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A note on h-convex functions

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Abstract. In this work, we discuss the continuity of h-convex functions by introducing the concepts of h-convex curves (h-cord). Geometric interpretation of h-convexity is given. The fact that for a h-continuous function f, is being h-convex if and only if is h-midconvex is proved. Generally, we prove that if f is h-convex then f is h-continuous. A discussion regarding derivative characterization of h-convexity is also proposed.

Keywords. h-Convex function, Hölder continuous.

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1 Introduction

Let I be a real interval. A function $f : I \to \mathbb{R}$ is called convex iff

$$f(t\alpha + (1-t)\beta) \le tf(\alpha) + (1-t)f(\beta), \tag{1}$$

for all points $\alpha, \beta \in I$ and all $t \in [0,1]$. If $-f$ is convex then we say that f is concave. Moreover, if f is both convex and concave, then f is said to be affine.

In 1979, Breckner [3] introduced the class of s-convex functions (in the second sense), as follows:

Definition 1.1. Let $I \subseteq [0,\infty)$ and $s \in (0,1]$, a function $f : I \to [0,\infty)$ is s-convex function or that f belongs to the class $K_s^2(I)$ if for all $x, y \in I$ and $t \in [0,1]$ we have

$$f(tx + (1-t)y) \le t^s f(x) + (1-t)^s f(y). \tag{2}$$

In the last years, among others, the notion of s-convex functions is discriminated and starred. In literature a few papers devoted to study this type of convexity. The building theories of s-convexity as geometric and analytic tools are still under consideration, development and examine. Due to Hudzik and Maligranda (1994) [15], two senses of s-convexity $(0 < s \le 1)$ of real-valued functions are known in the literature, and given below.

Definition 1.2. A function $f : \mathbb{R}_+ \to \mathbb{R}$, where $\mathbb{R}_+ = [0,\infty)$, is said to be s-convex in the first sense if

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y) \tag{3}$$

for all $x, y \in [0,\infty)$, $\alpha, \beta \geq 0$ with $\alpha^s + \beta^s = 1$ and for some fixed $s \in (0,1]$. This class of functions is denoted by K_s^1.

This definition of s-convexity, for so called φ-functions, was introduced by Orlicz in 1961 and was used in the theory of Orlicz spaces. A function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a φ-function if $f(0) = 0$ and f is nondecreasing and continuous. The symbol φ stands for an Orlicz function, i.e., φ is defined on the real line $\mathbb{R}$ with values in $[0,+\infty]$ and is convex, even, vanishing and continuous at zero. For further details see [15, 17, 18, 32].

Remark 1.3. We note that, it can be easily seen that for $s = 1$, s-convexity (in both senses) reduces to the ordinary convexity of functions defined on $[0,\infty)$.

In general, a real-valued function f defined on an open convex subset C of a linear space is called Breckner s-convex if (2) holds for every $x, y \in C$, $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$, where $s \in (0,1)$ is fixed. More preciously, Breckner considered an open convex subset $\mathbb{M}$ of a linear space $\mathbb{L}$ and defined $f : \mathbb{M} \subseteq \mathbb{L} \to \mathbb{R}$, to be s-convex if (2) holds, for all $x, y \in \mathbb{M}$, $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$, where $s \in (0,1)$ is fixed. Also, Breckner considered a special case of s-convex functions which is so called rational s-convex, that is for all rational $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$ and points $x, y \in \mathbb{M}$, the inequality (2) holds. Furthermore, Breckner proved that for locally bounded above s-convex functions defined on open subsets of linear topological spaces are continuous and nonnegative.

In 1978, Breckner and Orbán [4] studied functions defined on a convex subset of complex Hausdorff topological linear space of dimension greater than 1 into an ordered topological linear space such that all its order–bounded subsets are bounded, and proved that Breckner s-convex functions with $s \in (0,1]$ are continuous on the interior of their domain.

