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Eigenvalues asymptotics for Stark operators

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Abstract. We give the eigenvalues asymptotics for the Stark operator of the form $-\Delta+Fx$, $F>0$ on $\mathrm{L}^2([0,d])$. This is given in the case when F is small enough or sufficiently large. We impose various boundary conditions. The proof is based on the asymptotics of the specialized Airy functions.

Keywords. Spectral theory, Schrödinger operator, Dirichlet and Neumann Laplacian, Stark operator, Eigenvalue problem, Airy functions.

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1 Introduction and results

Stark effect means the shifts viewed in atomic emission spectra after placing the particle in a constant electric field of strength F. In the non-relativistic quantum theory, this Stark effect is usually modeled by an Hamiltonian operator that (in appropriately scaled units and with the atomic units $2m=h=q=1$ to simplify the equation) has the form

$$H(F)=-\Delta+V(x)+Fx. \tag{1}$$

Hamiltonians that are parameterized by operators of the form (1) have been intensively studied in the last five decades [4, 5, 8, 9, 11] and references therein.
Asymptotic properties of eigenvalues is one of the most studied problems in the spectral theory. Unfortunately it is not possible to get simple analytic and exact expressions for solutions of the eigenvalue equation for the Stark operators.
The problem was first analyzed by the use of perturbation theory for $|F|$ small, and spectral properties of $H(F)$ are deduced from those of $H_0=-\Delta+V(x)$ [5, 11]. A comparison was also made with other approximation techniques, such as WKB and perturbation theory methods [7]. The other major approach is based on semiclassical methods where the Hamiltonian is parameterized by the strength

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of the external field.

In this note we give the asymptotics of the eigenvalues when the operator (1) is restricted to a finite interval. We give the result for two different type of boundaries conditions (Dirichlet and Dirichlet-Neumann). As we already mentioned, most earlier work, treats only the case $F \to 0$ and it is based on the comparison between Stark eigenvalues and free Laplacian eigenvalues. In our approach the result is based on the asymptotics of the Airy functions. Here one finds the expression of the eigenvalues energies directly and in a very simple way. Moreover this method can treat both the cases of a weak and a strong electric filed.

1.1 The model and the main result

We consider the Stark operator

$$H = -\frac{d^2}{dx^2} + Fx, F > 0. \tag{2}$$

We work in the Hilbert space $L^2(0,d)$ equipped with the scalar product

$$\langle f, g\rangle = \int_0^d f(x)\overline{g(x)}dx, \forall f, g \in L^2([0,d]).$$

The maximal domain in which H is well defined as a self-adjoint operator is denote by $\mathcal{D}_{max}$ [6]. Consider the Dirichlet boundary condition domain

$$\mathcal{D}^D = \{\psi \in \mathcal{D}_{max} \text{ and } \psi(0) = \psi(d) = 0\}.$$

and the Dirichlet-Neumann boundary condition

$$\mathcal{D}_N = \{\psi \in \mathcal{D}_{max} \text{ and } \psi(0) = \psi'(d) = 0\}.$$

We denote by H^D and H^{DN} the operator H with domain $\mathcal{D}^D$ and $\mathcal{D}^N$, respectively. H^D is associated to the infinite well potential. Indeed in this case (2) corresponds to (1), with $V(x) = 0$ if $x \in [0,d]$ and $V(x) = +\infty$ otherwise.

