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Holmgren problem for Helmholtz equation with the three singular coefficients

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Abstract. Fundamental solutions for a multidimensional Helmholtz equation with three singular coefficients have been constructed recently which are expressed in terms of the confluent hypergeometric function in four variables. In this paper, we study the Holmgren problem for a 3D elliptic equation with three singular coefficients. A unique solution of the problem is obtained in the explicit form.

Keywords. Holmgren problem, multidimensional elliptic equations with three singular coefficients, decomposition formulas, Lauricella hypergeometric function in three variables, confluent hypergeometric function in four variables.

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1 Introduction

In the monograph of Gilbert [8], by applying a method of complex analysis, integral representation of solutions of the generalized bi-axially Helmholtz equation

$$H^{\lambda}_{\alpha\beta}(u) = u_{xx} + u_{yy} + \frac{2\alpha}{x}u_x + \frac{2\beta}{y}u_y - \lambda^2 u = 0,\ 0 < 2\alpha,\ 2\beta < 1 \quad (1)$$

are constructed through analytic functions. The obtained formula contains rather bulky series and is inconvenient in applications. In article [2], the equation (1) was considered in two cases: when $\alpha = 0$, $\beta > 0$ and $\lambda = 0$, $\beta > 0$.

There are many works in which some problems for the modified equation (1) were studied [7, 15, 17, 20, 25]. In paper [9], for generalized bi-axially Helmholtz equation (1) four fundamental solutions in $R_2^+ = \{(x, y) : x > 0, y > 0\}$ were found and in works [21–23] the Neumann-Dirichlet type boundary value problems in the first quarter of the circle were solved.

Dirichlet and Dirichlet-Neumann (Holmgren) problems for elliptic equation with one singular coefficient in some part of ball were investigated by Agostinelli [1] and Olevskii [19]. Fundamental solutions for the following three (and more)-

dimensional elliptic equations with two and three singular coefficients

$$u_{xx}+u_{yy}+u_{zz}+\frac{2\alpha}{x}u_x+\frac{2\beta}{y}u_y=0,\ 0<2\alpha,\,2\beta<1, \tag{2}$$

$$u_{xx}+u_{yy}+u_{zz}+\frac{2\alpha}{x}u_x+\frac{2\beta}{y}u_y+\frac{2\gamma}{z}u_z=0,\ 0<2\alpha,\,2\beta,2\gamma<1 \tag{3}$$

and

$$\sum_{i=1}^{p}u_{x_ix_i}+\frac{2\alpha}{x_1}u_{x_1}+\frac{2\beta}{x_2}u_{x_2}+\frac{2\gamma}{x_3}u_{x_3}-\lambda^2u=0,$$
$$p\geq 3,\ 0<2\alpha,\,2\beta,2\gamma<1,\ \lambda\in R$$

were constructed, respectively, in [14], [10] and [6], where $\alpha,\ \beta,\gamma$ and λ are real numbers. For equations (2) and (3), the Dirichlet, Neumann and Holmgren problems were solved in some parts of the ball [13, 18, 24].

In this paper, we study the Holmgren problem for the equation

$$H^{\lambda}_{\alpha\beta\gamma}(u)=u_{xx}+u_{yy}+u_{zz}+\frac{2\alpha}{x}u_x+\frac{2\beta}{y}u_y+\frac{2\gamma}{z}u_z-\lambda^2u=0, \tag{4}$$

where $\alpha,\ \beta,\gamma$ and λ are real numbers with $0<2\alpha,\,2\beta,2\gamma<1$.

2 Preliminaries

Below, we give some formulas for Euler gamma-function, Gauss hypergeometric function, Lauricella hypergeometric function in three variables and a confluent hypergeometric function in four variables, which will be used in the next sections.

It is known that the Euler gamma-function $\Gamma(a)$ has the following properties [5, pp. 17-19, (2), (10), (15)]:

$$\Gamma(a+m)=\Gamma(a)\,(a)_m\,,\quad \Gamma\left(a+\frac{1}{2}\right)=\frac{\sqrt{\pi}\Gamma(2a)}{2^{2a-1}\Gamma(a)},\quad \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}.$$

Here $(a)_m$ is a Pochhammer symbol, for which the equality

$$(a)_{m+n}=(a)_m\,(a+m)_n$$

is true [5, p.67, (5)].

A function

$$F(a,b;c;x)\equiv F\left[\begin{array}{c}a,b;\\ c;\end{array}x\right]:=\sum_{m=0}^{\infty}\frac{(a)_m\,(b)_m}{(c)_m\,m!}x^m,\quad |x|<1$$

is known as the Gauss hypergeometric function and the equality

$$F(a,b;c;1)=\frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)},$$
$$[c\neq 0,-1,-2,...;\;\; Re\,(c-a-b)>0] \tag{5}$$

holds [5, c.73, (14)]. Moreover, the following autotransformer formula [5, p.76, (22)]

$$F(a,b;c;x)=(1-x)^{-b}\,F\left(c-a,b;c;\frac{x}{x-1}\right) \tag{6}$$

is valid.

