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On approximation of two-dimensional potential and singular operators

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Abstract. The purpose of this paper is the construction of second-order of accuracy quadrature formulas for the numerical calculation of the Vekua types two-dimensional potential and singular integral operators in the unit disk of complex plane. We propose quadrature formulas for these integrals which based on first-order spline approximation of two-dimensional function. MATLAB programs are used for numerical experiments in test examples.

Keywords. Spline approximation, potential integral, singular integral, quadrature formula, numeric integrals.

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1 Introduction

Two-dimensional potential and singular integrals in the circle of complex plane are used to solve widely range of models in applied mechanics and various branches of physics (see [1–13] and references therein).

In paper [3], the first-order of accuracy quadrature formulas for the numerical calculation of the following two-dimensional potential and singular integral operators in the unit disk of complex plane

$$T\left(\rho|z\right) = -\frac{1}{\pi}\iint\limits_{K} \frac{\rho(\zeta)}{\zeta - z} d\zeta,\ S\left(\rho|z\right) = -\frac{1}{\pi}\iint\limits_{K} \frac{\rho(\zeta)}{\left(\zeta - z\right)^2} d\zeta, z \in K,$$

where $K = \{z \in C : |z| \leq 1\}$ are considered. These operators were introduced by I. Vekua [13]. Some properties of operators $T\left(\rho|z\right)$ and $S\left(\rho|z\right)$ are described in [10, p. 198-210]. In particular,

$$\frac{\partial T\left(\rho|z\right)}{\partial z} = S\left(\rho|z\right), \frac{\partial T\left(\rho|z\right)}{\partial \overline{z}} = \rho(z), z \in K \tag{1}$$

are given. Our purpose in this paper is the construction of second-order of accuracy quadrature formulas for the numerical calculation of the operators $T(\rho|z)$ and $S(\rho|z)$.

Following [3], we will use grid sets for the unit disk K in the complex plane. First, we introduce the following notations:

$$\begin{aligned}
&[0,1]_\tau = \{r_k \mid r_k = k\tau,\ 1 \le k \le N,\ N\tau = 1\},\\
&[-\pi,\pi]_{h_k} = \{\theta_{k,m} \mid \theta_{k,m} = -\pi + mh_k,\ 0 \le m \le M_k,\\
&M_k h_k = 2\pi, M_k = 2k+1, 1 \le k \le N\}; K_\tau = \{z \in C : |z| < \tau\},\\
&K_{r_k} = \{z \in C : |z| < r_k\}, 1 \le k \le N; H_{r_k} = K_{r_{k+1}} - K_{r_k}.
\end{aligned}$$

These notations permit us to write

$$K = K_\tau \bigcup \bigcup_{k=1}^{N-1} H_{r_k} = K_\tau \bigcup \bigcup_{k=1}^{N-1} \bigcup_{m=0}^{M_k-1} D_{k,m},$$

where domain $D_{k,m}$ is defined by

$$\begin{aligned}
D_{k,m} &= \{\zeta | \zeta = re^{i\theta},\ r_k \le r \le r_{k+1},\ 1 \le k \le N-1,\\
&\quad \theta_{k,m} \le \theta \le \theta_{k,m+1}, 0 \le m \le M_k - 1\}
\end{aligned}$$

with boundary $\partial D_{k,m}$:

$$\begin{aligned}
\partial D_{k,m} &= \Gamma_{km} = \Gamma^1_{km} \cup \Gamma^2_{km} \cup \Gamma^3_{km} \cup \Gamma^4_{km},\\
\Gamma^1_{km} &= \left\{\zeta \,\middle|\, \zeta = re^{i\theta_{k,m+1}},\ r_k \le r \le r_{k+1}\right\},\\
\Gamma^2_{km} &= \left\{\zeta \,\middle|\, \zeta = r_{k+1}e^{i\theta},\ \theta_{k,m+1} \ge \theta \ge \theta_{k,m}\right\},\\
\Gamma^3_{km} &= \left\{\zeta \,\middle|\, \zeta = re^{i\theta_{k,m}},\ r_{k+1} \ge r \ge r_k\right\},\\
\Gamma^4_{km} &= \left\{\zeta \,\middle|\, \zeta = r_k e^{i\theta},\ \theta_{k,m} \le \theta \le \theta_{k,m+1}\right\}
\end{aligned}$$

and its four corner points are

$$\begin{aligned}
z^{k,m}_{0,0} &= r_k e^{i\theta_{k,m}},\ z^{k,m}_{1,0} = r_{k+1}e^{i\theta_{k,m}},\\
z^{k,m}_{1,1} &= r_{k+1}e^{i\theta_{k,m+1}},\ z^{k,m}_{0,1} = r_k e^{i\theta_{k,m+1}}
\end{aligned} \tag{2}$$

for any $0 \le m \le M_k, 1 \le k \le N$.

Second, we introduce grid points

$$z^{*k,m}_{0,0} = \left(r_k + \tfrac{\tau}{2}\right) e^{i\left(\theta_{k,m} + \frac{h_k}{2}\right)},\ z^{*k,m}_{1,0} = \left(r_{k+1} + \tfrac{\tau}{2}\right) e^{i\left(\theta_{k,m} + \frac{h_k}{2}\right)},$$

$$z^{*k,m}_{1,1} = \left(r_{k+1} + \tfrac{\tau}{2}\right) e^{i\left(\theta_{k,m+1} + \frac{h_k}{2}\right)},\ z^{*k,m}_{0,1} = \left(r_k + \tfrac{\tau}{2}\right) e^{i\left(\theta_{k,m+1} + \frac{h_k}{2}\right)}$$

in which the approximate values of T and S will be calculated and denote by $D^*_{k,m}$ corresponding small domains with four corner points. Thus, we have two sets of gridpoints $\Omega^{(1)}$ and $\Omega^{(2)}$ defined by

$$\Omega^{(1)} = \Omega^{(1)}_{\tau,h} = \left\{ z^{k,m}_{0,0},\, z^{k,m}_{1,0},\, z^{k,m}_{1,1},\, z^{k,m}_{0,1} \;|1 \le k \le N-1\,,\, 0 \le m \le M_k - 1 \right\},$$

$$\Omega^{(2)} = \Omega^{(2)}_{\tau,h} = \left\{ z^{*k,m}_{0,0},\, z^{*k,m}_{1,0},\, z^{*k,m}_{1,1},\, z^{*k,m}_{0,1} \;|1 \le k \le N-1\,,\, 0 \le m \le M_k - 1 \right\}.$$

2 Spline approximation

In this section, we describe approximation of complex valued function ρ by the first-order splines in two variables z and $\overline{z}$. From $z = x + iy$, $\overline{z} = x - iy$, it follows $x = \frac{z+\overline{z}}{2}, y = \frac{z-\overline{z}}{2i}$. So, applying first-order two dimensional splines in variables x and y, we have the following approximation (see [2, p.12, formula (1.16)]):

$$\rho(z) \approx \widehat{\rho}(z) = a_{km}\, z + b_{km}\, \overline{z} + c_{km}\,(z^2 - \overline{z}^2) + d_{km} \quad , \quad z \in D_{k,m}, \tag{3}$$

where unknown coefficients a_{km}, b_{km}, c_{km}, d_{km} are defined by given values of ρ at four corner points (2). Hence, we get linear system equations

$$\begin{aligned}
\rho\left(z^{k,m}_{0,0}\right) &= a_{km}\, z^{k,m}_{0,0} + b_{km}\, \overline{z^{k,m}_{0,0}} + c_{km}\left[\left(z^{k,m}_{0,0}\right)^2 - \overline{z^{k,m}_{0,0}}^2\right] + d_{km},\\
\rho\left(z^{k,m}_{1,0}\right) &= a_{km}\, z^{k,m}_{1,0} + b_{km}\, \overline{z^{k,m}_{1,0}} + c_{km}\left[\left(z^{k,m}_{1,0}\right)^2 - \overline{z^{k,m}_{1,0}}^2\right] + d_{km},\\
\rho\left(z^{k,m}_{1,1}\right) &= a_{km}\, z^{k,m}_{1,1} + b_{km}\, \overline{z^{k,m}_{1,1}} + c_{km}\left[\left(z^{k,m}_{1,1}\right)^2 - \overline{z^{k,m}_{1,1}}^2\right] + d_{km},\\
\rho\left(z^{k,m}_{0,1}\right) &= a_{km}\, z^{k,m}_{0,1} + b_{km}\, \overline{z^{k,m}_{0,1}} + c_{km}\left[\left(z^{k,m}_{0,1}\right)^2 - \overline{z^{k,m}_{0,1}}^2\right] + d_{km}.
\end{aligned} \tag{4}$$