Theorem 1.1. *Let H^D (respectively H^N) has a sequence of discrete eigenvalues, and let us denote the n-th eigenvalue by λ_n^D (respectively H^{DN}). Then, they have the following asymptotics*

(i) *When, $F \to 0$,*

a.

$$\lambda_n^D = \left(\frac{n\pi + \sqrt{n^2\pi^2 + d^3F}}{2d}\right)^2 + o(F), \quad n \in \mathbb{Z}_+^*. \tag{3}$$

b.

$$\lambda_{n+1}^{DN} = \left(\frac{(2n+1)\frac{\pi}{2} + \sqrt{(2n+1)^2(\frac{\pi}{2})^2 + d^3F}}{2d} \right)^2$$
$$+o(F);\ n \in \mathbb{Z}_+. \tag{4}$$

(ii) *When,* $F \to +\infty$*,*

a.
$$\lambda_n^D = -\alpha_n F^{\frac{2}{3}}, \quad n \in \mathbb{Z}_+^*, \tag{5}$$

where α_n *is the* n*-th negative zero of the Airy function* Ai*.*

b.
$$\lambda^{DN} = -\alpha'_n F^{\frac{2}{3}}, \quad n \in \mathbb{Z}_+^*, \tag{6}$$

where α'_n *is the* n*-th negative zero of the derivative of the Airy function* Ai'*.*

Remark 1.2. For $F = 0$, in (3) and (4) we get the known eigenvalues of the free Laplacian.

The proof of the last theorem is the subject of the following section.

2 The proof of Theorem 1.1

2.1 Form of solutions

The spectral equation associated with the Stark operator is given by

$$-\frac{d^2\psi}{dx^2}(x) + Fx\psi(x) = E\psi(x). \tag{7}$$

Using the change of variable

$$\xi = \frac{E}{F\rho}; \rho = F^{-\frac{1}{3}},\ x = \rho z,$$

we get the new equation

$$\psi''(z) = (z - \xi)\psi(z). \tag{8}$$

The solutions of equation (8) are two linearly independent Airy functions $Ai(z - \xi)$ and $Bi(z - \xi)$. The eigenfunctions associated with the equation (8) are given as a superposition of two linearly independent functions and have the form

$$\phi(z) = A \cdot Ai(z - \xi) + B \cdot Bi(z - \xi);\ \mathbf{\Phi} = \begin{pmatrix} A \\ B \end{pmatrix} \in \mathbb{C}^2. \tag{9}$$

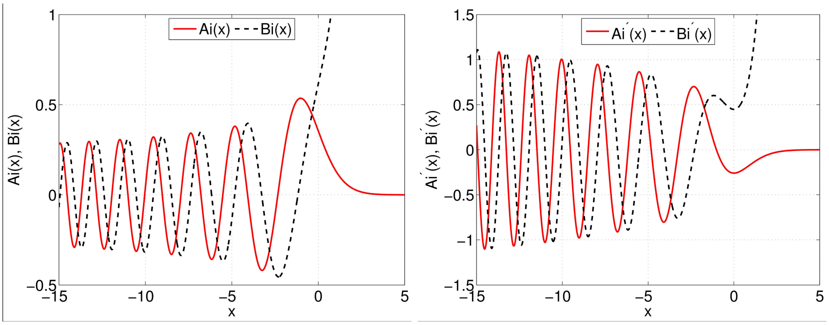

Figure 1. Airy functions and the corresponding derivatives.

Remark 2.1. If we consider a half line domain, i.e, with a potential $V(x) = 0$ for $x \geq 0$ and $V(x) = +\infty$ for $x < 0$ then in (9) we get just $Ai(\cdot)$, and then the quantized energies are given in terms of zeros of the well-behaved Airy functions $Ai(\cdot)$. So, the eigenvalues of the operator are given by $\lambda_n^D = F^{\frac{2}{3}}\xi_n$, where $-\xi_n$ is the n-th zero of Ai.

The equation (9) in the x variable is written as

$$\phi(x) = A \cdot Ai(F^{\frac{1}{3}}(x - \frac{E}{F})) + B \cdot Bi(F^{\frac{1}{3}}(x - \frac{E}{F})). \tag{10}$$

We denote the spectral parameter by λ and set:

$$u_{\lambda_\cdot}(z) := Ai\left(F^{1/3}\left(x - \frac{\lambda_\cdot}{F}\right)\right); \quad v_{\lambda_\cdot}(x) := Bi\left(F^{1/3}\left(x - \frac{\lambda_\cdot}{F}\right)\right). \tag{11}$$

We will calculate the eigenvalues of the operators H^D and H^{DN}.