The confluent hypergeometric function in four variables has a form [6]

$$\mathrm{H}_A^{(3,1)}(a,b_1,b_2,b_3;c_1,c_2,c_3;x,y,z,t)$$
$$=\sum_{m,n,k,l=0}^{\infty}\frac{(a)_{m+n+k-l}(b_1)_m(b_2)_n(b_3)_k}{(c_1)_m(c_2)_n(c_3)_k}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^k}{k!}\frac{t^l}{l!},\,|x|+|y|+|z|<1. \tag{7}$$

The three-dimensional analogue of the function

$$\mathrm{H}_A^{(3,1)}(a,b_1,b_2,b_3;c_1,c_2,c_3;x,y,z,t)$$

was first introduced and investigated by Hasanov [9]:

$$A_2(a,b_1,b_2;c_1,c_2;x,y,t)$$
$$=\sum_{m,n,l=0}^{\infty}\frac{(a)_{m+n-l}(b_1)_m(b_2)_n}{(c_1)_m(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{t^l}{l!},\,|x|+|y|<1. \tag{8}$$

For a given multiple hypergeometric function, it is useful to find a decomposition formula which would express the multivariable hypergeometric function in terms of products of several simpler hypergeometric functions involving fewer variables. Burchnall and Chaundy [3,4] systematically presented a number of expansion and decomposition formulas for some double hypergeometric functions in series of simpler hypergeometric functions. For example, a Lauricella function in three variables is defined by [16]

$$F_A^{(3)}(a,b_1,b_2,b_3;c_1,c_2,c_3;x,y,z)=\sum_{l,m,n=0}^{\infty}\frac{(a)_{l+m+n}\,(b_1)_l\,(b_2)_m\,(b_1)_n}{(c_1)_l\,(c_2)_m\,(c_1)_n}\frac{x^l}{l!}\frac{y^m}{m!}\frac{z^n}{n!}$$

and the following decomposition formula holds true [11,12]

$$F_A^{(3)}(a;b_1,b_2,b_3;c_1,c_2,c_3;x,y,z)$$

$$
\begin{aligned}
&= \sum_{l,m,n=0}^{\infty} \frac{(a)_{l+m+n}(b_1)_{l+m}(b_2)_{l+n}(b_3)_{m+n}}{(c_1)_{l+m}(c_2)_{l+n}(c_3)_{m+n}} \frac{x^{l+m}}{n!}\frac{y^{l+n}}{m!}\frac{z^{m+n}}{l!} \\
&\times F(a+l+m, b_1+l+m; c_1+l+m; x) \\
&\times F(a+l+m+n, b_2+l+n; c_2+l+n; y) \\
&\times F(a+l+m+n, b_3+m+n; c_3+m+n; z).
\end{aligned} \tag{9}
$$

Lemma 2.1. [6]. *The following decomposition formula holds true*

$$
\begin{aligned}
&H_A^{(3,1)}(a; b_1, b_2, b_3; c_1, c_2, c_3; x, , y, z, t) \\
&= \sum_{s=0}^{\infty}\sum_{q=0}^{s}\sum_{I(3,s)} A(s,q)\, C_s^q \frac{(-1)^{s+q}(b_1)_i (b_2)_j (b_3)_k}{(1-a)_q (c_1)_i (c_2)_j (c_3)_k} \frac{x^i}{i!}\frac{y^j}{j!}\frac{z^k}{k!}\frac{t^q}{q!} \\
&\times F_A^{(3)}(a+s; b_1+i, b_2+j, b_3+k; c_1+i, c_2+j, c_3+k; x, y, z) \\
&\times {}_0F_1(1-a+q; -t),
\end{aligned} \tag{10}
$$

where

$$
A(s,q) = \begin{cases} 1, & \text{if } s=0 \text{ and } q=0, \\ q/s, & \text{if } s \geq 1 \text{ and } q \geq 0, \end{cases}
$$

$$
I(3,s) = \{(i,j,k) : i \geq 0, j \geq 0, k \geq 0, i+j+k=s\}.
$$

${}_0F_1(a;x) = \sum_{n=0}^{\infty} \frac{x^n}{(a)_n n!}$ *is a generalized hypergeometric function* [5, Chapter IV].

By virtue of the formulas (9) and (6), the expansion (10) yields

$$
\begin{aligned}
&\mathrm{H}_A^{(3,1)}(a; b_1, b_2, b_3; c_1, c_2, c_3; x, y, z, t) = (1-x)^{-b_1}(1-y)^{-b_2}(1-z)^{-b_3} \\
&\times \sum_{l,m,n,s=0}^{\infty}\sum_{q=0}^{s}\sum_{I(3,s)} A(s,q)\, s!\, C_s^q \\
&\times \frac{(-1)^{s+q}(a)_{l+m+n+s}(b_1)_{i+l+m}(b_2)_{j+l+n}(b_3)_{k+m+n}}{(a)_s (1-a)_q (c_1)_{i+l+m}(c_2)_{j+l+n}(c_3)_{k+m+n}\, i!j!k!n!m!l!q!} t^q \\
&\times \left(\tfrac{x}{1-x}\right)^{i+l+m} F\left(c_1-a-j-k, b_1+i+l+m; c_1+i+l+m; \tfrac{x}{x-1}\right) \\
&\times \left(\tfrac{y}{1-y}\right)^{j+l+n} F\left(c_2-a-i-k-m, b_2+j+l+n; c_2+j+l+n; \tfrac{y}{y-1}\right) \\
&\times \left(\tfrac{z}{1-z}\right)^{k+m+n} F\left(c_3-a-i-j-l, b_3+k+m+n; c_3+k+m+n; \tfrac{z}{z-1}\right) \\
&\times {}_0F_1(1-a+q; -t).
\end{aligned} \tag{11}
$$

Expansion (11) will be used for solving various boundary value problems for equation (4).