Solving (2), we obtain coefficients a_{km}, b_{km}, c_{km}, and d_{km} as follows:

$$a_{k,m} = Q^{-1}_{k,m} \cdot \left(r^{k,m}_{0,0} \cdot \left(z^{k,m}_{0,1}\right)^2 \cdot \overline{z^{k,m}_{1,0}} - r^{k,m}_{0,0} \cdot \left(z^{k,m}_{1,0}\right)^2 \cdot \overline{z^{k,m}_{0,1}} \right.$$
$$-r^{k,m}_{0,1} \cdot \left(z^{k,m}_{0,0}\right)^2 \cdot \overline{z^{k,m}_{1,0}} + r^{k,m}_{0,1} \cdot \left(z^{k,m}_{1,0}\right)^2 \cdot \overline{z^{k,m}_{0,0}} + r^{k,m}_{1,0} \cdot \left(z^{k,m}_{0,0}\right)^2 \cdot \overline{z^{k,m}_{0,1}}$$

$$-r_{1,0}^{k,m}\cdot\left(z_{0,1}^{k,m}\right)^2\cdot\overline{z_{0,0}^{k,m}}-r_{0,0}^{k,m}\cdot\left(z_{0,1}^{k,m}\right)^2\cdot\overline{z_{1,1}^{k,m}}+r_{0,0}^{k,m}\cdot\left(z_{1,1}^{k,m}\right)^2\cdot\overline{z_{0,1}^{k,m}}$$

$$+r_{0,1}^{k,m}\cdot\left(z_{0,0}^{k,m}\right)^2\cdot\overline{z_{1,1}^{k,m}}-r_{0,1}^{k,m}\cdot\left(z_{1,1}^{k,m}\right)^2\cdot\overline{z_{0,0}^{k,m}}-r_{1,1}^{k,m}\cdot\left(z_{0,0}^{k,m}\right)^2\cdot\overline{z_{0,1}^{k,m}}$$

$$+r_{1,1}^{k,m}\cdot\left(z_{0,1}^{k,m}\right)^2\cdot\overline{z_{0,0}^{k,m}}+r_{1,1}^{k,m}\cdot\left(z_{0,1}^{k,m}\right)^2\cdot\overline{z_{0,0}^{k,m}}-r_{0,0}^{k,m}\cdot\left(z_{1,0}^{k,m}\right)^2\cdot\overline{z_{1,1}^{k,m}}$$

$$-r_{0,0}^{k,m}\cdot\left(z_{1,1}^{k,m}\right)^2\cdot\overline{z_{1,1}^{k,m}}+r_{1,0}^{k,m}\cdot\left(z_{1,1}^{k,m}\right)^2\cdot\overline{z_{0,0}^{k,m}}+r_{1,1}^{k,m}\cdot\left(z_{0,0}^{k,m}\right)^2\cdot\overline{z_{1,0}^{k,m}}$$

$$-r_{1,1}^{k,m}\cdot\left(z_{1,0}^{k,m}\right)^2\cdot\overline{z_{1,1}^{k,m}}-r_{0,1}^{k,m}\cdot\left(z_{1,0}^{k,m}\right)^2\cdot\overline{z_{1,1}^{k,m}}+r_{0,1}^{k,m}\cdot\left(z_{1,1}^{k,m}\right)^2\cdot\overline{z_{1,0}^{k,m}}$$

$$+r_{1,0}^{k,m}\cdot\left(z_{0,1}^{k,m}\right)^2\cdot\overline{z_{1,1}^{k,m}}-r_{1,0}^{k,m}\cdot\left(z_{1,1}^{k,m}\right)^2\cdot\overline{z_{0,1}^{k,m}}-r_{1,1}^{k,m}\cdot\left(z_{0,1}^{k,m}\right)^2\cdot\overline{z_{1,0}^{k,m}}$$

$$+r_{1,1}^{k,m}\cdot\left(z_{1,0}^{k,m}\right)^2\cdot\overline{z_{0,1}^{k,m}}+r_{0,0}^{k,m}\cdot\overline{z_{0,1}^{k,m}}.\left(\overline{z_{1,0}^{k,m}}\right)^2-r_{0,0}^{k,m}\cdot\left(\overline{z_{0,1}^{k,m}}\right)^2\cdot\overline{z_{1,0}^{k,m}}$$

$$-r_{0,1}^{k,m}\cdot\overline{z_{0,0}^{k,m}}\cdot\left(\overline{z_{1,0}^{k,m}}\right)^2+r_{0,1}^{k,m}\cdot\left(\overline{z_{0,0}^{k,m}}\right)^2\cdot\overline{z_{1,0}^{k,m}}+r_{1,0}^{k,m}\cdot\overline{z_{0,0}^{k,m}}\cdot\left(\overline{z_{0,1}^{k,m}}\right)^2$$

$$-r_{1,0}^{k,m}\cdot\left(\overline{z_{0,0}^{k,m}}\right)^2\cdot\left(\overline{z_{0,1}^{k,m}}\right)^2-r_{0,0}^{k,m}\cdot\overline{z_{0,1}^{k,m}}\left(\overline{z_{1,1}^{k,m}}\right)^2+r_{0,0}^{k,m}\cdot\left(\overline{z_{0,1}^{k,m}}\right)^2.\overline{z_{1,1}^{k,m}}$$

$$+r_{0,1}^{k,m}\cdot\overline{z_{0,0}^{k,m}}.\left(\overline{z_{1,1}^{k,m}}\right)^2-r_{0,1}^{k,m}\cdot\left(\overline{z_{0,0}^{k,m}}\right)^2\cdot\overline{z_{1,1}^{k,m}}-r_{1,1}^{k,m}\cdot\overline{z_{0,0}^{k,m}}\cdot\left(\overline{z_{0,1}^{k,m}}\right)^2$$

$$+r_{1,1}^{k,m}\cdot\left(\overline{z_{0,0}^{k,m}}\right)^2\cdot\overline{z_{0,1}^{k,m}}+r_{0,0}^{k,m}\cdot\overline{z_{1,0}^{k,m}}.\left(\overline{z_{1,1}^{k,m}}\right)^2-r_{0,0}^{k,m}\cdot\left(\overline{z_{1,0}^{k,m}}\right)^2.\overline{z_{1,1}^{k,m}}$$

$$-r_{1,0}^{k,m}\cdot\overline{z_{0,0}^{k,m}}.\left(\overline{z_{1,1}^{k,m}}\right)^2+r_{1,0}^{k,m}\cdot\left(\overline{z_{0,0}^{k,m}}\right)^2.\overline{z_{1,1}^{k,m}}+r_{1,1}^{k,m}\cdot\overline{z_{0,0}^{k,m}}.\left(\overline{z_{1,0}^{k,m}}\right)^2$$