2.2 The Dirichlet boundary conditions

The Dirichlet boundary conditions at points 0 and d yield

$$\begin{cases} Au_{\lambda^D}(0) + Bv_{\lambda^D}(0) = 0 \\ Au_{\lambda^D}(d) + Bv_{\lambda^D}(d) = 0, \end{cases}$$

with $A, B \in \mathbb{C}$. Or more simply as a matrix equation:

$$\begin{pmatrix} u_{\lambda^D}(0) & v_{\lambda^D}(0) \\ u_{\lambda^D}(d) & v_{\lambda^D}(d) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0.$$

The condition for the solution to such an equation is

$$\begin{vmatrix} u_{\lambda^D}(0) & v_{\lambda^D}(0) \\ u_{\lambda^D}(d) & v_{\lambda^D}(d) \end{vmatrix} = u_{\lambda^D}(0)v_{\lambda^D}(d) - u_{\lambda^D}(d)v_{\lambda^D}(0) = 0.$$

So we get that

$$\begin{aligned} & Ai\left(-\lambda^D F^{-\frac{2}{3}}\right) Bi\left(F^{\frac{1}{3}}\left(d - \frac{\lambda^D}{F}\right)\right) \\ & -Bi\left(-\lambda^D F^{-\frac{2}{3}}\right) Ai\left(F^{\frac{1}{3}}\left(d - \frac{\lambda^D}{F}\right)\right) = 0. \end{aligned} \tag{12}$$

Weak electric field

If $F \to 0$, for a $\lambda^D \in [a,b]$, with a and b independent of F , $-\lambda^D F^{-\frac{2}{3}}$ and $F^{\frac{1}{3}}\left(d - \frac{\lambda^D}{F}\right)$ tends to $-\infty$. We recall the asymptotic properties of Airy functions given in $[1,10]$

$$Ai(-t) \quad \sim \quad \frac{\sin(\frac{2}{3}t^{\frac{3}{2}} + \frac{\pi}{4})}{\sqrt{\pi}t^{\frac{1}{4}}}; \qquad t \to +\infty \tag{13}$$

$$Bi(-t) \quad \sim \quad \frac{\cos(\frac{2}{3}t^{\frac{3}{2}} + \frac{\pi}{4})}{\sqrt{\pi}t^{\frac{1}{4}}}; \qquad t \to +\infty. \tag{14}$$

Here and from now on, we use the standard convention for the asymptotic formula $f(x) \sim g(x)$ as $x \to x_0$ when $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1$ if $g(x) \neq 0$ or when $f(x) = g(x) + o(g(x))$.
We set

$$w_{D,1} = \frac{2}{3}(\lambda^D)^{\frac{3}{2}} F^{-1}, \tag{15}$$

and

$$\begin{aligned} w_{D,2} &= \frac{2}{3}\left[F^{\frac{1}{3}}\left(\frac{\lambda^D}{F}-d\right)\right]^{\frac{3}{2}} \\ &= \frac{2}{3}\left[\lambda^D F^{\frac{-2}{3}}\left(1-\frac{dF}{\lambda^D}\right)\right]^{\frac{3}{2}} \\ &= \frac{2}{3}(\lambda^D)^{\frac{3}{2}}F^{-1}\left(1-\frac{dF}{\lambda^D}\right)^{\frac{3}{2}} \\ &\sim \frac{2}{3}(\lambda^D)^{\frac{3}{2}}F^{-1}\left(1-\frac{3}{2}\frac{dF}{\lambda^D}+\frac{3}{8}\left(\frac{dF}{\lambda^D}\right)^2+o(F^2)\right); F\to 0 \\ &\sim \frac{2}{3}(\lambda^D)^{\frac{3}{2}}F^{-1}-d(\lambda^D)^{\frac{1}{2}}+\frac{1}{4}d^2(\lambda^D)^{-\frac{1}{2}}F+o(F); F\to 0. \quad (16) \end{aligned}$$