In the present paper, R_3^{3+} denotes $1/8$ part of the Euclidean space R^3:

$$R_3^{3+} := \{(x,y,z): \ x>0, y>0, z>0,\}.$$

All the fundamental solutions of equation (4) in the domain R_3^{3+} were found in [6], and we will use one of these solutions in the study of the problem:

$$q_0\left(x,y,z;\xi,\eta,\zeta\right) = \gamma_0\, r^{-2\alpha_0}\mathrm{H}_{\mathrm{A}}^{(3,1)}\left(\alpha_0,\alpha,\beta,\gamma;2\alpha,2\beta,2\gamma;\sigma\right), \qquad (12)$$

where

$$\sigma := \left(\sigma_1,\sigma_2,\sigma_3,\sigma_4\right),$$

$$0<2\alpha,2\beta,2\gamma<1, \quad \alpha_0 = \frac{1}{2}+\alpha+\beta+\gamma,$$

$$\gamma_0 = 2^{2\alpha_0-3}\frac{\Gamma\left(\alpha_0\right)}{\pi^{3/2}}\frac{\Gamma\left(\alpha\right)}{\Gamma\left(2\alpha\right)}\frac{\Gamma\left(\beta\right)}{\Gamma\left(2\beta\right)}\frac{\Gamma\left(\gamma\right)}{\Gamma\left(2\gamma\right)}, \qquad (13)$$

$$r^2 = (x-\xi)^2+(y-\eta)^2+(z-\zeta)^2, \quad r_1^2 = (x+\xi)^2+(y-\eta)^2+(z-\zeta)^2,$$

$$r_2^2 = (x-\xi)^2+(y+\eta)^2+(z-\zeta)^2, \quad r_3^2 = (x-\xi)^2+(y-\eta)^2+(z+\zeta)^2,$$

and

$$\sigma_k = 1-\frac{r_k^2}{r^2}(k=1,2,3), \sigma_4 = -\frac{\lambda^2}{4}r^2.$$

Here $\mathrm{H}_{\mathrm{A}}^{(3,1)}\left(\mathrm{a},\mathrm{b}_1,\mathrm{b}_2,\mathrm{b}_3;\mathrm{c}_1,\mathrm{c}_2,\mathrm{c}_3;\mathrm{x},\mathrm{y},\mathrm{z},\mathrm{t}\right)$ is a confluent hypergeometric function, defined in (7).

It is easy to verify that the fundamental solution $q_0\left(x,y,z;\xi,\eta,\zeta\right)$ has the property

$$\left.\left(x^{2\alpha}\frac{\partial q_0}{\partial x}\right)\right|_{x=0} = 0, \ \left.\left(y^{2\alpha}\frac{\partial q_0}{\partial y}\right)\right|_{y=0} = 0, \ \left.\left(z^{2\alpha}\frac{\partial q_0}{\partial z}\right)\right|_{z=0} = 0.$$

3 Formulation of the problem and uniqueness of the solution

Let $\Omega \subset R_3^+$ be a domain, bounded by planes

$$S_1 = \{(x,y,z): \ x=0, 0<y<b, \ 0<z<c\},$$

$$S_2 = \{(x,y,z): \ y=0, 0<x<a, \ 0<z<c\},$$

$$S_3 = \{(x,y,z): \ z=0, 0<x<a, \ 0<y<b\}$$

and by surface S which intersects domains $S_i(i=\overline{1,3})$. Lines of intersections are designated as $\Gamma_1 = S\cap S_1, \Gamma_2 = S\cap S_2, \Gamma_3 = S\cap S_3$ and $a,b,c = const. > 0$.

Holmgren problem. Find a function $u\left(\mathrm{x}\right)\in C\left(\bar{\Omega}\right)\cap C^{2}\left(\Omega\right)$, satisfying equation (4) in Ω and conditions

$$\left.\left(x^{2\alpha}\frac{\partial u}{\partial x}\right)\right|_{x=0}=\nu_1\left(y,z\right),\quad (y,z)\in S_1,\tag{14}$$

$$\left.\left(y^{2\beta}\frac{\partial u}{\partial y}\right)\right|_{y=0}=\nu_2\left(x,z\right),\quad (x,z)\in S_2,\tag{15}$$

$$\left.\left(z^{2\gamma}\frac{\partial u}{\partial z}\right)\right|_{z=0}=\nu_3\left(x,y\right),\quad (x,y)\in S_3,\tag{16}$$

$$\left.u\right|_{S}=\varphi\left(x,y,z\right),\quad (x,y,z)\in\bar{S},\tag{17}$$

where ν_k and φ are given functions, and, moreover, ν_k can reduce to an infinity of the order less than $1-2\alpha-2\beta-2\gamma$ on the boundaries of S_k.

One can readily check that an equality

$$x^{2\alpha}y^{2\beta}z^{2\gamma}\left[uH^{\lambda}_{\alpha\beta\gamma}(w)-wH^{\lambda}_{\alpha\beta\gamma}(u)\right]=\frac{\partial}{\partial x}\left[x^{2\alpha}y^{2\beta}z^{2\gamma}\left(uw_x-wu_x\right)\right]$$
$$+\frac{\partial}{\partial y}\left[x^{2\alpha}y^{2\beta}z^{2\gamma}\left(uw_y-wu_y\right)\right]+\frac{\partial}{\partial z}\left[x^{2\alpha}y^{2\beta}z^{2\gamma}\left(uw_z-wu_z\right)\right]$$

is true. Integrating both sides of above given equality on the domain Ω_ε and using the classical formula of Gauss-Ostrogradsky, we get

$$\int_{\Omega_\varepsilon}x^{2\alpha}y^{2\beta}z^{2\gamma}\left[uH^{\lambda}_{\alpha\beta\gamma}(w)-wH^{\lambda}_{\alpha\beta\gamma}(u)\right]dxdydz$$
$$=\int_{\partial\Omega_\varepsilon}x^{2\alpha}y^{2\beta}z^{2\gamma}(uw_x-wu_x)\cos\left(\mathbf{n},x\right)d\vartheta$$
$$+\int_{\partial\Omega_\varepsilon}x^{2\alpha}y^{2\beta}z^{2\gamma}(uw_y-wu_y)\cos\left(\mathbf{n},y\right)d\vartheta$$
$$+\int_{\partial\Omega_\varepsilon}x^{2\alpha}y^{2\beta}z^{2\gamma}(uw_z-wu_z)\cos\left(\mathbf{n},z\right)d\vartheta.$$