In 1994, Breckner [5] (see also [6]) proved that for a rationally s-convex function continuity and local s-Hölder continuity are equivalent at each interior point of the domain of definition of the function. Furthermore, it is shown that a rationally s-convex function which is bounded on a nonempty open convex set is s-Hölder continuous on every compact subset of this set. Indeed, Breckner [4], showed that if a real-valued function defined on a convex subset of a linear space endowed with topology generated by a direct pseudonorm is continuous and rationally Breckner s-convex for an $s \in (0,1]$, then it is locally s-Hölder.

In 1994, Hudzik and Maligranda [15], realized the importance and undertook a systematic study of s-convex functions in both senses. They compared the notion of Breckner s-convexity with a similar one of [18]. A function f is Orlicz s-convex if the inequality (3) is satisfied for all α, β such that $\alpha^s + \beta^s = 1$. Hudzik and Maligranda, among others, gave an example of a non-continuous Orlicz s-convex function, which is not Breckner s-convex.

In 2001, Pycia [24] established a direct proof of Breckner's result that Breckner s-convex real-valued functions on finite dimensional normed spaces are locally s-Hölder. The same result was proved in [1] where different context was considered. For the same result regarding convexity see [7, 8].

In the 2008, Pinheiro [25] studied the class of K_s^1 of s-convex functions and explained why the first s-convexity sense was abandoned by the literature in the field. In fact, Pinheiro , proposed some criticisms to the current way of presenting the definition of s-convex functions. We may summarize Pinheiro criticisms in the following points:

(i) What is the 'true' difference between convex and s-convex in both senses.

(ii) So far, Pinheiro did not find references, in the literature, to the geometry of an s-convex function, what, once more, makes it less clear to understand the difference between an s-convex and a convex function whilst there are clear references to the geometry of the convex functions.

In the same paper [25], Pinheiro revised the class of s–convexity in the first sense. In [26], Pinheiro proposed a geometric interpretation for this type of functions.

Definition 1.4. Let U be any subset of $[0, \infty)$. A function $f : X \to \mathbb{R}$, is said to be s–convex in the first sense if

$$f\left(\lambda x + (1-\lambda^s)^{1/s} y\right) \leq \lambda^s f(x) + (1-\lambda^s) f(y) \tag{4}$$

for all $x, y \in U$ and $\lambda \in [0, 1]$.

The presented reason from Pinheiro to why s-convexity in the first sense got abandoned in the literature, is that, if one takes $x = y = \frac{1}{4}$ with $\alpha = \frac{1}{2}$ and $\beta = 1$ for example, one gets that $\alpha x + \beta y = 0.125 + 0.25 = 0.375$. So that, if $s = \frac{1}{2}$, then the value of $\alpha x + \beta y$ would lie outside of the interval $[x, y]$, on the contrary of this, the value of $\alpha x + \beta y$ would lie inside of the interval $[x, y]$ in case of convexity. With this the first sense of s-convexity becomes a close to the meaning of convexity and so the geometric explanation of s-convex function is easy to be

compared with the geometry of convex function if some further restrictions are imposed to it.
The proposed geometric description for s-convex curve in the first sense stated by Pinheiro [25–30] as follows:

Definition 1.5. A function $f : X \subset \mathbb{R}_+ \to \mathbb{R}$ is called s-convex in the first sense if and only if one in two situations occur:

- $0 < s_1 < 1$, f then belonging to K_s^1 , for $0 < s \leq s_1$: The graph of f lies below (L), which is a convex curve between any two domain points with minimum distance of $(2^{-1} - 2^{-1/s})$ (domain points distance), that is, for every compact interval $J \subset I$, where length of J is greater than, or equal to $(2^{-1} - 2^{-1/s})$ interval with boundary ∂J, it is true that

$$\sup_J (L - f) \geq \sup_{\partial J} (L - f)$$

  and L is such that it is continuous, smooth, and, for each point x of L, defined in terms of ninety degrees intercepts with the straight line between the two points of the function, it is true that $1 \leq x \leq 2^{-1} + 2^{-s}$, where 1 corresponds to the straight line height;
- f is convex.