The equation (12) becomes

$$\sin\left(w_{D,1}+\frac{\pi}{4}\right)\cos\left(w_{D,2}+\frac{\pi}{4}\right)-\sin\left(w_{D,2}+\frac{\pi}{4}\right)\cos\left(w_{D,1}+\frac{\pi}{4}\right)=0.$$

This can be written as

$$\sin(w_{D,1}-w_{D,2})=0.$$

So, we get

$$w_{D,1}-w_{D,2}=n\pi; \quad n\in\mathbb{Z}.$$

One computes

$$\begin{aligned} w_{D,1}-w_{D,2} &= \frac{2}{3}(\lambda^D)^{\frac{3}{2}}F^{-1}-\left(\frac{2}{3}(\lambda^D)^{\frac{3}{2}}F^{-1}\right. \\ &- \left. d(\lambda^D)^{\frac{1}{2}}+\frac{1}{4}d^2(\lambda^D)^{-\frac{1}{2}}F+o(F)\right) \\ &= d(\lambda^D)^{\frac{1}{2}}-\frac{1}{4}d^2(\lambda^D)^{-\frac{1}{2}}F+o(F). \end{aligned}$$

Taking into account the sign of $w_{D,1}-w_{D,2}$, we get

$$w_{D,1}-w_{D,2}=n\pi;\ \text{for } n\in\mathbb{Z}_+^*.$$

So, we get the following condition:

$$d\lambda_D^{\frac{1}{2}}-\frac{1}{4}d^2\lambda_D^{-\frac{1}{2}}F+o(F)=n\pi; \quad n\in\{1,2,3,\cdots\}. \quad (17)$$

Multiply the last equation by $(\lambda^D)^{\frac{1}{2}}$

$$d\lambda^D - \frac{1}{4}d^2F + o(F) = n\pi(\lambda^D)^{\frac{1}{2}}; \quad n \in \{1, 2, 3, \cdots\}. \tag{18}$$

So,

$$d\lambda^D - n\pi(\lambda^D)^{\frac{1}{2}} - \frac{1}{4}d^2F = o(F); \quad n \in \{1, 2, 3, \cdots\}.$$

We set $(\lambda^D)^{\frac{1}{2}} = \nu$, and consider the equation

$$d\nu^2 - n\pi\nu - \frac{1}{4}d^2F = o(F); \quad n \in \{1, 2, 3, \cdots\}. \tag{19}$$

We notice that the left side of equation (19) is an equation of degree two and its solutions are given by

$$\nu_1 = \frac{n\pi - \sqrt{n^2\pi^2 + d^3F}}{2d} + o(F); \quad n \in \{1, 2, 3, \cdots\}, \tag{20}$$

$$\nu_2 = \frac{n\pi + \sqrt{n^2\pi^2 + d^3F}}{2d} + o(F); \quad n \in \{1, 2, 3, \cdots\}. \tag{21}$$

As we are interested for eigenvalues in some fixed interval we consider only ν_2, and we obtain

$$\lambda_D = \nu_2^2 = \left(\frac{n\pi + \sqrt{n^2\pi^2 + d^3F}}{2d}\right)^2 + o(F); \quad n \in \{1, 2, 3, \cdots\}. \tag{22}$$

There is a sequence of solutions, so we re-index λ^D by λ_n^D for $n \in \{1, 2, 3, \cdots\}$. So the eigenvalues of the operator H^D are given by:

$$\lambda_n^D = \left(\frac{n\pi + \sqrt{n^2\pi^2 + d^3F}}{2d}\right)^2 + o(F); \quad n \in \{1, 2, 3, \cdots\}. \tag{23}$$

