\alpha_n; \end{array} \sigma \right] \sum_{i=1}^{n} \frac{1-2\alpha_i}{x_i} \cos(\mathbf{n}; x_i) dC_\rho.$$

We use the following generalization spherical system of coordinates:

$$x_1 = \xi_1 + \rho \cos \varphi_1,$$

$$x_2 = \xi_2 + \rho \sin \varphi_1 \cos \varphi_2,$$

$$x_3 = \xi_3 + \rho \sin \varphi_1 \sin \varphi_2 \cos \varphi_3,$$

$$\vdots$$

$$x_{m-1} = \xi_{m-1} + \rho \sin \varphi_1 \sin \varphi_2 \cdots \sin \varphi_{m-2} \cos \varphi_{m-1},$$

$$x_m = \xi_m + \rho \sin \varphi_1 \sin \varphi_2 \cdots \sin \varphi_{m-2} \sin \varphi_{m-1},$$

$$[\rho \geq 0, \ 0 \leq \varphi_1 \leq \pi, \cdots, 0 \leq \varphi_{m-2} \leq \pi, \ 0 \leq \varphi_{m-1} \leq 2\pi \ ].$$

Then we have

$$I_{11} = 2\bar{\alpha}_n \gamma_n \prod_{i=1}^{n} \left[\xi_i^{1-2\alpha_i} \right] \rho^{2\alpha_1 + \cdots + 2\alpha_n - 2n} \int_0^{2\pi} d\varphi_{m-1}$$

$$\int_0^{\pi} \sin\varphi_{m-2} d\varphi_{m-2} \cdots \int_0^{\pi} \tilde{u}\, v\, F_A^{(n)}(\tilde{\sigma}) \sin^{m-2}\varphi_1 d\varphi_1,$$

where

$$\tilde{u} := u\,(\xi_1 + \rho\cos\varphi_1, \xi_2 + \rho\sin\varphi_1\cos\varphi_2, \xi_3 + \rho\sin\varphi_1\sin\varphi_2\cos\varphi_3, \cdots,$$
$$\xi_{m-1} + \rho\sin\varphi_1\sin\varphi_2\cdots\sin\varphi_{m-2}\cos\varphi_{m-1},$$
$$\xi_m + \rho\sin\varphi_1\sin\varphi_2\cdots\sin\varphi_{m-2}\sin\varphi_{m-1})\,,$$

$$v := (\xi_1 + \rho\cos\varphi_1)\,(\xi_2 + \rho\sin\varphi_1\cos\varphi_2)\,(\xi_3 + \rho\sin\varphi_1\sin\varphi_2\cos\varphi_3)\cdots$$
$$\cdots(\xi_{m-1} + \rho\sin\varphi_1\sin\varphi_2\cdots\sin\varphi_{m-2}\cos\varphi_{m-1})$$
$$\times (\xi_m + \rho\sin\varphi_1\sin\varphi_2\cdots\sin\varphi_{m-2}\sin\varphi_{m-1})\,,$$

$$F_A^{(n)}(\tilde{\sigma}) := F_A^{(n)}\,(\bar{\alpha}_n + 1, 1 - \alpha_1, \cdots, 1 - \alpha_n; 2 - 2\alpha_1, \cdots, 2 - 2\alpha_n; \tilde{\sigma})\,,$$

$$\tilde{\sigma} := (\sigma_{1\rho}, \cdots, \sigma_{n\rho})\,; r_k^2 = (x_k + \xi_k)^2 + \sum_{i=1, i \neq k}^{m} (x_i - \xi_i)^2, \ \sigma_{k\rho} = 1 - \frac{r_{k\rho}^2}{\rho^2}.$$

First we evaluate $F_A^{(n)}(\tilde{\sigma})$. For this aim we use decomposition formula (13) and then auto-transformation formula (7):

$$F_A^{(n)}(\tilde{\sigma}) = \sum_{\substack{m_{i,j}=0 \\ (2 \leq i \leq j \leq n)}}^{\infty} \frac{(\bar{\alpha}_n + 1)_{A(n,n)}}{m_{ij}!} \prod_{k=1}^{n} \left[\frac{(1 - \alpha_k)_{B(k,n)}}{(2 - 2\alpha_k)_{B(k,n)}} \left(\frac{r_{k\rho}^2}{\rho^2} \right)^{\alpha_k - 1 - B(k,n)} \right.$$

$$\times \left. \left(1 - \frac{r_{k\rho}^2}{\rho^2} \right)^{B(k,n)} \right] \prod_{k=1}^{n} [F\,(1 - 2\alpha_k - \bar{\alpha}_n + B(k,n) - A(k,n),$$

$$1 - \alpha_k + B(k,n); 2 - 2\alpha_k + B(k,n); \frac{\sigma_{k\rho}}{\sigma_{k\rho} - 1})],$$

where $A(k,n)$ and $B(k,n)$ are an expressions defined in (14).