In general, the class of s-convex functions in the second sense would incomplete concept without a geometric interpretations for it is behavior. Recently, Pinheiro devoted her efforts to give a clear geometric definition for s-convexity in second sense. In [27] Pinheiro successfully proposed a geometric description for s-convex curve, as follows:

Definition 1.6. f is called s-convex in the second sense if and only if one in two situations occur:

- $0 < s_1 < 1$, f then belonging to K_s^2 , for $0 < s \leq s_1$: The graph of f lies below (L), which is a convex curve between any two domain points with minimum distance of $(2^{-s} - 2^{-1})$ (domain points distance), that is, for every compact interval $J \subset I$, where length of J is greater than, or equal to $(2^{-s} - 2^{-1})$ interval with boundary ∂J, it is true that

$$\sup_J (L - f) \geq \sup_{\partial J} (L - f)$$

  and L is such that it is continuous, smooth, and, for each point x of L, defined in terms of ninety degrees intercepts with the straight line between the two points of the function, it is true that $1 \leq x \leq 2^{1-s}$, where 1 corresponds to the straight line height;

- f is convex.

More geometrically, an interpretation of s-convex functions is introduced as follows:

Definition 1.7. f is called s–convex, $0 < s < 1$, $f \geq 0$, if the graph of f lies below a 'bent chord' L between any two points. That is, for every compact interval $J \subset I$, with boundary ∂J, it is true that

$$\sup_{J}(L-f) \geq \sup_{\partial J}(L-f).$$

Indeed the geometric view for s-convex mapping of second sense is going through which Pinheiro called it '*limiting curve*', which is going to distinguish curves that are s-convex of second sense from those that are not. After that, Pinheiro obtained how the choice of 's' affects the limiting curve. In general a 'limiting curve' may be described by a *bent chord* joining $f(x)$ to $f(y)$-corresponding to the verification of the s-convexity property of the function f in the interval $[x, y]$-forms representing the limiting height for the curve f to be at, limit included, in case f is s-convex. This curve is represented by $\lambda^s f(x) + (1-\lambda)^s f(y)$, for each $0 < s < 1$.
Some properties of the limiting curve such as: maximum height, length, and local inclination are considered in [26–29].

- **Height.** The maximum of the limiting s-curve is 2^{1-s}.
- **Length.** Let $f(\lambda) = \lambda^s X + (1-\lambda)^s Y$, with $X = f(x)$, and $Y = f(y)$. The size of the limiting curve from $f(x)$ to $f(y)$ is

$$L(\lambda) = \int_0^1 \sqrt{1 + s^2\lambda^{2s-2} + s^2(1-\lambda)^{2s-2} - 2s^2\lambda^{s-1}(1-\lambda)^{s-1}}d\lambda$$

  which shows that how bent is the limiting curve.

- **Local inclination.** The local inclination of the limiting curve may be founded by means of the first derivative, consider $f(\lambda) = \lambda^s f(x) + (1-\lambda)^s f(y)$, Therefore, the inclination is $f'(\lambda) = s\lambda^{s-1} f(x) - s(1-\lambda)^{s-1} f(y)$ and varies accordingly to the value of λ.

In 1985, E. K. Godunova and V. I. Levin (see [13] or [20, pp. 410-433]) introduced the following class of functions:

Definition 1.8. We say that $f : I \to \mathbb{R}$ is a Godunova-Levin function or that f belongs to the class $Q(I)$ if for all $x, y \in I$ and $t \in (0,1)$ we have

$$f(tx + (1-t)y) \le \frac{f(x)}{t} + \frac{f(y)}{1-t}.$$

In the same work, the authors proved that all nonnegative monotonic and nonnegative convex functions belong to this class. For related works see [12, 19].