Strong electric field

When $F \to +\infty$, $F^{\frac{1}{3}}\left(d - \frac{\lambda^D}{F}\right)$ tends to $+\infty$. So, by asymptotic properties of the Airy functions as t goes to $+\infty$ (see [1])

$$Ai(t) \sim \frac{\exp(-\frac{2}{3}t^{\frac{3}{2}})}{2\sqrt{\pi}t^{\frac{1}{4}}}, \tag{24}$$

$$Bi(t) \sim \frac{\exp(\frac{2}{3}t^{\frac{3}{2}})}{\sqrt{\pi}t^{\frac{1}{4}}}, \tag{25}$$

and by the properties of Airy functions for a real negative x

$$Ai(t) = M(t)\sin\theta(t), \tag{26}$$
$$Bi(t) = M(t)\cos\theta(t), \tag{27}$$

where

$$M(t) = \sqrt{Ai^2(t) + Bi^2(t)}, \tag{28}$$
$$\theta(t) = \arctan\left(Ai(t)/Bi(t)\right). \tag{29}$$

The equation (12) takes the following form:

$$\sin\theta\left(-\lambda^D F^{-\frac{2}{3}}\right)\exp\left(\frac{2}{3}\left[F^{\frac{1}{3}}\left(d-\frac{\lambda^D}{F}\right)\right]^{\frac{3}{2}}\right)-\frac{1}{2}\cos\theta\left(-\lambda^D F^{-\frac{2}{3}}\right)$$
$$\times\exp\left(-\frac{2}{3}\left[F^{\frac{1}{3}}\left(d-\frac{\lambda^D}{F}\right)\right]^{\frac{3}{2}}\right)=0.$$

So, we get

$$\tan\theta\left(-\lambda^D F^{-\frac{2}{3}}\right)=\frac{1}{2}\exp\left(-\frac{4}{3}\left[F^{\frac{1}{3}}\left(d-\frac{\lambda_D}{F}\right)\right]^{\frac{3}{2}}\right). \tag{30}$$

Since $\exp\left(-\frac{4}{3}\left[F^{\frac{1}{3}}\left(d-\frac{\lambda^D}{F}\right)\right]^{\frac{3}{2}}\right) \sim 0$ when $F \to +\infty$, the last equation becomes

$$\tan\theta\left(-\lambda^D F^{-\frac{2}{3}}\right)\sim 0.$$

So using equation (29) and by applying tan in the two sides one gets,

$$Ai\left(-\lambda^D F^{-\frac{2}{3}}\right)=0.$$

Consequently, we obtain

$$-\lambda^D F^{-\frac{2}{3}}=\alpha_n, \quad n\in\{1,2,3,\cdots\},$$

where α_n is the n-th negative zero of the Airy function Ai. Therefore, we have a sequence of eigenvalues

$$\lambda^D=-\alpha_n F^{\frac{2}{3}}, \quad n\in\{1,2,3,\cdots\}. \tag{31}$$

2.3 The Dirichlet-Neumann boundary conditions

As in the previous case, the use of boundary conditions of type Dirichlet at d and Neumann at 0, yields to

$$u'_{\lambda^{DN}}(0)v_{\lambda^{DN}}(d) - v'_{\lambda^{DN}}(0)u_{\lambda^{DN}}(d) = 0. \tag{32}$$

This implies that

$$Ai'\left(-\lambda^{DN}F^{-\frac{2}{3}}\right)Bi\left(F^{\frac{1}{3}}\left(d-\frac{\lambda^{DN}}{F}\right)\right) - Bi'\left(-\lambda^{DN}F^{-\frac{2}{3}}\right) \times Ai\left(F^{\frac{1}{3}}\left(d-\frac{\lambda^{DN}}{F}\right)\right) = 0. \tag{33}$$