$$-r_{1,1}^{k,m}\cdot\left(\overline{z_{0,0}^{k,m}}\right)^2.\overline{z_{1,0}^{k,m}}-r_{0,1}^{k,m}\cdot\overline{z_{1,0}^{k,m}}.\left(\overline{z_{1,1}^{k,m}}\right)^2+r_{0,1}^{k,m}\cdot\left(\overline{z_{1,0}^{k,m}}\right)^2.\overline{z_{1,1}^{k,m}}$$

$$+r_{1,0}^{k,m}\cdot\overline{z_{0,1}^{k,m}}.\left(\overline{z_{1,1}^{k,m}}\right)^2-r_{1,0}^{k,m}\cdot\left(\overline{z_{0,1}^{k,m}}\right)^2.\overline{z_{1,1}^{k,m}}-r_{1,1}^{k,m}\cdot\overline{z_{0,1}^{k,m}}.\left(\overline{z_{1,0}^{k,m}}\right)^2$$

$$+r_{1,1}^{k,m}\cdot\left(\overline{z_{0,1}^{k,m}}\right)^2.\overline{z_{1,0}^{k,m}}\Bigg), \qquad (5)$$

$$b_{k,m}=Q_{k,m}^{-1}\cdot\left(r_{0,0}^{k,m}\cdot z_{0,1}^{k,m}\cdot\left(z_{1,0}^{k,m}\right)^2-r_{0,0}^{k,m}\cdot\left(z_{0,1}^{k,m}\right)^2\cdot z_{1,0}^{k,m}\right.$$

$$-r_{0,1}^{k,m}\cdot z_{0,0}^{k,m}\cdot\left(z_{1,0}^{k,m}\right)^2+r_{0,1}^{k,m}\cdot\left(z_{0,0}^{k,m}\right)^2\cdot z_{1,0}^{k,m}+r_{1,0}^{k,m}\cdot z_{0,0}^{k,m}\cdot\left(z_{0,1}^{k,m}\right)^2$$

$$-r_{1,0}^{k,m}\cdot\left(z_{0,0}^{k,m}\right)^2\cdot z_{0,1}^{k,m}-r_{0,0}^{k,m}\cdot z_{0,1}^{k,m}\cdot\left(z_{1,1}^{k,m}\right)^2+r_{0,0}^{k,m}\cdot\left(z_{0,1}^{k,m}\right)^2\cdot z_{1,1}^{k,m}$$

$$+r_{0,1}^{k,m}\cdot z_{0,0}^{k,m}\cdot\left(z_{1,1}^{k,m}\right)^2-r_{0,1}^{k,m}\cdot\left(z_{0,0}^{k,m}\right)^2\cdot z_{1,1}^{k,m}-r_{1,1}^{k,m}\cdot z_{0,0}^{k,m}\cdot\left(z_{0,1}^{k,m}\right)^2$$

$$
\begin{aligned}
&+r_{1,1}^{k,m}\cdot\left(z_{0,0}^{k,m}\right)^2\cdot z_{0,1}^{k,m}+r_{0,0}^{k,m}\cdot z_{1,0}^{k,m}\cdot\left(z_{1,1}^{k,m}\right)^2-r_{0,0}^{k,m}\cdot\left(z_{1,0}^{k,m}\right)^2\cdot z_{1,1}^{k,m}\\
&-r_{1,0}^{k,m}\cdot z_{0,0}^{k,m}\cdot\left(z_{1,1}^{k,m}\right)^2+r_{1,0}^{k,m}\cdot\left(z_{0,0}^{k,m}\right)^2\cdot z_{1,1}^{k,m}+r_{1,1}^{k,m}\cdot z_{0,0}^{k,m}\cdot\left(z_{1,0}^{k,m}\right)^2\\
&-r_{1,1}^{k,m}\cdot\left(z_{0,0}^{k,m}\right)^2\cdot z_{1,0}^{k,m}-r_{0,1}^{k,m}\cdot z_{1,0}^{k,m}\cdot\left(z_{1,1}^{k,m}\right)^2+r_{0,1}^{k,m}\cdot\left(z_{1,0}^{k,m}\right)^2\cdot z_{1,1}^{k,m}\\
&+r_{1,0}^{k,m}\cdot z_{0,1}^{k,m}\cdot\left(z_{1,1}^{k,m}\right)^2-r_{1,0}^{k,m}\cdot\left(z_{0,1}^{k,m}\right)^2\cdot z_{1,1}^{k,m}-r_{1,1}^{k,m}\cdot z_{0,1}^{k,m}\cdot\left(z_{1,0}^{k,m}\right)^2\\
&+r_{1,1}^{k,m}\cdot\left(z_{0,1}^{k,m}\right)^2\cdot z_{1,0}^{k,m}-r_{0,0}^{k,m}\cdot z_{0,1}^{k,m}\cdot\left(\overline{z_{1,0}^{k,m}}\right)^2+r_{0,0}^{k,m}\cdot z_{1,0}^{k,m}\cdot\left(\overline{z_{0,1}^{k,m}}\right)^2\\
&+r_{0,1}^{k,m}\cdot z_{0,0}^{k,m}\cdot\left(\overline{z_{1,0}^{k,m}}\right)^2-r_{0,1}^{k,m}\cdot z_{1,0}^{k,m}\cdot\left(\overline{z_{0,0}^{k,m}}\right)^2-r_{1,0}^{k,m}\cdot z_{0,0}^{k,m}\left(\overline{z_{0,1}^{k,m}}\right)^2\\
&+r_{1,0}^{k,m}\cdot z_{0,1}^{k,m}\cdot\left(\overline{z_{0,0}^{k,m}}\right)^2+r_{0,0}^{k,m}\cdot z_{0,1}^{k,m}\cdot\left(\overline{z_{1,1}^{k,m}}\right)^2-r_{0,0}^{k,m}\cdot z_{1,1}^{k,m}\cdot\left(\overline{z_{0,1}^{k,m}}\right)^2\\
&-r_{0,1}^{k,m}\cdot z_{0,0}^{k,m}\cdot\left(\overline{z_{1,1}^{k,m}}\right)^2+r_{0,1}^{k,m}\cdot z_{1,1}^{k,m}\cdot\left(\overline{z_{0,0}^{k,m}}\right)^2+r_{1,1}^{k,m}\cdot z_{0,0}^{k,m}\cdot\left(\overline{z_{0,1}^{k,m}}\right)^2\\
&-r_{1,1}^{k,m}\cdot z_{0,1}^{k,m}\cdot\left(\overline{z_{0,0}^{k,m}}\right)^2-r_{0,0}^{k,m}\cdot z_{1,0}^{k,m}\cdot\left(\overline{z_{1,1}^{k,m}}\right)^2+r_{0,0}^{k,m}\cdot z_{1,1}^{k,m}\cdot\left(\overline{z_{1,0}^{k,m}}\right)^2\\
&+r_{1,0}^{k,m}\cdot z_{0,0}^{k,m}\cdot\left(\overline{z_{1,1}^{k,m}}\right)^2-r_{1,0}^{k,m}\cdot z_{1,1}^{k,m}\cdot\left(\overline{z_{0,0}^{k,m}}\right)^2-r_{1,1}^{k,m}\cdot z_{0,0}^{k,m}\cdot\left(\overline{z_{1,0}^{k,m}}\right)^2\\
&+r_{1,1}^{k,m}\cdot z_{1,0}^{k,m}\cdot\left(\overline{z_{0,0}^{k,m}}\right)^2+r_{0,1}^{k,m}\cdot z_{1,0}^{k,m}\cdot\left(\overline{z_{1,1}^{k,m}}\right)^2-r_{0,1}^{k,m}\cdot z_{1,1}^{k,m}\cdot\left(\overline{z_{1,0}^{k,m}}\right)^2\\
&-r_{1,0}^{k,m}\cdot z_{0,1}^{k,m}\cdot\left(\overline{z_{1,1}^{k,m}}\right)^2+r_{1,0}^{k,m}\cdot z_{1,1}^{k,m}\cdot\left(\overline{z_{0,1}^{k,m}}\right)^2+r_{1,1}^{k,m}\cdot z_{0,1}^{k,m}\cdot\left(\overline{z_{1,0}^{k,m}}\right)^2\\
&-r_{1,1}^{k,m}\cdot z_{1,0}^{k,m}\cdot\left(\overline{z_{0,1}^{k,m}}\right)^2\Bigg),
\end{aligned}
\tag{6}
$$