After the elementary evaluations we find

$$F_A^{(n)}(\tilde{\sigma}) = \rho^{2n-2\alpha_1-\cdots-2\alpha_n} \prod_{k=1}^{n} \left[r_{k\rho}^{2\alpha_k-2} \right] \cdot \aleph, \tag{31}$$

where

$$\aleph := \sum_{\substack{m_{i,j}=0 \\ (2\leq i\leq j\leq n)}}^{\infty} \frac{(\bar{\alpha}_n+1)_{A(n,n)}}{m_{ij}!} \prod_{k=1}^{n} \left[\frac{(1-\alpha_k)_{B(k,n)}}{(2-2\alpha_k)_{B(k,n)}} \left(\frac{\rho^2}{r_{k\rho}^2} - 1 \right)^{B(k,n)} \right]$$

$$\times \prod_{k=1}^{n} [F\left(1 - 2\alpha_k - \bar{\alpha}_n + B(k,n) - A(k,n),\right.$$

$$\left.1 - \alpha_k + B(k,n); 2 - 2\alpha_k + B(k,n); 1 - \frac{\rho^2}{r_{k\rho}^2})\right].$$

It is easy to see that when $\rho \to 0$ the function $\aleph$ becomes an expression that does not depend on x and ξ. Indeed, taking into account the equality (15), we have

$$\lim_{\rho\to 0}\aleph = \sum_{\substack{m_{i,j}=0 \\ (2\leq i\leq j\leq n)}}^{\infty} \frac{(\bar{\alpha}_n+1)_{A(n,n)}}{m_{ij}!} \prod_{k=1}^{n} \left[\frac{(1-\alpha_k)_{B(k,n)}}{(2-2\alpha_k)_{B(k,n)}} \right] \prod_{k=1}^{n} [F\left(1 - 2\alpha_k\right.$$

$$\left.-\bar{\alpha}_n + B(k,n) - A(k,n), 1 - \alpha_k + B(k,n); 2 - 2\alpha_k + B(k,n); 1)\right]. \tag{32}$$

Applying now the summation formula (6) to each hypergeometric function $F(a,b;c;1)$ in the sum (32), we get

$$\lim_{\rho\to 0}\aleph = \frac{1}{\Gamma(\bar{\alpha}_n+1)} \sum_{\substack{m_{i,j}=0 \\ (2\leq i\leq j\leq n)}}^{\infty} \frac{\Gamma(\bar{\alpha}_n+1+A(n,n))}{m_{ij}!}$$

$$\times \prod_{k=1}^{n} \left[\frac{\Gamma(1-\alpha_k+B(k,n))\Gamma(2-2\alpha_k)\Gamma(\bar{\alpha}_n+\alpha_k+A(k,n)-B(k,n))}{\Gamma(\bar{\alpha}_n+1+A(k,n))\Gamma^2(1-\alpha_k)} \right].$$

Taking into account the identity

$$\sum_{\substack{m_{i,j}=0 \\ (2\leq i\leq j\leq n)}}^{\infty} \frac{(\bar{\alpha}_n+1)_{A(n,n)}}{m_{ij}!} \prod_{k=1}^{n} \left[\frac{(\bar{\alpha}_n+\alpha_k)_{A(k,n)-B(k,n)}(1-\alpha_k)_{B(k,n)}}{(\bar{\alpha}_n+1)_{B(k,n)}} \right]$$

$$= \Gamma\left(\frac{m}{2}\right) \frac{\Gamma^{n-1}\left(\bar{\alpha}_n + 1\right)}{\prod\limits_{k=1}^{n} \Gamma\left(\bar{\alpha}_n + \alpha_k\right)}, \tag{33}$$

$$\left[m \geq 2, 0 < n \leq m; \; 0 < 2\alpha_k < 1, \;\; k = \overline{1, n}\right]$$

we obtain

$$\lim_{\rho \to 0} \aleph = \frac{\Gamma\left(m/2\right)}{\Gamma\left(\bar{\alpha}_n + 1\right)} \prod_{i=1}^{n} \frac{\Gamma\left(2 - 2\alpha_k\right)}{\Gamma\left(1 - \alpha_k\right)}, \tag{34}$$

where $A(k, n)$, $B(k, n)$ and $\bar{\alpha}_n$ are expressions defined in (14) and (17).

Using the properties (4) of gamma-function $\Gamma(z)$, property (5) of the Pochhammer symbol and summation formula (6) for hypergeometric function $F(a, b; c; z)$, the formula (33) is proved by the method mathematical induction.