Weak electric field

When $F \to 0$, for $\lambda^{DN} \in [a,b]$, with a and b independent of F $-\lambda^{DN}F^{-\frac{2}{3}}$ and $F^{\frac{1}{3}}\left(d-\frac{\lambda^{DN}}{F}\right)$ tends to $-\infty$. Then, using the asymptotic properties of Airy functions (13) and (14) as t goes to $+\infty$, we get

$$\begin{aligned} -Ai'(-t) &\sim \frac{1}{\sqrt{\pi}}\left[t^{\frac{1}{4}}\cos\left(\frac{2}{3}t^{\frac{3}{2}}+\frac{\pi}{4}\right) - \frac{t^{-\frac{5}{4}}}{4}\sin\left(\frac{2}{3}t^{\frac{3}{2}}+\frac{\pi}{4}\right)\right], \\ -Bi'(-t) &\sim \frac{1}{\sqrt{\pi}}\left[-t^{\frac{1}{4}}\sin\left(\frac{2}{3}t^{\frac{3}{2}}+\frac{\pi}{4}\right) - \frac{t^{-\frac{5}{4}}}{4}\cos\left(\frac{2}{3}t^{\frac{3}{2}}+\frac{\pi}{4}\right)\right]. \end{aligned}$$

We set

$$w_1^{DN} = \frac{2}{3}(\lambda^{DN})^{\frac{3}{2}}F^{-1}, \tag{34}$$

and

$$\begin{aligned} w_2^{DN} &= \frac{2}{3}\left[F^{\frac{1}{3}}\left(\frac{\lambda^{DN}}{F}-d\right)\right]^{\frac{3}{2}} \\ &= \frac{2}{3}\left[\lambda^{DN}F^{\frac{-2}{3}}\left(1-\frac{dF}{\lambda^{DN}}\right)\right]^{\frac{3}{2}} \\ &\sim \frac{2}{3}(\lambda^{DN})^{\frac{3}{2}}F^{-1} - d(\lambda^{DN})^{\frac{1}{2}} + \frac{1}{4}d^2(\lambda^{DN})^{-\frac{1}{2}}F + o(F). \end{aligned} \tag{35}$$

The equation (12) yields

$$-\left[(\lambda^{DN}F^{-\frac{2}{3}})^{\frac{1}{4}}\cos\left(w_1^{DN}+\frac{\pi}{4}\right)-\frac{(\lambda^{DN}F^{-\frac{2}{3}})^{-\frac{5}{4}}}{4}\sin\left(w_1^{D}+\frac{\pi}{4}\right)\right]$$
$$\times\cos\left(w_2^{DN}+\frac{\pi}{4}\right)+\left[-(\lambda^{DN}F^{-\frac{2}{3}})^{\frac{1}{4}}\sin\left(w_1^{DN}+\frac{\pi}{4}\right)-\frac{(\lambda^{DN}F^{-\frac{2}{3}})^{-\frac{5}{4}}}{4}\right.$$
$$\left.\times\cos\left(w_1^{DN}+\frac{\pi}{4}\right)\right]\sin\left(w_2^{DN}+\frac{\pi}{4}\right)=0. \quad (36)$$

Since the sine and cosine functions are bounded functions and since we have $(\lambda^{DN}F^{-\frac{2}{3}})^{-\frac{5}{4}}\sim 0$ if $F\to 0$, the equation (36) becomes

$$-(\lambda^{DN}F^{-\frac{2}{3}})^{\frac{1}{4}}\left[\cos\left(w_1^{DN}+\frac{\pi}{4}\right)\cos\left(w_2^{DN}+\frac{\pi}{4}\right)+\sin\left(w_1^{DN}+\frac{\pi}{4}\right)\right.$$
$$\left.\times\sin\left(w_2^{DN}+\frac{\pi}{4}\right)\right]=0. \quad (37)$$