$$
\begin{aligned}
c_{k,m}=-Q_{k,m}^{-1}\cdot\Big(&r_{0,0}^{k,m}\cdot z_{0,1}^{k,m}\cdot\overline{z_{1,0}^{k,m}}-r_{0,0}^{k,m}\cdot z_{1,0}^{k,m}\cdot\overline{z_{0,1}^{k,m}}-r_{0,1}^{k,m}\cdot z_{0,0}^{k,m}\cdot\overline{z_{1,0}^{k,m}}\\
&+r_{0,1}^{k,m}\cdot z_{1,0}^{k,m}\cdot\overline{z_{0,0}^{k,m}}+r_{1,0}^{k,m}\cdot z_{0,0}^{k,m}\cdot\overline{z_{0,1}^{k,m}}-r_{0,1}^{k,m}\cdot z_{0,1}^{k,m}\cdot\overline{z_{0,0}^{k,m}}-r_{0,0}^{k,m}\cdot z_{0,1}^{k,m}\cdot\overline{z_{1,1}^{k,m}}\\
&+r_{0,0}^{k,m}\cdot z_{1,1}^{k,m}\cdot\overline{z_{0,1}^{k,m}}+r_{0,1}^{k,m}\cdot z_{0,0}^{k,m}\cdot\overline{z_{1,1}^{k,m}}-r_{0,1}^{k,m}\cdot z_{1,1}^{k,m}\cdot\overline{z_{0,0}^{k,m}}-r_{1,1}^{k,m}\cdot z_{0,0}^{k,m}\cdot\overline{z_{0,1}^{k,m}}\\
&+r_{1,1}^{k,m}\cdot z_{0,1}^{k,m}\cdot\overline{z_{0,0}^{k,m}}+r_{0,0}^{k,m}\cdot z_{1,0}^{k,m}\cdot\overline{z_{1,1}^{k,m}}-r_{0,0}^{k,m}\cdot z_{1,1}^{k,m}\cdot\overline{z_{1,0}^{k,m}}-r_{1,0}^{k,m}\cdot z_{0,0}^{k,m}\cdot\overline{z_{0,1}^{k,m}}\\
&+r_{1,0}^{k,m}\cdot z_{1,1}^{k,m}\cdot\overline{z_{0,0}^{k,m}}+r_{1,1}^{k,m}\cdot z_{0,0}^{k,m}\cdot\overline{z_{1,0}^{k,m}}-r_{1,1}^{k,m}\cdot z_{1,0}^{k,m}\cdot\overline{z_{0,0}^{k,m}}-r_{0,1}^{k,m}\cdot z_{1,0}^{k,m}\cdot\overline{z_{1,0}^{k,m}}\\
&+r_{0,1}^{k,m}\cdot z_{1,1}^{k,m}\cdot\overline{z_{1,0}^{k,m}}+r_{1,0}^{k,m}\cdot z_{0,1}^{k,m}\cdot\overline{z_{1,1}^{k,m}}-r_{1,0}^{k,m}\cdot z_{1,1}^{k,m}\cdot\overline{z_{0,1}^{k,m}}-r_{1,1}^{k,m}\cdot z_{0,1}^{k,m}\cdot\overline{z_{1,0}^{k,m}}\\
&+r_{1,1}^{k,m}\cdot z_{1,0}^{k,m}\cdot\overline{z_{0,1}^{k,m}}\Big),
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
d_{k,m} = Q_{k,m}^{-1} \cdot \Big(& r_{0,0}^{k,m} \cdot z_{0,1}^{k,m} \cdot \left(z_{1,0}^{k,m}\right)^2 \cdot \overline{z_{1,1}^{k,m}} - r_{0,0}^{k,m} \cdot z_{0,1}^{k,m} \cdot \left(z_{1,1}^{k,m}\right)^2 \cdot \overline{z_{1,0}^{k,m}} \\
& + r_{0,0}^{k,m} \cdot z_{1,0}^{k,m} \cdot \left(z_{1,1}^{k,m}\right)^2 \cdot \overline{z_{0,1}^{k,m}} - r_{0,0}^{k,m} \cdot \left(z_{0,1}^{k,m}\right)^2 \cdot z_{1,0}^{k,m} \cdot \overline{z_{1,1}^{k,m}} \\
& + r_{0,0}^{k,m} \cdot \left(z_{0,1}^{k,m}\right)^2 \cdot z_{1,1}^{k,m} \cdot \overline{z_{1,0}^{k,m}} - r_{0,0}^{k,m} \cdot \left(z_{1,0}^{k,m}\right)^2 \cdot z_{1,1}^{k,m} \cdot \overline{z_{0,1}^{k,m}} \\
& - r_{0,1}^{k,m} \cdot z_{0,0}^{k,m} \cdot \left(z_{1,0}^{k,m}\right)^2 \cdot \overline{z_{1,1}^{k,m}} + r_{0,1}^{k,m} \cdot z_{0,0}^{k,m} \cdot \left(z_{1,1}^{k,m}\right)^2 \cdot \overline{z_{1,0}^{k,m}} \\
& - r_{0,1}^{k,m} \cdot z_{1,0}^{k,m} \cdot \left(z_{1,1}^{k,m}\right)^2 \cdot \overline{z_{0,0}^{k,m}} + r_{0,1}^{k,m} \cdot \left(z_{0,0}^{k,m}\right)^2 \cdot z_{1,0}^{k,m} \cdot \overline{z_{1,1}^{k,m}} \\
& - r_{0,1}^{k,m} \cdot \left(z_{0,0}^{k,m}\right)^2 \cdot z_{1,1}^{k,m} \cdot \overline{z_{1,0}^{k,m}} + r_{0,1}^{k,m} \cdot \left(z_{1,0}^{k,m}\right)^2 \cdot z_{1,1}^{k,m} \cdot \overline{z_{0,0}^{k,m}} \\
& + r_{1,0}^{k,m} \cdot z_{0,0}^{k,m} \cdot \left(z_{0,1}^{k,m}\right)^2 \cdot \overline{z_{1,1}^{k,m}} - r_{1,0}^{k,m} \cdot z_{0,0}^{k,m} \cdot \left(z_{1,1}^{k,m}\right)^2 \cdot \overline{z_{0,1}^{k,m}} \\
& + r_{1,0}^{k,m} \cdot z_{0,1}^{k,m} \cdot \left(z_{1,1}^{k,m}\right)^2 \cdot \overline{z_{0,0}^{k,m}} - r_{1,0}^{k,m} \cdot \left(z_{0,0}^{k,m}\right)^2 \cdot z_{0,1}^{k,m} \cdot \overline{z_{1,1}^{k,m}} \\
& + r_{1,0}^{k,m} \cdot \left(z_{0,0}^{k,m}\right)^2 \cdot z_{1,1}^{k,m} \cdot \overline{z_{0,1}^{k,m}} - r_{1,0}^{k,m} \cdot \left(z_{0,1}^{k,m}\right)^2 \cdot z_{1,1}^{k,m} \cdot \overline{z_{0,0}^{k,m}} \\