In 1999, Pearce and Rubinov [23], established a new type of convex functions which is called P-functions.

Definition 1.9. We say that $f : I \to \mathbb{R}$ is P-function or that f belongs to the class $P(I)$ if for all $x, y \in I$ and $t \in [0,1]$ we have

$$f(tx + (1-t)y) \le f(x) + f(y).$$

Indeed, $Q(I) \supseteq P(I)$ and for applications it is important to note that $P(I)$ also consists only of nonnegative monotonic, convex and quasi-convex functions. A related work was considered in [12, 34].

In 2007, Varošanec [35] introduced the class of h-convex functions which generalize convex, s-convex, Godunova-Levin functions and P-functions. Namely, the h-convex function is defined as a non-negative function $f : I \to \mathbb{R}$ which satisfies

$$f(t\alpha + (1-t)\beta) \le h(t) f(\alpha) + h(1-t) f(\beta), \qquad (5)$$

where h is a non-negative function, $t \in (0,1) \subseteq J$ and $x, y \in I$, where I and J are real intervals such that $(0,1) \subseteq J$. Accordingly, some properties of h-convex functions were discussed in the same work of Varošanec. For more results; generalization, counterparts and inequalities regarding h-convexity see [2, 9–11, 14, 16, 22].

2 On h–convex functions

Throughout this work, I and J are two intervals subset of $(0,\infty)$ such that $(0,1) \subseteq J$ and $[a,b] \subseteq I$ with $0 < a < b$.

Definition 2.1. The h-cord joining any two points $(x, f(x))$ and $(y, f(y))$ on the graph of f is defined to be

$$L(t;h) := [f(y) - f(x)]\, h\left(\frac{t-x}{y-x}\right) + f(x), \qquad (6)$$

for all $t \in [x, y] \subseteq \mathcal{I}$. In particular, if $h(t) = t$ then we obtain the well known form of chord, which is

$$L(t;t) := \frac{f(y) - f(x)}{y - x}(t - x) + f(x).$$

It's worth to mention that, if $h(0) = 0$ and $h(1) = 1$, then $L(x;h) = f(x)$ and $L(y;h) = f(y)$, so that the h-cord L agrees with f at endpoints x, y, and this true for all such $x, y \in I$.

The h-convexity of a function $f : I \to \mathbb{R}$ means geometrically that the points of the graph of f are on or below the h-chord joining the endpoints $(x, f(x))$ and $(y, f(y))$ for all $x, y \in I$, $x < y$. In symbols, we write

$$f(t) \leq [f(y) - f(x)]\, h\left(\frac{t - x}{y - x}\right) + f(x) = L(t;h),$$

for any $x \leq t \leq y$ and $x, y \in I$.

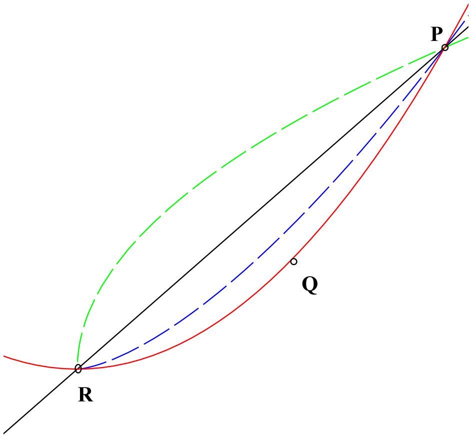

Figure 1. The graph of $h_k(t) = t^k$, $k = \frac{1}{2}, 1, \frac{3}{2}$ (green, black, blue), respectively, and $f(t) = t^2$ (red), $t \in [0, 1]$.

Hence, (5) means geometrically that for a given three non-collinear points P, Q and R on the graph of f with Q between P and R (say $P < Q < R$). Let h is super(sub)multiplicative and $h(\alpha) \geq (\leq)\, \alpha$, for $\alpha \in (0, 1) \subset J$. A function f is h–convex (concave) if Q is on or below (above) the h-chord $\widehat{PR}$ (see Figure 1).