Here Ω_ε be a sub-domain of Ω at a distance $\varepsilon>0$ from its boundary $\partial\Omega_\varepsilon=S_1\cup S_2\cup S_3\cup S$ and

$$\frac{\partial}{\partial\mathbf{n}}=\cos\left(\mathbf{n},x\right)\cdot\frac{\partial}{\partial x}+\cos\left(\mathbf{n},y\right)\cdot\frac{\partial}{\partial y}+\cos\left(\mathbf{n},y\right)\cdot\frac{\partial}{\partial z},$$

where $\mathbf{n}$ is outer normal to $\partial\Omega$.

The following equality

$$\int_{\Omega_\varepsilon} x^{2\alpha}y^{2\beta}z^{2\gamma}uH^{\lambda}_{\alpha\beta\gamma}(u)dxdydz$$

$$+\int_{\Omega_\varepsilon} x^{2\alpha}y^{2\beta}z^{2\gamma}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2+\lambda^2u^2\right]$$

$$= \int_{\Omega_\varepsilon}\left[\frac{\partial}{\partial x}\left(x^{2\alpha}y^{2\beta}z^{2\gamma}u\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(x^{2\alpha}y^{2\beta}z^{2\gamma}u\frac{\partial u}{\partial y}\right)\right.$$
$$\left.+\frac{\partial}{\partial z}\left(x^{2\alpha}y^{2\beta}z^{2\gamma}u\frac{\partial u}{\partial z}\right)\right]dxdydz$$

is valid. Applying the formula of Gauss-Ostrogradsky to this equality and letting $\varepsilon\to 0$, we get

$$\int_{\Omega_\varepsilon} x^{2\alpha}y^{2\beta}z^{2\gamma}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2+\lambda^2u^2\right]dxdydz$$

$$=-\int_{S_1} y^{2\beta}z^{2\gamma}\tau_1\nu_1dS_1-\int_{S_2} x^{2\alpha}z^{2\gamma}\tau_2\nu_2dS_2$$
$$-\int_{S_3} x^{2\alpha}y^{2\beta}\tau_3\nu_3dS_3-\int_{S} x^{2\alpha}y^{2\beta}z^{2\gamma}\varphi\frac{\partial u}{\partial \mathbf{n}}dS, \qquad (18)$$

where

$$\tau_1 := u(x,y,z)|_{x=0},\ \tau_2 := u(x,y,z)|_{y=0},\ \tau_3 := u(x,y,z)|_{z=0}.$$

To prove the uniqueness of the solution, as usual, we suppose that the problem has two solutions v and w. Denoting $u=v-w$, it satisfies homogeneous Holmgren problem ($\nu_1=0$, $\nu_2=0$, $\nu_3=0, \varphi=0$). Further we have to prove that the homogeneous problem has only trivial solution. In this case, from (18) one can easily get

$$\int_{\Omega_\varepsilon} x^{2\alpha}y^{2\beta}z^{2\gamma}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2+\lambda^2u^2\right]dxdydz=0.$$

Hence, it follows that $u_x=u_y=u_z=u=0$. We conclude that $u\,(x,y,z)\equiv 0$ in $\overline{\Omega}$.

4 Existence of the solution

We prove the existence of the solution in a special case of the domain Ω in order to get the solution in an explicit form. With this aim, suppose that $a = b = c$ and S is a $1/8$ part of a sphere with radius $R = a$ and origin at the point $O(0,0,0)$. Let

$$\Omega = \left\{(x,y,z) : x^2 + y^2 + z^2 < R^2,\ x > 0, y > 0, z > 0\right\}.$$

We find a solution of the considered problem using Green's function method [20]. Therefore, first we give the definition of Green's function for the formulated problem.

Definition 4.1. We call the function $G\left(x,y,z;\xi,\eta,\zeta\right)$ as Green's function of the Holmgren problem, if it satisfies the following conditions:

- this function is a regular solution of equation (4) in the domain Ω, expect at the point (ξ,η,ζ), which is any fixed point of Ω,
- it satisfies boundary conditions

$$\left(x^{2\alpha}\frac{\partial G\left(x,y,z;\xi,\eta,\zeta\right)}{\partial x}\right)\bigg|_{x=0} = 0,\ \left(y^{2\beta}\frac{\partial G\left(x,y,z;\xi,\eta,\zeta\right)}{\partial y}\right)\bigg|_{y=0} = 0,$$

$$\left(z^{2\gamma}\frac{\partial G\left(x,y,z;\xi,\eta,\zeta\right)}{\partial z}\right)\bigg|_{z=0} = 0, \quad G\left(x,y,z;\xi,\eta,\zeta\right)\big|_S = 0,$$