As $(\lambda^{DN}F^{-\frac{2}{3}})^{\frac{1}{4}}\neq 0$, we get that

$$\cos\left(w_1^{DN}+\frac{\pi}{4}\right)\cos\left(w_2^{DN}+\frac{\pi}{4}\right)$$
$$+\sin\left(w_1^{DN}+\frac{\pi}{4}\right)\sin\left(w_2^{DN}+\frac{\pi}{4}\right)=0. \quad (38)$$

So, equation (38) reduces to

$$\cos(w_1^{DN}-w_2^{DN})=0.$$

Thus, we have

$$w_1^{DN}-w_2^{DN}=\frac{\pi}{2}+n\pi;\quad n\in\mathbb{Z}.$$

Since

$$\begin{aligned} w_1^{DN}-w_2^{DN} &= \frac{2}{3}(\lambda^{DN})^{\frac{3}{2}}F^{-1} \\ &\quad -\Big(\frac{2}{3}(\lambda^{DN})^{\frac{3}{2}}F^{-1}-d(\lambda^{DN})^{\frac{1}{2}}+\frac{1}{4}d^2(\lambda^{DN})^{-\frac{1}{2}}F+o(F)\Big) \\ &= d(\lambda^{DN})^{\frac{1}{2}}-\frac{1}{4}d^2(\lambda^{DN})^{-\frac{1}{2}}F+o(F), \end{aligned}$$

and taking into account the sign of $w_1^{DN} - w_2^{DN}$, we conclude that

$$w_1^{DN} - w_2^{DN} = \frac{\pi}{2} + n\pi; \quad n \in \mathbb{Z}_+.$$

So,

$$d(\lambda^{DN})^{\frac{1}{2}} - \frac{1}{4}d^2(\lambda^{DN})^{-\frac{1}{2}}F + o(F) = (2n+1)\frac{\pi}{2}; \quad n \in \{0, 1, 2, \cdots\}.$$

We multiply the last equation by $(\lambda^{DN})^{\frac{1}{2}}$

$$d\lambda^{DN} - \frac{1}{4}d^2F + o(F) = (2n+1)\frac{\pi}{2}(\lambda^{DN})^{\frac{1}{2}}; \quad n \in \mathbb{Z}_+. \tag{39}$$

We set $(\lambda^{DN})^{\frac{1}{2}} = \mu$, and get

$$d\mu^2 - (2n+1)\frac{\pi}{2}\mu - \frac{1}{4}d^2F = o(F); \quad n \in \mathbb{Z}_+. \tag{40}$$

The left side of equation (40) is a second degree equation whose solutions are

$$\begin{aligned} \mu_1 &= \frac{(2n+1)\frac{\pi}{2} - \sqrt{(2n+1)^2(\frac{\pi}{2})^2 + d^3F}}{2d} + o(F); \quad n \in \mathbb{Z}_+, \\ \mu_2 &= \frac{(2n+1)\frac{\pi}{2} + \sqrt{(2n+1)^2(\frac{\pi}{2})^2 + d^3F}}{2d} + o(F); \quad n \in \mathbb{Z}_+. \end{aligned}$$

As we are interested for eigenvalues in some fixed interval we consider only μ_2, we finally obtain that

$$\begin{aligned} \lambda^{DN} &= \mu_2^2 \\ &= \left(\frac{(2n+1)\frac{\pi}{2} + \sqrt{(2n+1)^2(\frac{\pi}{2})^2 + d^3F}}{2d}\right)^2 + o(F); n \in \mathbb{Z}_+. \end{aligned} \tag{41}$$

We get that there is a sequence of solutions, so we re-index λ^{DN} by λ_{n+1}^{DN} for $n \in \{1, 2, 3, \cdots\}$. Therefore the eigenvalues of the operator H^{DN} are given by

$$\lambda_{n+1}^{DN} = \left(\frac{(2n+1)\frac{\pi}{2} + \sqrt{(2n+1)^2(\frac{\pi}{2})^2 + d^3F}}{2d}\right)^2 + o(F); n \in \mathbb{Z}_+. \tag{42}$$