Caution: In special case, for $h_k(t) = t^k$, $t \in (0,1)$ the proposed geometric interpretation is valid for $k \in (-1,0) \cup (0,\infty)$. In the case that $k \leq -1$ or $k = 0$ the geometric meaning is inconclusive so we exclude this case (and (and similar cases) from our proposal above.

Definition 2.2. Let $h : J \to (0,\infty)$ be a non-negative function. Let $f : I \to \mathbb{R}$ be any function. We say f is h-midconvex (h-midconcave) if

$$f\left(\frac{x+y}{2}\right) \leq (\geq)\, h\left(\frac{1}{2}\right)[f(x) + f(y)]$$

for all $x, y \in I$.

In particular, f is locally h-midocnvex if and only if

$$h\left(\frac{1}{2}\right)[f(x+p) + f(x-p)] - f(x) \geq 0,$$

for all $x \in (x-p, x+p)$, $p > 0$.

A generalization of Jensen characterization of convex functions could be stated as follows:

Theorem 2.3. *Let $h : J \to (0,\infty)$ be a non-negative function such that $h(\alpha) \geq \alpha$, for all $\alpha \in (0,1)$. Let $f : I \to \mathbb{R}_+$ be a nonnegative continuous function. f is h-convex if and only if it is h-midconvex; i.e., the inequality*

$$f\left(\frac{x+y}{2}\right) \leq h\left(\frac{1}{2}\right)[f(x) + f(y)],$$

holds for all $x, y \in I$.

Proof. The first direction follows directly by definition of h-convexity. To prove the second direction, suppose on the contrary that f is not h-convex. Then, there exists a subinterval $[x, y]$ such that the graph of f is not under the chord joining $(x, f(x))$ and $(y, f(y))$; that is,

$$f(t) \geq [f(y) - f(x)]\, h\left(\frac{t-x}{y-x}\right) + f(x) = L(t;h),$$

for all such $x, y \in I \cap J$. In other words, the function

$$g(t) = f(t) - [f(y) - f(x)]\, h\left(\frac{t-x}{y-x}\right) - f(x), \qquad t \in I$$

satisfies $M = \sup\{g(t) : t \in [x, y]\} > 0$. Since $h(0) = 0$ and $h(1) = 1$, then $L(x; h) = f(x)$ and $L(y; h) = f(y)$, so that the h-cord L agrees with f at endpoints x, y. Thus, g is continuous and $g(x) = g(y) = 0$, direct computation shows that g is also mid h-convex. Setting $c = \inf\{t \in [x, y] : g(t) = M\}$, then necessarily $g(c) = M$ and $c \in (x, y)$. By the definition of c, for every $p > 0$ for which $c \pm p \in (a, b)$, we have $g(c - p) < g(c)$ and $g(c + p) < g(c)$, so that since $h(\alpha) \geq \alpha$, for all $\alpha \in (0, 1)$ we have

$$g(c - p) + g(c + p) < 2g(c) = \frac{1}{\frac{1}{2}} g(c) \leq \frac{1}{h\left(\frac{1}{2}\right)} g(c),$$

which contradicts the fact that g is mid h-convex. □

Corollary 2.4. *Let $h : J \to (0, \infty)$ be a non-negative function such that $h(\alpha) \leq \alpha$, for all $\alpha \in (0, 1)$. Let $f : I \to \mathbb{R}_+$ be a nonnegative continuous function. f is h-concave if and only if it is h-midconcave.*

We often need to know how fast limits are converging, and this allows us to control the remainder of a given function in a neighborhood of some point x_0. So that, we need to extend the concept of continuity. Fortunately, in control theory and numerical analysis, a function $h : J \subseteq [0, \infty) \to [0, \infty]$ is called a control function if

(i) h is nondecreasing,

(ii) $\inf_{\delta > 0} h(\delta) = 0$.