& - r_{1,1}^{k,m} \cdot z_{0,0}^{k,m} \cdot \left(z_{0,1}^{k,m}\right)^2 \cdot \overline{z_{1,0}^{k,m}} + r_{1,1}^{k,m} \cdot z_{0,0}^{k,m} \cdot \left(z_{1,0}^{k,m}\right)^2 \cdot \overline{z_{0,1}^{k,m}} \\
& - r_{1,1}^{k,m} \cdot z_{0,1}^{k,m} \cdot \left(z_{1,0}^{k,m}\right)^2 \cdot \overline{z_{0,0}^{k,m}} + r_{1,1}^{k,m} \cdot \left(z_{0,0}^{k,m}\right)^2 \cdot z_{0,1}^{k,m} \cdot \overline{z_{1,0}^{k,m}} \\
& - r_{1,1}^{k,m} \cdot \left(z_{0,0}^{k,m}\right)^2 \cdot z_{1,0}^{k,m} \cdot \overline{z_{0,1}^{k,m}} + r_{1,1}^{k,m} \cdot \left(z_{0,1}^{k,m}\right)^2 \cdot z_{1,0}^{k,m} \cdot \overline{z_{0,0}^{k,m}} \\
& + r_{0,0}^{k,m} \cdot z_{0,1}^{k,m} . \overline{z_{1,0}^{k,m}} \cdot \left(\overline{z_{1,1}^{k,m}}\right)^2 - r_{0,0}^{k,m} \cdot z_{0,1}^{k,m} \cdot \left(\overline{z_{1,0}^{k,m}}\right)^2 \cdot \overline{z_{1,1}^{k,m}} \\
& - r_{0,0}^{k,m} \cdot z_{1,0}^{k,m} \cdot \overline{z_{0,1}^{k,m}} \cdot \left(\overline{z_{1,1}^{k,m}}\right)^2 + r_{0,0}^{k,m} \cdot z_{1,0}^{k,m} \cdot \left(\overline{z_{0,1}^{k,m}}\right)^2 \cdot \overline{z_{1,1}^{k,m}} \\
& + r_{0,0}^{k,m} \cdot z_{1,1}^{k,m} \cdot \overline{z_{0,1}^{k,m}} \cdot \left(\overline{z_{1,0}^{k,m}}\right)^2 - r_{0,0}^{k,m} \cdot z_{1,1}^{k,m} \cdot \left(\overline{z_{0,1}^{k,m}}\right)^2 \cdot \overline{z_{1,0}^{k,m}} \\
& - r_{0,1}^{k,m} \cdot z_{0,0}^{k,m} . \overline{z_{1,0}^{k,m}} \cdot \left(\overline{z_{1,1}^{k,m}}\right)^2 + r_{0,1}^{k,m} \cdot z_{0,0}^{k,m} \cdot \left(\overline{z_{1,0}^{k,m}}\right)^2 \cdot \overline{z_{1,1}^{k,m}} \\
& + r_{0,1}^{k,m} \cdot z_{1,0}^{k,m} \cdot \overline{z_{0,0}^{k,m}} \cdot \left(\overline{z_{1,1}^{k,m}}\right)^2 - r_{0,1}^{k,m} \cdot z_{1,0}^{k,m} \cdot \left(\overline{z_{0,0}^{k,m}}\right)^2 \cdot \overline{z_{1,1}^{k,m}} \\
& - r_{0,1}^{k,m} \cdot z_{1,1}^{k,m} . \overline{z_{0,0}^{k,m}} \cdot \left(\overline{z_{1,0}^{k,m}}\right)^2 + r_{0,1}^{k,m} \cdot z_{1,1}^{k,m} \cdot \left(\overline{z_{0,0}^{k,m}}\right)^2 \cdot \overline{z_{1,0}^{k,m}} \\
& + r_{1,0}^{k,m} \cdot z_{0,0}^{k,m} \cdot \overline{z_{0,1}^{k,m}} \cdot \left(\overline{z_{1,1}^{k,m}}\right)^2 - r_{1,0}^{k,m} \cdot z_{0,0}^{k,m} \cdot \left(\overline{z_{0,1}^{k,m}}\right)^2 \cdot \overline{z_{1,1}^{k,m}} \\
& - r_{1,0}^{k,m} \cdot z_{0,1}^{k,m} \cdot \overline{z_{0,0}^{k,m}} \cdot \left(\overline{z_{1,1}^{k,m}}\right)^2 + r_{1,0}^{k,m} \cdot z_{0,1}^{k,m} \cdot \left(\overline{z_{0,0}^{k,m}}\right)^2 \cdot \overline{z_{1,1}^{k,m}} \\
& + r_{1,0}^{k,m} \cdot z_{1,1}^{k,m} . \overline{z_{0,0}^{k,m}} \cdot \left(\overline{z_{0,1}^{k,m}}\right)^2 - r_{1,0}^{k,m} \cdot z_{1,1}^{k,m} \cdot \left(\overline{z_{0,0}^{k,m}}\right)^2 \cdot \overline{z_{0,1}^{k,m}} \\
& - r_{1,1}^{k,m} . z_{0,0}^{k,m} \cdot \overline{z_{0,1}^{k,m}} \cdot \left(\overline{z_{1,0}^{k,m}}\right)^2 + r_{1,1}^{k,m} \cdot z_{0,0}^{k,m} \cdot \left(\overline{z_{0,1}^{k,m}}\right)^2 \cdot \overline{z_{1,0}^{k,m}} \\
& + r_{1,1}^{k,m} . z_{0,1}^{k,m} \cdot \overline{z_{0,0}^{k,m}} \cdot \left(\overline{z_{0,1}^{k,m}}\right)^2 - r_{1,1}^{k,m} . z_{0,1}^{k,m} \cdot \left(\overline{z_{0,0}^{k,m}}\right)^2 \cdot \overline{z_{1,0}^{k,m}} \\
& - r_{1,1}^{k,m} \cdot z_{1,0}^{k,m} \cdot \overline{z_{0,0}^{k,m}} \cdot \left(\overline{z_{0,1}^{k,m}}\right)^2 + r_{1,1}^{k,m} . z_{1,0}^{k,m} \cdot \left(\overline{z_{0,0}^{k,m}}\right)^2 \cdot \overline{z_{0,1}^{k,m}} \Big),
\end{aligned} \tag{8}
$$