- it can be represented as

$$G\left(x,y,z;\xi,\eta,\zeta\right) = q_0\left(x,y,z;\xi,\eta,\zeta\right) + q_0^*\left(x,y,z;\xi,\eta,\zeta\right), \tag{19}$$

  where $q_0\left(x,y,z;\xi,\eta,\zeta\right)$ is the fundamental solution found earlier (see formula (12)) and the function

$$q_0^*\left(x,y,z;\xi,\eta,\zeta\right) = -\left(\frac{a}{R_0}\right)^{2\alpha_0} q_0\left(x,y,z;\bar{\xi},\bar{\eta},\bar{\zeta}\right)$$

  is a regular solution of equation (4) in the domain Ω. Here,

$$\bar{\xi} = \frac{a^2}{R_0^2}\xi, \quad \bar{\eta} = \frac{a^2}{R_0^2}\eta, \quad \bar{\zeta} = \frac{a^2}{R_0^2}\zeta, \quad R_0^2 = \xi^2 + \eta^2 + \zeta^2.$$

Excise a small ball with its center at (ξ,η,ζ) and with radius $\rho > 0$ from the domain Ω. Designate the sphere of the excised ball as C_ρ and by Ω_ρ denote the

remaining part of Ω.
Applying formula (18), we obtain

$$\int_{C_\rho} x^{2\alpha}y^{2\beta}z^{2\gamma}\left[u\frac{\partial G}{\partial \mathbf{n}} - G\frac{\partial u}{\partial \mathbf{n}}\right]dC_\rho$$

$$= -\int_{S_1} y^{2\beta}z^{2\gamma}G\left(0,y,z;\xi,\eta,\zeta\right)\nu_1 dS_1 - \int_{S_2} x^{2\alpha}z^{2\gamma}G\left(x,0,z;\xi,\eta,\zeta\right)\nu_2 dS_2$$

$$-\int_{S_3} x^{2\alpha}y^{2\beta}G\left(x,y,0;\xi,\eta,\zeta\right)\nu_3 dS_3 - \int_S x^{2\alpha}y^{2\beta}z^{2\gamma}\varphi\frac{\partial u}{\partial \mathbf{n}}dS. \qquad (20)$$

First, we consider the integral

$$\int_{C_\rho} x^{2\alpha}y^{2\beta}z^{2\gamma}u\frac{\partial G}{\partial \mathbf{n}}dC_\rho.$$

Taking (19) into account we rewrite it as follows

$$\int_{C_\rho} x^{2\alpha}y^{2\beta}z^{2\gamma}u\frac{\partial G}{\partial \mathbf{n}}dC_\rho \equiv \int_{C_\rho} x^{2\alpha}y^{2\beta}z^{2\gamma}u\frac{\partial q_0\left(x,y,z;\xi,\eta,\zeta\right)}{\partial \mathbf{n}}dC_\rho$$

$$+\int_{C_\rho} x^{2\alpha}y^{2\beta}z^{2\gamma}u\frac{\partial q_0^*\left(\mathrm{x},\mathrm{y},\mathrm{z};\xi,\eta,\zeta\right)}{\partial \mathbf{n}}dC_\rho \equiv I_1 + I_2.$$

Using the differentiation formula, we get

$$\frac{\partial^{i+j+k+l}}{\partial\sigma_1^i\partial\sigma_2^j\partial\sigma_3^k\partial\sigma_4^l}\mathrm{H}_A^{(3,1)}\left(a,b_1,b_2,b_3;c_1,c_2,c_3;\sigma\right) = \frac{(a)_{i+j+k-l}\left(b_1\right)_i\left(b_2\right)_j\left(b_3\right)_k}{\left(c_1\right)_i\left(c_2\right)_j\left(c_3\right)_k}$$

$$\times\mathrm{H}_A^{(3,1)}\left(a+i+j+k-l,b_1+i,b_2+j,b_3+k;;c_1+i,c_2+j,c_3+k;\sigma\right)$$

and the following an adjacent relation

$$\frac{ab_1}{c_1}\sigma_1\mathrm{H}_A^{(3,1)}(a+1,b_1+1,b_2,b_3;c_1+1,c_2,c_3;\,\sigma)$$

$$+\frac{ab_2}{c_2}\sigma_2\mathrm{H}_A^{(3,1)}(a+1,b_1,b_2+1,b_3;c_1,c_2+1,c_3;\,\sigma)$$

$$+\frac{ab_3}{c_3}\sigma_3\mathrm{H}_A^{(3,1)}(a+1,b_1,b_2,b_3+1;c_1,c_2,c_3+1;\,\sigma)$$

$$-\frac{1}{a-1}\sigma_4\mathrm{H}_A^{(3,1)}(a-1,b_1,b_2,b_3;c_1,c_2,c_3;\,\sigma)$$

$$= a\mathrm{H}_A^{(3,1)}(a+1,b_1,b_2,b_3;c_1,c_2,c_3;\,\sigma) - a\mathrm{H}_A^{(3,1)}(a,b_1,b_2,b_3;c_1,c_2,c_3;\,\sigma).$$

We define

$$\frac{\partial q_0}{\partial \mathbf{n}} = \frac{\partial q_0}{\partial x} \cdot \cos\left(\mathbf{n},\, x\right) + \frac{\partial q_0}{\partial y} \cdot \cos\left(\mathbf{n},\, y\right) + \frac{\partial q_0}{\partial z} \cdot \cos\left(\mathbf{n},\, z\right). \tag{21}$$