A function $f : I \to \mathbb{R}$ is h-continuous at x_0 if $|f(x) - f(x_0)| \leq h(|x - x_0|)$, for all $x \in I$. Furthermore, a function is continuous in x_0 if it is h-continuous for some control function h.

This approach leads us to refining the notion of continuity by restricting the set of admissible control functions.

For a given set of control functions $\mathcal{C}$ a function is $\mathcal{C}$-continuous if it is h-continuous for all $h \in \mathcal{C}$. For example the Hölder continuous functions of order $\alpha \in (0, 1]$ are defined by the set of control functions

$$\mathcal{C}_H^{(\alpha)}(h) = \{h | h(\delta) = H |\delta|^\alpha, H > 0\}.$$

In case $\alpha = 1$, the set $\mathcal{C}_H^{(1)}(h)$ contains all functions satisfying the Lipschitz condition.

Theorem 2.5. *Let $(0, 1) \subseteq J$, $h : J \to (0, \infty)$ be a control function which is super multiplicative such that $h(\alpha) \geq \alpha$ for each $\alpha \in (0, 1)$. Let I be a real interval, $a, b \in \mathbb{R}$ $(a < b)$ with a, b in I° (the interior of I). If $f : I \to \mathbb{R}$ is non-negative h-convex function on $[a, b]$, then f is h-continuous on $[a, b]$.*

Proof. Choose $\epsilon > 0$ be small enough such that $(a-\epsilon, b+\epsilon) \subseteq I$ and let

$$m_\epsilon := \inf \{f(x), x \in (a-\epsilon, b+\epsilon)\}$$

and

$$M_\epsilon := \sup \{f(x), x \in (a-\epsilon, b+\epsilon)\},$$

such that $h(\epsilon) = M_\epsilon - m_\epsilon$. If $x, y \in [a,b]$, such that $x = y + \frac{\epsilon}{|y-x|}(y-x)$ and $\lambda_\epsilon = \frac{|y-x|}{\epsilon+|y-x|}$. Then for $z \in [a-\epsilon, b+\epsilon]$, $y = \lambda_\epsilon z + (1-\lambda_\epsilon)x$, we have

$$\begin{aligned} f(y) &= f(\lambda_\epsilon z + (1-\lambda_\epsilon)x) \le \lambda_\epsilon f(z) + (1-\lambda_\epsilon) f(x) \\ &\le \lambda_\epsilon [f(z) - f(x)] + f(x) \le h(\lambda_\epsilon)[f(z) - f(x)] + f(x), \end{aligned}$$

which implies that $y = \lambda_\epsilon z + (1-\lambda_\epsilon)x$, we have

$$\begin{aligned} f(y) - f(x) \le h(\lambda_\epsilon)[f(z) - f(x)] &\le h(\lambda_\epsilon)(M_\epsilon - m_\epsilon) \\ &< h\left(\frac{|y-x|}{\epsilon}\right)(M_\epsilon - m_\epsilon) \\ &< \frac{h(|y-x|)}{h(\epsilon)}(M_\epsilon - m_\epsilon) \\ &= h(|y-x|). \end{aligned}$$

Since this is true for any $x, y \in [a,b]$, we conclude that

$$|f(y) - f(x)| \le h(|y-x|),$$

which shows that f is h-continuous on $[a,b]$ as desired. □

Another Proof. Alternatively, if one replaces the condition $h(\alpha) + h(1-\alpha) \le 1$ for each $\alpha \in (0,1)$ instead of $h(\alpha) \ge \alpha$ in Theorem 2.5. Then by repeating the same steps in the above proof, we have

$$\begin{aligned} f(y) = f(\lambda_\epsilon z + (1-\lambda_\epsilon)x) &\le h(\lambda_\epsilon) f(z) + h(1-\lambda_\epsilon) f(x) \\ &\le h(\lambda_\epsilon) f(z) + [1 - h(\lambda_\epsilon)] f(x) \\ (\text{since } \ h(1-\lambda_\epsilon) &\le 1 - h(\lambda_\epsilon)) \\ &= h(\lambda_\epsilon)[f(z) - f(x)] + f(x), \end{aligned}$$