Strong electric field

We know that the Dirichlet-Neumann boundary conditions at points d and 0 are described as

$$Ai'\left(-\lambda^{DN}F^{-\frac{2}{3}}\right)Bi\left(F^{\frac{1}{3}}\left(d-\frac{\lambda^{DN}}{F}\right)\right)-Bi'\left(-\lambda^{DN}F^{-\frac{2}{3}}\right)\times Ai\left(F^{\frac{1}{3}}\left(d-\frac{\lambda^{DN}}{F}\right)\right)=0. \quad (43)$$

When $F\to+\infty$, $-\lambda^{DN}F^{-\frac{2}{3}}$ tends to 0 and the term $F^{\frac{1}{3}}\left(d-\frac{\lambda^{DN}}{F}\right)$ tends to $+\infty$. So, by the asymptotic behavior of the Airy functions as t goes to $+\infty$ given in equations (24) and (25) and by the properties of the Airy function for a negative t

$$\begin{aligned} Ai'(t) &= N(t)\sin\phi(t), && (44)\\ Bi'(t) &= N(t)\cos\phi(t), && (45) \end{aligned}$$

with

$$\begin{aligned} N(t) &= \sqrt{Ai'^2(t)+Bi'^2(t)},\\ \phi(t) &= \arctan\left(Ai'(t)/Bi'(t)\right), \end{aligned}$$

and the equation (12) is written as

$$\sin\phi\left(-\lambda^{DN}F^{-\frac{2}{3}}\right)\exp\left(\frac{2}{3}\left[F^{\frac{1}{3}}\left(d-\frac{\lambda^{DN}}{F}\right)\right]^{\frac{3}{2}}\right)-\frac{1}{2}\cos\phi\left(-\lambda^{DN}F^{-\frac{2}{3}}\right)\times\exp\left(-\frac{2}{3}\left[F^{\frac{1}{3}}\left(d-\frac{\lambda^{DN}}{F}\right)\right]^{\frac{3}{2}}\right)=0.$$

So, we have

$$\tan\phi\left(-\lambda^{DN}F^{-\frac{2}{3}}\right)=\frac{1}{2}\exp\left(-\frac{4}{3}\left[F^{\frac{1}{3}}\left(d-\frac{\lambda^{DN}}{F}\right)\right]^{\frac{3}{2}}\right). \quad (46)$$

As $\exp\left(-\frac{4}{3}\left[F^{\frac{1}{3}}\left(d-\frac{\lambda^{DN}}{F}\right)\right]^{\frac{3}{2}}\right)$ has limit 0 when $F\to+\infty$, the last equation becomes

$$\tan\phi\left(-\lambda^{DN}F^{-\frac{2}{3}}\right)=0.$$

As a result, we get:

$$Ai'(-\lambda^{DN}F^{-\frac{2}{3}}) = 0.$$

Therefore, we obtain

$$-\lambda^{DN}F^{-\frac{2}{3}} = \alpha'_n, \quad n \in \{1, 2, 3, \cdots\},$$

where α'_n is the n-th negative zero of the Airy function derivative Ai'. This yields a sequence of eigenvalues given by

$$\lambda_n^{DN} = -\alpha'_n F^{\frac{2}{3}}, \quad n \in \{1, 2, 3, \cdots\}. \tag{47}$$

3 Conclusion

We obtain asymptotic representations for eigenvalues of the Stark operators with various boundary conditions. The formulas are presented in simple forms which highlights the dependence on F. The perturbation theory previously used in this context covers only the case $F \to 0$. The WKB method used to get the asymptotics of eigenvalues and eigenfunctions of differential equations have their inherent drawback because it gives good approximation only for large eigenvalues i.e, when $\lambda_n^{D,DN} \to +\infty$. So, here one gets another novelty of the paper which is that our method allows one to estimate the eigenvalues for both the strong ($F \to +\infty$) and weak electric filed $(F \to 0)$ and for any order.

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