Now we consider an integral

$$L_m = \int\limits_0^{2\pi} d\varphi_{m-1} \int\limits_0^{\pi} \sin\varphi_{m-2} d\varphi_{m-2} \int\limits_0^{\pi} \sin^2\varphi_{m-3} d\varphi_{m-3} \cdots \int\limits_0^{\pi} \sin^{m-2}\varphi_1 d\varphi_1.$$

with elementary transformations it is not difficult to establish that

$$L_{2m} = \frac{2\,\pi^m}{(m-1)!}, \;\; L_{2m+1} = \frac{2^{m+1}\,\pi^m}{(2m-1)!!}, \;\; m = 1, 2, 3, \cdots \tag{35}$$

If we take into account (31), (34), (35) and (18), then we will have

$$\lim_{\rho \to 0} I_{11} = u\left(\xi\right).$$

By similar evaluations one can get that

$$\lim_{\rho \to 0} I_{12} = \lim_{\rho \to 0} I_{13} = \lim_{\rho \to 0} I_2 = 0.$$

If we consider an integral

$$\int_{C_\rho} \mathrm{x}^{(2\alpha)} G\left(\mathrm{x}; \xi\right) \frac{\partial u\left(\mathrm{x}\right)}{\partial \mathbf{n}} dC_\rho,$$

using above given algorithm for evaluations (in this case calculations will be more simple), we can prove that

$$\lim_{\rho \to 0} \int_{C_\rho} \mathrm{x}^{(2\alpha)} G\left(\mathrm{x}; \xi\right) \frac{\partial u\left(\mathrm{x}\right)}{\partial \mathbf{n}} dC_\rho = 0.$$

Now from (25) we can write the solution of the Dirichlet problem as follows:

$$u(\xi) = \sum_{k=1}^{n} \int_{S_k} G_k^*(x_1, \cdots, x_{k-1}, 0, x_{k+1}, \cdots, x_m; \xi)\, \tau_k(\tilde{x}_k)\, dS_k$$

$$+ \int_S x^{(2\alpha)} \frac{\partial G(\mathbf{x}; \xi)}{\partial \mathbf{n}} \varphi(\mathbf{x})\, dS. \tag{36}$$

The particular values of Green's function are given by

$$G_k^*(x_1, \cdots, x_{k-1}, 0, x_{k+1}, \cdots, x_m; \xi)$$

$$= (1 - 2\alpha_k)\, \gamma_n \prod_{i=1}^{n} \left[\xi_i^{1-2\alpha_i}\right] \cdot \prod_{i=1, i\neq k}^{n} [x_i]$$

$$\times \left\{ \frac{F_A^{(n-1)}\left[\begin{array}{c} \bar{\alpha}_n, 1-\alpha_1, \cdots, 1-\alpha_{k-1}, 1-\alpha_{k+1}, \cdots, 1-\alpha_n; \\ 2-2\alpha_1, \cdots, 2-2\alpha_{k-1}, 2-2\alpha_{k+1}, \cdots, 2-2\alpha_n; \end{array} \sigma_0\right]}{\left[\xi_k^2 + \sum_{i=1, i\neq k}^{m}(\xi_i - x_i)^2\right]^{\bar{\alpha}_n}} \right.$$

$$\left. - \frac{F_A^{(n-1)}\left[\begin{array}{c} \bar{\alpha}_n, 1-\alpha_1, \cdots, 1-\alpha_{k-1}, 1-\alpha_{k+1}, \cdots, 1-\alpha_n; \\ 2-2\alpha_1, \cdots, 2-2\alpha_{k-1}, 2-2\alpha_{k+1}, \cdots, 2-2\alpha_n; \end{array} \bar{\sigma}_0\right]}{\left[\sum_{i=1, i\neq k}^{m}\left(a - \frac{x_i \xi_i}{a}\right)^2 + \frac{1}{a^2} \sum_{i=1, i\neq k}^{m} \sum_{j=1, j\neq i}^{m} x_i^2 \xi_j^2 - (m-2)a^2\right]^{\bar{\alpha}_n}} \right\},$$

where

$$\sigma_0 := \left(\sigma_1^0, \cdots, \sigma_{k-1}^0, \sigma_{k+1}^0, \cdots, \sigma_n^0\right), \quad \bar{\sigma}_0 := \left(\bar{\sigma}_1^0, \cdots, \bar{\sigma}_{k-1}^0, \bar{\sigma}_{k+1}^0, \cdots, \bar{\sigma}_n^0\right),$$

$$\bar{\sigma}_s^0 = -\frac{a^2}{R_0^2} \frac{4x_s \xi_s}{\sum_{i=1, i\neq k}^{m}\left(a - \frac{x_i \xi_i}{a}\right)^2 + \frac{1}{a^2}\sum_{i=1, i\neq k}^{m}\sum_{j=1, j\neq i}^{m} x_i^2 \xi_j^2 - (m-2)a^2},$$

$$\sigma_s^0 = -\frac{4x_s \xi_s}{\xi_k^2 + \sum_{i=1, i\neq k}^{m}(\xi_i - x_i)^2}, \quad s = \overline{1, n}, \ s \neq k.$$

Constant γ_n has the form (18).

Hence, the main result of the paper is formulated as the following theorem:

Theorem 4.2. *If $\tau_k(\tilde{x}_k) \in C^2(S_k)$ and $\varphi(\mathbf{x}) \in C^2(S)$ are given functions fulfilling the matching conditions (21), then the Dirichlet problem has unique solution represented by formula (36).*

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