which implies that $y = \lambda_\epsilon z + (1 - \lambda_\epsilon) x$, we have

$$\begin{aligned} f(y) - f(x) \le h(\lambda_\epsilon)[f(z) - f(x)] &\le h(\lambda_\epsilon)(M_\epsilon - m_\epsilon) \\ &< h\left(\frac{|y-x|}{\epsilon}\right)(M_\epsilon - m_\epsilon) \\ &< \frac{h(|y-x|)}{h(\epsilon)}(M_\epsilon - m_\epsilon) \\ &= h(|y-x|). \end{aligned}$$

Since this is true for any $x, y \in [a, b]$, we conclude that $|f(y) - f(x)| \le h(|y - x|)$, which shows that f is h-continuous on $[a, b]$. Surely, this is can be considered as an alternative proof of Theorem 2.5.

It's well known that if f is twice differentiable then f is convex if and only if $f'' \ge 0$. In a convenient way Pinheiro in [29] proposed that f is an s-convex (in the second sense) if and only if $f'' \ge 1 - 2^{1-s}$. Indeed, Pinheiro presented a “proof" to her result, however we can say without doubt that she introduced some good thoughts rather than formal mathematical proof. Following the same way in [29] and in viewing the presented discussion in the introduction we conjecture that:

Conjecture 2.6. Let $h : J \to (0, \infty)$ be a non-negative function such that $h(\alpha) \ge \alpha$, for all $\alpha \in (0, 1)$, and consider $f : I \to \mathbb{R}$ be a twice differentiable function. A function f is h-convex if and only if $f''(x) \ge 1 - 2h\left(\frac{1}{2}\right)$.

Bibliography

[1] M. W. Alomari, M. Darus, S. S. Dragomir and U. Kirmaci, On fractional differentiable s-convex functions, *Jordan J. Math and Stat.* **3(1)** (2010), 33–42.

[2] M. Bombardelli and S. Varošanec, Properties of h-convex functions related to the Hermite-Hadamard-Fejér inequalities, *Compute. Math. Applica.* **58(9)** (2009), 1869–1877.

[3] W. W. Breckner, Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer funktionen in topologischen linearen Räumen, *Publ. Inst. Math.* **23** (1978), 13–20 (in German).

[4] W. W. Breckner and G. Orban, Continuity Properties of Rationally s-convex Mappings with Values in an Ordered Topological Linear Space, Babes-Bolyai University, Cluj-Napocoi, 1978.

[5] W. W. Breckner, Hölder-continuity of certain generalized convex functions, *Optimization* **28** (1994), 201–209.

[6] W. W. Breckner, Rational s-convexity, a Generalized Jensen-convexity, Cluj University Press, Cluj-Napoca, 2011.

[7] S. Cobzas and I. Muntean, Continuous and locally Lipschitz convex functions, *Mathematica Rev. d'Anal. Numér. et de Théorie de I'Approx.*, Ser. Mathematica **18(41)** (1976), 41–51.

[8] S. Cobzas, On the Lipschitz properties of continuous convex functions, *Mathematlca Ret. d'Anal. Numér. et de Théorie de I'Approx.*, Ser. Mathemarica **21(44)** (1979), 123-125.

[9] M. V. Cortez, Relative strongly h-convex functions and integral inequalities, *Appl. Math. Inf. Sci. Lett.* **4(2)** (2016), 39–45.

[10] S. S. Dragomir, Inequalities of Jensen type for h-convex functions on linear spaces, *Math. Moravica* **19(1)** (2015), 107–121.

[11] S. S. Dragomir, Inequalities of Hermite-Hadamard type for h-convex functions on linear spaces, *Proyecciones J. Math.* **34(4)**, (2015) 323–341.