Indeed, substituting following three formulas of differentiation

$$\begin{aligned}
&\frac{\partial q_0\left(x,y,z;\xi,\eta,\zeta\right)}{\partial x}\\
&= -2\alpha_0\gamma_0\, \xi r^{-2\alpha_0-2}\mathrm{H}_A^{(3,1)}(\alpha_0+1, 1+\alpha, \beta, \gamma; 1+2\alpha, 2\beta, 2\gamma;\sigma)\\
&-2\alpha_0\gamma_0\, \left(x-\xi\right) r^{-2\alpha_0-2}\mathrm{H}_A^{(3,1)}(\alpha_0+1, \alpha, \beta, \gamma; 2\alpha, 2\beta, 2\gamma;\sigma),
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial q_0\left(x,y,z;\xi,\eta,\zeta\right)}{\partial y}\\
&= -2\alpha_0\gamma_0\, \eta r^{-2\alpha_0-2}\mathrm{H}_A^{(3,1)}(\alpha_0+1, \alpha, 1+\beta, \gamma; 2\alpha, 1+2\beta, 2\gamma;\sigma)\\
&-2\alpha_0\gamma_0\, \left(y-\eta\right) r^{-2\alpha_0-2}\mathrm{H}_A^{(3,1)}(\alpha_0+1, \alpha, \beta, \gamma; 2\alpha, 2\beta, 2\gamma;\sigma),
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial q_0\left(x,y,z;\xi,\eta,\zeta\right)}{\partial z}\\
&= -2\alpha_0\gamma_0\, \zeta r^{-2\alpha_0-2}\mathrm{H}_A^{(3,1)}(\alpha_0+1, \alpha, \beta, 1+\gamma; 2\alpha, 2\beta, 1+2\gamma;\sigma)\\
&-2\alpha_0\gamma_0\, \left(z-\zeta\right) r^{-2\alpha_0-2}\mathrm{H}_A^{(3,1)}(\alpha_0+1, \alpha, \beta, \gamma; 2\alpha, 2\beta, 2\gamma;\sigma)
\end{aligned}$$

in (21), we get

$$\begin{aligned}
&\frac{\partial q_0}{\partial \mathbf{n}} = -\alpha_0\gamma_0\, r^{-2\alpha_0}\mathrm{H}_A^{(3,1)}(\alpha_0+1, \alpha, \beta, \gamma; 2\alpha, 2\beta, 2\gamma;\sigma)\frac{\partial}{\partial \mathbf{n}}\left[\ln r^2\right]\\
&+\Phi\left(x,y,z;\xi,\eta,\zeta\right),
\end{aligned}$$

where

$$\begin{aligned}
&\mathbf{\Phi}\left(x,y,z;\xi,\eta,\zeta\right)\\
&= -2\alpha_0\gamma_0\, \xi r^{-2\alpha_0-2}\mathrm{H}_A^{(3,1)}(\alpha_0+1, 1+\alpha, \beta, \gamma; 1+2\alpha, 2\beta, 2\gamma;\sigma)\cos\left(\mathbf{n};x\right)\\
&-2\alpha_0\gamma_0\, \eta r^{-2\alpha_0-2}\mathrm{H}_A^{(3,1)}(\alpha_0+1, \alpha, 1+\beta, \gamma; 2\alpha, 1+2\beta, 2\gamma;\sigma)\cos\left(\mathbf{n};y\right)\\
&-2\alpha_0\gamma_0\, \zeta r^{-2\alpha_0-2}\mathrm{H}_A^{(3,1)}(\alpha_0+1, \alpha, \beta, 1+\gamma; 2\alpha, 2\beta, 1+2\gamma;\sigma)\cos\left(\mathbf{n};z\right).
\end{aligned}$$

We separate the left part of identity (20) on three integrals

$$\int_{C_\rho} x^{2\alpha}y^{2\beta}z^{2\gamma}u\frac{\partial G}{\partial \mathbf{n}}dC_\rho = J_1 + J_2 + J_3,$$

where

$$\begin{aligned}
J_1 \;=\; &-\alpha_0\gamma_0\int_{C_\rho} x^{2\alpha}y^{2\beta}z^{2\gamma}u\, r^{-2\alpha_0}\\
&\times\mathrm{H}_A^{(3,1)}(\alpha_0+1, \alpha, \beta, \gamma; 2\alpha, 2\beta, 2\gamma;\sigma)\frac{\partial}{\partial \mathbf{n}}\left[\ln r^2\right]dC_\rho,
\end{aligned}$$

$$J_2 = \int_{C_\rho} x^{2\alpha} y^{2\beta} z^{2\gamma} u \Phi(x,y,z;\xi,\eta,\zeta)\, dC_\rho,$$

$$J_3 = \int_{C_\rho} x^{2\alpha} y^{2\beta} z^{2\gamma} u \frac{\partial q_1^*}{\partial \mathbf{n}} dC_\rho.$$

We use the following spherical system of coordinates [20]:

$$x = \xi + \rho\cos\varphi, y = \eta + \rho\sin\varphi\cos\psi, z = \zeta + \rho\sin\varphi\sin\psi,$$

$$[\rho \geq 0, 0 \leq \varphi \leq \pi, 0 \leq \psi \leq 2\pi].$$

Then, we have

$$J_1 = 2\alpha_0\gamma_0\rho^{-2\alpha_0+1} \int\limits_0^{2\pi} d\psi \int\limits_0^{\pi} (\xi + \rho\cos\varphi)^{2\alpha}$$

$$\times(\eta + \rho\sin\varphi\cos\psi)^{2\beta}(\zeta + \rho\sin\varphi\sin\psi)^{2\gamma}$$

$$\times u(\xi + \rho\cos\varphi, \eta + \rho\sin\varphi\cos\psi, \zeta + \rho\sin\varphi\sin\psi)$$

$$\times \mathrm{H}_{\mathrm{A}}^{(3,1)}(\alpha_0+1,\alpha,\beta,\gamma;2\alpha,2\beta,2\gamma;\sigma_{1\rho},\sigma_{2\rho},\sigma_{3\rho},\sigma_{4\rho})\,\sin\varphi \mathrm{d}\varphi,$$

where

$$\sigma_{1\rho} = -\frac{4(\xi+\rho\cos\varphi)\xi}{\rho^2}, \sigma_{2\rho} = -\frac{4(\eta+\rho\sin\varphi\cos\psi)\eta}{\rho^2},$$

$$\sigma_{3\rho} = -\frac{4(\zeta+\rho\sin\varphi\sin\psi)}{\rho^2}, \sigma_{4\rho} = -\frac{\lambda^2}{4}\rho^2.$$