where

$$
\begin{aligned}
Q_{k,m} = {} & z_{0,0}^{k,m} \cdot \left(z_{0,1}^{k,m}\right)^2 \cdot \overline{z_{1,0}^{k,m}} - z_{0,0}^{k,m} \cdot \left(z_{1,0}^{k,m}\right)^2 \cdot \overline{z_{0,1}^{k,m}} + z_{0,1}^{k,m} \cdot \left(z_{1,0}^{k,m}\right)^2 \cdot \overline{z_{0,0}^{k,m}} \\
& - \left(z_{0,0}^{k,m}\right)^2 \cdot z_{0,1}^{k,m} \cdot \overline{z_{1,0}^{k,m}} + \left(z_{0,0}^{k,m}\right)^2 \cdot z_{1,0}^{k,m} \cdot \overline{z_{0,1}^{k,m}} - \left(z_{0,1}^{k,m}\right)^2 \cdot z_{1,0}^{k,m} \cdot \overline{z_{0,0}^{k,m}} \\
& - z_{0,0}^{k,m} \cdot \left(z_{0,1}^{k,m}\right)^2 \cdot \overline{z_{1,1}^{k,m}} + z_{0,0}^{k,m} \left(z_{1,1}^{k,m}\right)^2 \cdot \overline{z_{0,1}^{k,m}} - z_{0,1}^{k,m} \cdot \left(z_{1,1}^{k,m}\right)^2 \cdot \overline{z_{0,0}^{k,m}} \\
& + \left(z_{0,0}^{k,m}\right)^2 \cdot z_{0,1}^{k,m} \cdot \overline{z_{1,1}^{k,m}} - \left(z_{0,0}^{k,m}\right)^2 \cdot z_{1,1}^{k,m} \cdot \overline{z_{0,1}^{k,m}} + \left(z_{0,1}^{k,m}\right)^2 \cdot z_{1,1}^{k,m} \cdot \overline{z_{0,0}^{k,m}} \\
& + z_{0,0}^{k,m} \cdot \left(z_{1,0}^{k,m}\right)^2 \cdot \overline{z_{1,1}^{k,m}} - z_{0,0}^{k,m} \cdot \left(z_{1,1}^{k,m}\right)^2 \cdot \overline{z_{1,0}^{k,m}} + z_{1,0}^{k,m} \cdot \left(z_{1,1}^{k,m}\right)^2 \cdot \overline{z_{0,0}^{k,m}} \\
& - \left(z_{0,0}^{k,m}\right)^2 \cdot z_{1,0}^{k,m} \cdot \overline{z_{1,1}^{k,m}} + \left(z_{0,0}^{k,m}\right)^2 \cdot z_{1,1}^{k,m} \cdot \overline{z_{1,0}^{k,m}} - \left(z_{1,0}^{k,m}\right)^2 \cdot z_{1,1}^{k,m} \cdot \overline{z_{0,0}^{k,m}} \\
& - z_{0,1}^{k,m} \cdot \left(z_{1,0}^{k,m}\right)^2 \cdot \overline{z_{1,1}^{k,m}} + z_{0,1}^{k,m} \cdot \left(z_{1,1}^{k,m}\right)^2 \cdot \overline{z_{1,0}^{k,m}} - z_{1,0}^{k,m} \cdot \left(z_{1,1}^{k,m}\right)^2 \cdot \overline{z_{0,1}^{k,m}} \\
& + \left(z_{0,1}^{k,m}\right)^2 \cdot z_{1,0}^{k,m} \cdot \overline{z_{1,1}^{k,m}} - \left(z_{0,1}^{k,m}\right)^2 \cdot z_{1,1}^{k,m} \cdot \overline{z_{1,0}^{k,m}} \\
& + \left(z_{1,0}^{k,m}\right)^2 \cdot z_{1,1}^{k,m} \cdot \overline{z_{0,1}^{k,m}} + z_{0,0}^{k,m} \cdot \overline{z_{0,1}^{k,m}} \cdot \left(\overline{z_{1,0}^{k,m}}\right)^2 - z_{0,0}^{k,m} \cdot \left(\overline{z_{0,1}^{k,m}}\right)^2 \cdot \overline{z_{1,0}^{k,m}} \\
& - z_{0,1}^{k,m} \cdot \overline{z_{0,0}^{k,m}} \cdot \left(\overline{z_{1,0}^{k,m}}\right)^2 + z_{0,1}^{k,m} \cdot \left(\overline{z_{0,0}^{k,m}}\right)^2 \cdot \overline{z_{1,0}^{k,m}} + z_{1,0}^{k,m} \cdot \overline{z_{0,0}^{k,m}} \cdot \left(\overline{z_{0,1}^{k,m}}\right)^2 \\
& - z_{1,0}^{k,m} \cdot \left(\overline{z_{0,0}^{k,m}}\right)^2 \cdot \overline{z_{0,1}^{k,m}} - z_{0,0}^{k,m} \cdot \overline{z_{0,1}^{k,m}} \cdot \left(\overline{z_{1,1}^{k,m}}\right)^2 + z_{0,0}^{k,m} \cdot \left(\overline{z_{0,1}^{k,m}}\right)^2 \cdot \overline{z_{1,1}^{k,m}} \\
& + z_{0,1}^{k,m} \cdot \overline{z_{0,0}^{k,m}} \cdot \left(\overline{z_{1,1}^{k,m}}\right)^2 - z_{0,1}^{k,m} \cdot \left(\overline{z_{0,0}^{k,m}}\right)^2 \cdot \overline{z_{1,1}^{k,m}} - z_{1,1}^{k,m} \cdot \overline{z_{0,0}^{k,m}} \cdot \left(\overline{z_{0,1}^{k,m}}\right)^2 \\
& z_{1,1}^{k,m} \cdot \left(\overline{z_{0,0}^{k,m}}\right)^2 \cdot \overline{z_{0,1}^{k,m}} + z_{0,0}^{k,m} \cdot \overline{z_{1,0}^{k,m}} \cdot \left(\overline{z_{1,1}^{k,m}}\right)^2 - z_{0,0}^{k,m} \cdot \left(\overline{z_{1,0}^{k,m}}\right)^2 \cdot \overline{z_{1,1}^{k,m}} \\
& - z_{1,0}^{k,m} \cdot \overline{z_{0,0}^{k,m}} \cdot \left(\overline{z_{1,1}^{k,m}}\right)^2 + z_{1,0}^{k,m} \cdot \left(\overline{z_{0,0}^{k,m}}\right)^2 \cdot \overline{z_{1,1}^{k,m}} + z_{1,1}^{k,m} \cdot \overline{z_{0,0}^{k,m}} \cdot \left(\overline{z_{1,0}^{k,m}}\right)^2 \\
& - z_{1,1}^{k,m} \cdot \left(\overline{z_{0,0}^{k,m}}\right)^2 \cdot \overline{z_{1,0}^{k,m}} - z_{0,1}^{k,m} \cdot \overline{z_{1,0}^{k,m}} \cdot \left(\overline{z_{1,1}^{k,m}}\right)^2 + z_{0,1}^{k,m} \cdot \left(\overline{z_{1,0}^{k,m}}\right)^2 \cdot \overline{z_{1,1}^{k,m}} \\
& + z_{1,0}^{k,m} \cdot \overline{z_{0,1}^{k,m}} \cdot \left(\overline{z_{1,1}^{k,m}}\right)^2 - z_{1,0}^{k,m} \cdot \left(\overline{z_{0,1}^{k,m}}\right)^2 \cdot \overline{z_{1,1}^{k,m}} - z_{1,1}^{k,m} \cdot \overline{z_{0,1}^{k,m}} \cdot \left(\overline{z_{1,0}^{k,m}}\right)^2 \\
& \left. + z_{1,1}^{k,m} \cdot \left(\overline{z_{0,1}^{k,m}}\right)^2 \cdot \overline{z_{1,0}^{k,m}} \right).
\end{aligned}
$$

3 Approximation of T

In this section, we give quadrature formula for potential integral T. By using spline approximation of ρ in the domain $D_{k,m}$, we have

$$
\begin{aligned}
T\left(\widehat{\rho}|z\right) = {} & -\frac{1}{\pi} \sum_{k=1}^{N-1} \sum_{m=0}^{M_k-1} \left(a_{k,m} \iint\limits_{D_{k,m}} \frac{\zeta d\zeta}{\zeta - z} + b_{k,m} \iint\limits_{D_{k,m}} \frac{\overline{\zeta} d\zeta}{\zeta - z} \right. \\
& \left. + c_{k,m} \iint\limits_{D_{k,m}} \frac{\left(\zeta^2 - \overline{\zeta}^2\right) d\zeta}{\zeta - z} + d_{k,m} \iint\limits_{D_{k,m}} \frac{d\zeta}{\zeta - z} \right)
\end{aligned}
$$

$$= \sum_{k=1}^{N-1} \sum_{m=0}^{M_k-1} \Big(a_{k,m} T^a_{k,m}\,(z) + b_{k,m} T^b_{k,m}\,(z) \\ + c_{k,m} T^c_{k,m}\,(z) + d_{k,m} T^d_{k,m}\,(z) \Big), \tag{9}$$

where

$$T^a_{k,m}\,(z) = -\frac{1}{\pi} \iint\limits_{D_{k,m}} \frac{\zeta d\zeta}{\zeta - z}, \quad T^b_{k,m}\,(z) = -\frac{1}{\pi} \iint\limits_{D_{k,m}} \frac{\overline{\zeta} d\zeta}{\zeta - z},$$
$$T^c_{k,m}\,(z) = -\frac{1}{\pi} \iint\limits_{D_{k,m}} \frac{\left(\zeta^2 - \overline{\zeta}^2\right) d\zeta}{\zeta - z}, T^d_{k,m}\,(z) = -\frac{1}{\pi} \iint\limits_{D_{k,m}} \frac{d\zeta}{\zeta - z}.$$