[12] S. S. Dragomir, J. Pečarić and L. E. Persson, Some inequalities of Hadamard type, *Soochow J. Math.* **21** (1995), 335–341.

[13] E. K. Godunova and V. I. Levin, Neravenstva dlja funkcii širokogo klassa, soderžaščego vypuklye, monotonnye i nekotorye drugie vidy funkcii, Vyčislitel. Mat. i. Mat. Fiz. Mežvuzov. Sb. Nauk. Trudov, MGPI, Moskva, 1985, pp. 138–142 (in Russian).

[14] A. Házy, Bernstein-doetsch type results for h-convex functions, *Math. Inequal. Appl.* **14(3)** (2011), 499–508.

[15] H. Hudzik and L. Maligranda, Some remarks on s-convex functions, *Aequationes Math.* **48** (1994), 100–111.

[16] M. Matłoka, On Hadamard's inequality for h-convex function on a disk, *Appl. Math. Comp.* **235** (2014), 118–123.

[17] J. Musielak, *Orlicz Spaces and Modular Spaces*, Lecture Notes in Mathematics, Springer-Verlag, Berlin Heidelberg, 1983.

[18] W. Matuszewska and W. Orlicz, A note on the theory of s-normed spaces of ψ-integrable functions, *Studia Math.* **21** (1981), 107–115.

[19] D. S. Mitrinović and J. Pečarić, Note on a class of functions of Godunova and Levin, *C. R. Math. Rep. Acad. Sci. Can.* **12** (1990), 33–36.

[20] D.S. Mitrinović, J. Pečarić and A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic, Dordrecht, 1993.

[21] C. P. Niculescu and L. E. Persson, *Convex Functions and Their Applications. A Contemporary Approach*, CMS Books Math., Vol. 23, Springer-Verlag, New York, 2006.

[22] A. Olbryś, Representation theorems for h-convexity, *J. Math. Anal. Appl* **426(2)** (2015), 986–994.

[23] C. E. M. Pearce and A. M. Rubinov, P-functions, quasi-convex functions and Hadamard-type inequalities, *J. Math. Anal. Appl.* **240** (1999), 92–104.

[24] M. Pycia, A direct proof of the s-Hölder continuity of Breckner s-convex functions, *Aequationes Math.* **61(1-2)** (2001), 128–130.

[25] M. R. Pinheiro, *Convexity Secrets*, Trafford Publishing, 2008.

[26] M. R. Pinheiro, Exploring the concept of s-convexity, *Aequationes Mathematicae* **74(3)** (2007), 201–209.

[27] M. R. Pinheiro,*Hudzik and Maligranda's s-convexity as a local approximation to convex functions*, Preprint, 2008.

[28] M. R. Pinheiro, *Hudzik and Maligranda's s-convexity as a local approximation to convex functions II*, Preprint, 2008.

[29] M. R. Pinheiro, *Hudzik and Maligranda's s-convexity as a local approximation to convex functions III*, Preprint, 2008.

[30] M. R. Pinheiro, H–H Inequality for s-Convex Functions, *Inter. J. P. Appl. Math.* **44(4)** (2008), 563–579.

[31] A. W. Roberts and D. E. Varberg, *Convex Functions*, Academic Press, New York, 1973.

[32] S. Rolewicz, *Metric Linear Spaces*, 2nd ed., PWN, Warsaw, 1984.

[33] T. Trif, Hölder continuity of generalized convex set-valued mappings, *J. Math. Anal. Appl.* **255** (2001), 44–57.

[34] K.-L. Tseng, G.-S. Yang and S. S. Dragomir, On quasi convex functions and Hadamard's inequality, *Demonsrtatio Mathematics* **XLI(2)** (2008), 323–335.

[35] S. Varošanec, On h-convexity, *J. Math. Anal. Appl.* **326** (2007), 303–311.

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