First we evaluate $\mathrm{H}_{\mathrm{A}}^{(3,1)}$. Using the decomposition formula (11), we find

$$\mathrm{H}_{\mathrm{A}}^{(3,1)}(\alpha_0+1,\alpha,\beta,\gamma;2\alpha,2\beta,2\gamma;\sigma_{1\rho},\sigma_{2\rho},\sigma_{3\rho},\sigma_{4\rho})$$

$$= \rho^{2\alpha+2\beta+2\gamma} r_{1\rho}^{-2\alpha} r_{2\rho}^{-2\beta} r_{3\rho}^{-2\gamma} \cdot \aleph,$$

where

$$r_{1\rho}^2 = (2\xi + \rho\cos\varphi)^2 + \rho^2\sin^2\varphi,$$

$$r_{2\rho}^2 = \rho^2\cos^2\varphi + (2\eta + \rho\sin\varphi\cos\psi)^2 + \rho^2\sin^2\varphi\sin^2\psi,$$

$$r_{3\rho}^2 = \rho^2\cos^2\varphi + \rho^2\sin^2\varphi\cos^2\psi + (2\zeta + \rho\sin\varphi\cos\psi)^2,$$

$$\aleph = \sum_{l,m,n,s=0}^{\infty} \sum_{q=0}^{s} \sum_{I(3.s)} A(s,q)\, s! C_s^q$$

$$\times \frac{(-1)^{s+q}(1+\alpha_0)_{l+m+n+s}(\alpha)_{i+l+m}(\beta)_{j+l+n}(\gamma)_{k+m+n}}{(1+\alpha_0)_s(-\alpha_0)_q(2\alpha)_{i+l+m}(2\beta)_{j+l+m}(2\gamma)_{k+m+n}k!n!m!q!}$$

$$\times \left(\frac{\rho^2}{r_{1\rho}^2}-1\right)^{i+l+m} F\left[\begin{matrix} 2\alpha-\alpha_0-1-j-k, \alpha+i+l+m; \\ 2\alpha+i+l+m; \end{matrix} 1-\frac{\rho^2}{r_{1\rho}^2}\right]$$

$$\times \left(\frac{\rho^2}{r_{2\rho}^2}-1\right)^{j+l+n} F\left[\begin{matrix} 2\beta-\alpha_0-1-i-k-m, \beta+j+l+n; \\ 2\beta+j+l+n; \end{matrix} 1-\frac{\rho^2}{r_{2\rho}^2}\right]$$

$$\times \left(\frac{\rho^2}{r_{3\rho}^2}-1\right)^{k+m+n} F\left[\begin{matrix} 2\gamma-\alpha_0-1-i-j-l, \gamma+k+m+n; \\ 2\gamma+k+m+n; \end{matrix} 1-\frac{\rho^2}{r_{3\rho}^2}\right]$$

$$\times {}_0F_1\left(-\alpha_0+q; -\frac{\lambda^2}{4}\rho\right).$$

It is easy to see that when $\rho \to 0$, the function $\aleph$ becomes an expression that does not depend on x, y, z and ξ, η, ζ :

$$\lim_{\rho\to 0} \aleph = \sum_{l,m,n=0}^{\infty} \frac{(\alpha_0+1)_{l+m+n}(\alpha)_{l+m}(\beta)_{l+n}(\gamma)_{m+n}}{(2\alpha)_{l+m}(2\beta)_{l+n}(2\gamma)_{m+n}m!n!l!}$$

$$\times F(2\alpha-\alpha_0-1, \alpha+l+m; 2\alpha+l+m; 1)$$

$$\times F(2\beta-\alpha_0-1-m, \beta+l+n; 2\beta+l+n; 1)$$

$$\times F(2\gamma-\alpha_0-1-l, \gamma+m+n; 2\gamma+m+n; 1).$$

Now, applying summation formula (5) to each hypergeometric function $F(a, b, c; 1)$ in the last sum, we get

$$\lim_{\rho\to 0} \aleph = \Gamma\left(\frac{3}{2}\right) \frac{\Gamma(2\alpha)\Gamma(2\beta)\Gamma(2\gamma)}{\Gamma(\alpha_0+1)\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}.$$

If we choose γ_0 as (13), we then have

$$\lim_{\rho\to 0} J_1 = u(\xi, \eta, \zeta).$$

By similar evaluations, one can get that

$$\lim_{\rho\to 0} J_2 = \lim_{\rho\to 0} J_3 = 0.$$

If we consider the integral

$$\int\limits_{C_\rho} x^{2\alpha}y^{2\beta}z^{2\gamma}\cdot G(x,\xi)\frac{\partial u}{\partial n}dC_\rho,$$

and use the algorithm given above for evaluations, we can prove that

$$\lim_{\rho\to 0}\int\limits_{C_\rho} x^{2\alpha}y^{2\beta}z^{2\gamma}\cdot G(x,\xi)\frac{\partial u}{\partial n}dC_\rho = 0.$$