Let us evaluate $T^a_{k,m}$. By using Pompeu formula (see [13, p. 26, formula (4.15)]), two-dimensional integral can be reduced to one dimensional integral on boundary $\Gamma_{k,m}$. From identity $\frac{\partial(\zeta\overline{\zeta})}{\partial\overline{\zeta}} = \zeta$ on $\Gamma^1_{k,m}$, it follows that

$$T^a_{k,m}\,(z) = \int\limits_{\Gamma_{k,m}} \frac{\zeta\overline{\zeta}}{\zeta - z}\, d\zeta = \sum_{j=1}^{4} \int\limits_{\Gamma^j_{k,m}} \frac{\zeta\overline{\zeta}}{\zeta - z}\, d\zeta. \tag{10}$$

In sequel,

$$T^{a,1}_{k,m}\,(z) = \int\limits_{\Gamma^1_{k,m}} \frac{\zeta\overline{\zeta}}{\zeta - z}\, d\zeta = e^{-2i\theta_{k,m}} \int\limits_{\Gamma^1_{k,m}} \frac{\zeta^2}{\zeta - z}\, d\zeta \\
= e^{-2i\theta_{k,m}} \int\limits_{\Gamma^1_{k,m}} \frac{\zeta^2 - z^2 + z^2}{\zeta - z}\, d\zeta \\
= e^{-2i\theta_{k,m}} \left[\int\limits_{\Gamma^1_{k,m}} \frac{\zeta^2 - z^2}{\zeta - z}\, d\zeta + z^2 \int\limits_{\Gamma^1_{k,m}} \frac{d\zeta}{\zeta - z} \right] \\
= e^{-2i\theta_{k,m}} \left[\frac{\zeta^2}{2} + z\zeta + z^2\ \ln(\zeta - z) \right]_{z^{k,m}_{0,0}}^{z^{k,m}_{1,0}} \\
= \frac{\overline{z^{k,m}_{0,0}}}{z^{k,m}_{0,0}} \left[\frac{\left(z^{k,m}_{1,0}\right)^2}{2} - \frac{\left(z^{k,m}_{0,0}\right)^2}{2} + z z^{k,m}_{1,0} - z z^{k,m}_{0,0} + z^2 \ln \left| \frac{z^{k,m}_{1,0} - z}{z^{k,m}_{0,0} - z} \right| \right]. \tag{11}$$

We have $\zeta\overline{\zeta} = r_{k+1} e^{i\theta}\, r_{k+1} e^{-i\theta} = r^2_{k+1}$ on $\Gamma^2_{k,m}$. Thus,

$$T^{a,2}_{k,m}\,(z) = \int\limits_{\Gamma^2_{k,m}} \frac{\zeta\overline{\zeta}}{\zeta - z}\, d\zeta = r^2_{k+1} \int\limits_{\Gamma^2_{k,m}} \frac{d\zeta}{\zeta - z} \\
= r^2_{k+1}\ \ln(\zeta - z)|_{z^{k,m}_{1,0}}^{z^{k,m}_{1,1}} = z^{k,m}_{1,0} \overline{z^{k,m}_{1,0}}\, ln \frac{z^{k,m}_{1,1} - z}{z^{k,m}_{1,0} - z}. \tag{12}$$

For any $\zeta \in \Gamma^3_{k,m}$, the relation $\zeta\overline{\zeta} = re^{i\theta_{k,m}}\, re^{-i\theta_{k,m}} = r^2 = \zeta^2\, e^{-2i\theta_{k,m}}$ is true. So, we get

$$\begin{aligned} T^{a,3}_{k,m}(z) &= \int\limits_{\Gamma^3_{k,m}} \frac{\zeta\overline{\zeta}}{\zeta-z}\, d\zeta = e^{-2i\theta_{k,m}} \int\limits_{\Gamma^3_{k,m}} \frac{\zeta^2-z^2+z^2}{\zeta-z}\, d\zeta \\ &= e^{-2i\theta_{k,m}} \left(\int\limits_{\Gamma^3_{k,m}} (\zeta+z)\, d\zeta + \int\limits_{\Gamma^3_{k,m}} \frac{z^2}{\zeta-z}\, d\zeta \right) \\ &= \frac{\overline{z^{k,m}_{1,1}}}{z^{k,m}_{1,1}} \left[\frac{\left(z^{k,m}_{0,1}\right)^2}{2} - \frac{\left(z^{k,m}_{1,1}\right)^2}{2} + zz^{k,m}_{0,1} - zz^{k,m}_{1,1} + z^2 \ln\left|\frac{z^{k,m}_{0,1}-z}{z^{k,m}_{1,1}-z}\right| \right]. \end{aligned} \tag{13}$$

Since $\zeta\overline{\zeta} = r_k e^{i\theta} r_k e^{-i\theta} = r_k^2 = z_{k,m}\overline{z_{k,m}}$ on $\Gamma^4_{k,m}$, we get

$$\begin{aligned} T^{a,4}_{k,m}(z) &= \int\limits_{\Gamma^4_{k,m}} \frac{\zeta\overline{\zeta}}{\zeta-z} d\zeta = r_k^2 \int\limits_{\Gamma^4_{k,m}} \frac{d\zeta}{\zeta-z} \\ &= r_k^2 \left(\ln(\zeta-z)\right)^{z^{k,m}_{0,0}}_{z^{k,m}_{0,1}} = z^{k,m}_{0,0}\overline{z^{k,m}_{0,0}} \ln \frac{z^{k,m}_{0,0}-z}{z^{k,m}_{0,1}-z}. \end{aligned} \tag{14}$$

By virtue (10)- (14), we have formula

$$\begin{aligned} T^a_{k,m}(z) = \frac{\overline{z^{k,m}_{0,0}}}{z^{k,m}_{0,0}} &\left[\frac{\left(z^{k,m}_{1,0}\right)^2}{2} - \frac{\left(z^{k,m}_{0,0}\right)^2}{2} + zz^{k,m}_{1,0} - zz^{k,m}_{0,0} + z^2 \ln\left|\frac{z^{k,m}_{1,0}-z}{z^{k,m}_{0,0}-z}\right| \right] \\ &+ z^{k,m}_{1,0}\overline{z^{k,m}_{1,0}}\, ln \frac{z^{k,m}_{1,1}-z}{z^{k,m}_{1,0}-z} + z^{k,m}_{0,0}\overline{z^{k,m}_{0,0}} \ln \frac{z^{k,m}_{0,0}-z}{z^{k,m}_{0,1}-z} \\ &+ \frac{\overline{z^{k,m}_{1,1}}}{z^{k,m}_{1,1}} \left[\frac{z^{k,m^2}_{0,1}}{2} - \frac{\left(z^{k,m}_{1,1}\right)^2}{2} + zz^{k,m}_{0,1} - zz^{k,m}_{1,1} + z^2 \ln\left|\frac{z^{k,m}_{0,1}-z}{z^{k,m}_{1,1}-z}\right| \right]. \end{aligned} \tag{15}$$