From (20), we can write the solution of the Holmgren problem as follows:

$$u(\xi,\eta,\zeta) = -\int\limits_{S_1} y^{2\beta}z^{2\gamma}G_1^*(y,z;\xi,\eta,\zeta)\nu_1(y,z)dS_1$$

$$-\int\limits_{S_2} x^{2\alpha}z^{2\gamma}G_2^*(x,z;\xi,\eta,\zeta)\nu_2(x,z)dS_2 - \int\limits_{S_3} x^{2\alpha}y^{2\beta}G_2^*(x,y;\xi,\eta,\zeta)\nu_3(x,y)dS_3$$

$$-\int\limits_{S} x^{2\alpha}y^{2\beta}z^{2\gamma}\frac{\partial G(x,y,z;\xi,\eta,\zeta)}{\partial n}\varphi(x,y,z)dS. \tag{22}$$

The particular values of Green's function are given by

$$G_1^*(y,z;\xi,\eta,\zeta) = 2\alpha_0\gamma_1\left\{\frac{A_2\left(\alpha_0,\beta,\gamma,2\beta,2\gamma;\eta_{01}^{(x)},\zeta_{01}^{(x)},\theta_{01}^{(x)}\right)}{\left[\xi^2+(\eta-y)^2+(\zeta-z)^2\right]^{\alpha_0}}\right.$$

$$\left.-\frac{A_2\left(\alpha_0,\beta,\gamma,2\beta,2\gamma;\eta_{02}^{(x)},\zeta_{02}^{(x)},\theta_{02}^{(x)}\right)}{\left[\left(a-\frac{y\eta}{a}\right)^2+\left(a-\frac{z\zeta}{a}\right)^2+\frac{\xi^2+\zeta^2}{a^2}y^2+\frac{\xi^2+\eta^2}{a^2}z^2-a^2\right]^{\alpha_0}}\right\},$$

$$G_2^*(x,z;\xi,\eta,\zeta) = 2\alpha_0\gamma_1\left\{\frac{A_2\left(\alpha_0,\alpha,\gamma,2\alpha,2\gamma;\xi_{01}^{(y)},\zeta_{01}^{(y)},\theta_{01}^{(y)}\right)}{\left[(\xi-x)^2+\eta^2+(\zeta-z)^2\right]^{\alpha_0}}\right.$$

$$\left.-\frac{A_2\left(\alpha_0,\alpha,\gamma,2\alpha,2\gamma;\xi_{02}^{(y)},\zeta_{02}^{(y)},\theta_{02}^{(y)}\right)}{\left[\left(a-\frac{x\xi}{a}\right)^2+\left(a-\frac{z\zeta}{a}\right)^2+\frac{\eta^2+\zeta^2}{a^2}x^2+\frac{\xi^2+\eta^2}{a^2}z^2-a^2\right]^{\alpha_0}}\right\},$$

$$G_3^*(x,y;\xi,\eta,\zeta) = 2\alpha_0\gamma_1 \left\{ \frac{A_2\left(\alpha_0,\alpha,\beta,2\alpha,2\beta;\xi_{01}^{(z)},\eta_{01}^{(z)},\theta_{01}^{(z)}\right)}{\left[(\xi-x)^2+(\eta-y)^2+\zeta^2\right]^{\alpha_0}} \right.$$

$$\left. - \frac{A_2\left(\alpha_0,\alpha,\beta,2\alpha,2\beta;\xi_{02}^{(z)},\eta_{02}^{(z)},\theta_{02}^{(z)}\right)}{\left[\left(a-\frac{x\xi}{a}\right)^2+\left(a-\frac{y\eta}{a}\right)^2+\frac{\eta^2+\zeta^2}{a^2}x^2+\frac{\xi^2+\zeta^2}{a^2}y^2-a^2\right]^{\alpha_0}} \right\},$$

where

$$\xi_{0i}^{(y)} = \xi_i|_{y=0},\ \xi_{0i}^{(z)} = \xi_i|_{z=0},\ \eta_{0i}^{(x)} = \eta_i|_{x=0},\ \eta_{0i}^{(z)} = \eta_i|_{z=0},$$

$$\zeta_{0i}^{(x)} = \zeta_i|_{x=0},\ \zeta_{0i}^{(y)} = \zeta_i|_{y=0},\ \xi_i = \mu_i x\xi,\ \eta_i = \mu_i y\eta,\ \zeta_i = \mu_i z\zeta,$$

$$\theta_{0i}^{(x)} = \theta_i|_{x=0},\ \theta_{0i}^{(y)} = \theta_i|_{y=0},\ \theta_{0i}^{(z)} = \theta_i|_{z=0},\ i=1,2,$$

$$\mu_1 = -\frac{4}{r^2},\ \mu_2 = -\frac{a^2}{R_0^2}\frac{4}{R^2},\ \theta_1 = -\frac{\lambda^2}{4}r^2,\ \theta_2 = -\frac{\lambda^2}{4}\frac{a^2}{R_0^2}R^2,$$

$$r^2 = (\xi-x)^2+(\eta-y)^2+(\zeta-z)^2,$$

$$\begin{aligned} R^2 \quad = \quad & \left(a-\frac{x\xi}{a}\right)^2+\left(a-\frac{y\eta}{a}\right)^2+\left(a-\frac{z\zeta}{a}\right)^2 \\ & +\frac{\eta^2+\zeta^2}{a^2}x^2+\frac{\xi^2+\zeta^2}{a^2}y^2+\frac{\xi^2+\eta^2}{a^2}z^2-2a^2. \end{aligned}$$

Here, $A_2\left(a,b_1,b_2;c_1,c_2;x,y,t\right)$ is confluent hypergeometric function, defined in (8).

Thus, the following theorem is proved.

Theorem 4.2. *The solution of the Holmgren problem with conditions (*14*)–(*17*) exists and is defined by formula (*22*).*

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