In a similar manner, the following formulas for
$T^b_{k,m}(z)\,, T^c_{k,m}(z)\,, T^d_{k,m}(z)$ can be obtained respectively:

$$\begin{aligned} T^b_{k,m}(z) &= \frac{1}{2}\left(\frac{\overline{z^{k,m}_{0,0}}}{z^{k,m}_{0,0}}\right)^2 \\ &\times \left[\frac{\left(z^{k,m}_{1,0}\right)^2}{2} - \frac{\left(z^{k,m}_{0,0}\right)^2}{2} + zz^{k,m}_{1,0} - zz^{k,m}_{0,0} + z^2 \ln\left|\frac{z^{k,m}_{1,0}-z}{z^{k,m}_{0,0}-z}\right| \right] \end{aligned}$$

$$+\frac{1}{2}\left(\overline{z_{1,1}^{k,m}}z_{1,1}^{k,m}\right)^2\left[\frac{1}{z^2}\ \ln\frac{\left(z_{1,1}^{k,m}-z\right)z_{1,0}^{k,m}}{\left(z_{1,0}^{k,m}-z\right)z_{1,1}^{k,m}}+\frac{1}{z}\ \left(\frac{1}{z_{1,1}^{k,m}}-\frac{1}{z_{1,0}^{k,m}}\right)\right]$$
$$+\frac{1}{2}\left(\frac{\overline{z_{0,1}^{k,m}}}{z_{0,1}^{k,m}}\right)^2\ \left[\frac{\left(z_{0,1}^{k,m}\right)^2}{2}-\frac{\left(z_{1,1}^{k,m}\right)^2}{2}+zz_{0,1}^{k,m}-zz_{1,1}^{k,m}+z^2\ \ \ln\frac{z_{0,1}^{k,m}-z}{z_{1,1}^{k,m}-z}\right] \quad (16)$$
$$+\frac{1}{2}\left(\overline{z_{0,0}^{k,m}}z_{0,0}^{k,m}\right)^2\left[\frac{1}{z^2}\ \ln\frac{\left(z_{0,0}^{k,m}-z\right)z_{0,1}^{k,m}}{\left(z_{0,1}^{k,m}-z\right)z_{0,0}^{k,m}}+\frac{1}{z}\ \left(\frac{1}{z_{0,0}^{k,m}}-\frac{1}{z_{0,1}^{k,m}}\right)\right],$$

$$T_{k,m}^{c}(z)=\left(\left(\frac{\overline{z_{0,0}^{k,m}}}{z_{0,0}^{k,m}}\right)-\frac{1}{3}\left(\frac{\overline{z_{0,0}^{k,m}}}{z_{0,0}^{k,m}}\right)^3\right)\left[\frac{\left(z_{1,0}^{k,m}\right)^3}{3}-\frac{\left(z_{0,0}^{k,m}\right)^3}{3}+z\frac{\left(z_{1,0}^{k,m}\right)^2}{2}\right.$$
$$\left.-z\frac{\left(z_{0,0}^{k,m}\right)^2}{2}\ +\ z^2\left(z_{1,0}^{k,m}-z_{0,0}^{k,m}\right)\ +\ z^3\ln\frac{z_{1,0}^{k,m}-z}{z_{0,0}^{k,m}-z}\right]$$
$$+\left(z_{1,0}^{k,m}\overline{z_{1,0}^{k,m}}\right)\left\{z_{1,1}^{k,m}-z_{1,0}^{k,m}+z\ln\left(\frac{z_{1,1}^{k,m}-z}{z_{1,0}^{k,m}-z}\right)\right\}-\frac{\left(z_{1,0}^{k,m}\overline{z_{1,0}^{k,m}}\right)^3}{3}$$
$$\times\left\{-\frac{1}{z^3}\cdot\ln\left(\frac{z_{1,1}^{k,m}}{z_{1,0}^{k,m}}\right)+\frac{1}{z^2}\cdot\left(\frac{1}{z_{1,1}^{k,m}}-\frac{1}{z_{1,0}^{k,m}}\right)\right.$$
$$\left.+\frac{1}{2z}\cdot\left(\frac{1}{\left(z_{1,1}^{k,m}\right)^2}-\frac{1}{\left(z_{1,0}^{k,m}\right)^2}\right)+\frac{1}{z^3}\cdot\ln\left(\frac{z_{1,1}^{k,m}-z}{z_{1,0}^{k,m}-z}\right)\right\}$$
$$+\left(\left(\frac{\overline{z_{0,1}^{k,m}}}{z_{0,1}^{k,m}}\right)-\frac{1}{3}\left(\frac{\overline{z_{0,1}^{k,m}}}{z_{0,1}^{k,m}}\right)^3\right)\left[\frac{\left(z_{0,1}^{k,m}\right)^3}{3}-\frac{\left(z_{1,1}^{k,m}\right)^3}{3}+z\frac{\left(z_{0,1}^{k,m}\right)^2}{2}\right. \quad (17)$$
$$\left.-z\frac{\left(z_{1,1}^{k,m}\right)^2}{2}+z^2\left(z_{0,1}^{k,m}-z_{1,1}^{k,m}\right)\ +z^3\ln\left|\frac{z_{0,1}^{k,m}-z}{z_{1,1}^{k,m}-z}\right|\right]$$
$$+\left(z_{0,0}^{k,m}\overline{z_{0,0}^{k,m}}\right)\left\{z_{0,0}^{k,m}-z_{0,1}^{k,m}+z\ln\left(\frac{z_{0,0}^{k,m}-z}{z_{0,1}^{k,m}-z}\right)\right\}-\frac{\left(z_{0,0}^{k,m}\overline{z_{0,0}^{k,m}}\right)^3}{3}$$
$$\times\left\{-\frac{1}{z^3}\cdot\ln\left(\frac{z_{0,0}^{k,m}}{z_{0,1}^{k,m}}\right)+\frac{1}{z^2}\cdot\left(\frac{1}{z_{0,0}^{k,m}}-\frac{1}{z_{0,1}^{k,m}}\right)\right.$$
$$\left.+\frac{1}{2z}\cdot\left(\frac{1}{\left(z_{0,0}^{k,m}\right)^2}-\frac{1}{\left(z_{0,1}^{k,m}\right)^2}\right)+\frac{1}{z^3}\cdot\ln\left(\frac{z_{0,0}^{k,m}-z}{z_{0,1}^{k,m}-z}\right)\right\},$$

$$T_{k,m}^{d}(z)=\frac{\overline{z_{0,0}^{k,m}}}{z_{0,0}^{k,m}}\left[z_{1,0}^{k,m}-z_{0,0}^{k,m}+z\ln\frac{z_{1,0}^{k,m}-z}{z_{0,0}^{k,m}-z}\right]$$
$$+z_{1,1}^{k,m}\overline{z_{1,1}^{k,m}}\frac{1}{z}\ln\frac{\left(z_{1,1}^{k,m}-z\right)z_{1,0}^{k,m}}{\left(z_{1,0}^{k,m}-z\right)z_{1,1}^{k,m}} \quad (18)$$
$$+\frac{\overline{z_{1,1}^{k,m}}}{z_{1,1}^{k,m}}\left[z_{0,1}^{k,m}-z_{1,1}^{k,m}+z\ln\frac{z_{0,1}^{k,m}-z}{z_{1,1}^{k,m}-z}\right]+\frac{z_{0,0}^{k,m}\overline{z_{0,0}^{k,m}}}{z}\ln\left(\frac{\left(z_{0,0}^{k,m}-z\right)z_{0,1}^{k,m}}{\left(z_{0,1}^{k,m}-z\right)z_{0,0}^{k,m}}\right).$$

Finally, we get an approximate formula

$$T(\rho|z) \approx \sum_{k=1}^{N-1}\sum_{m=0}^{M_k-1}\Big(a_{k,m}T^a_{k,m}(z) + b_{k,m}T^b_{k,m}(z) + c_{k,m}T^c_{k,m}(z) + d_{k,m}T^d_{k,m}(z)\Big), z \in \Omega^{(2)},$$

where coefficients $a_{k,m},\ b_{k,m},\ c_{k,m},\ d_{k,m}$ and integrals $T^a_{k,m}$, $T^b_{k,m}$, $T^c_{k,m}$, and $T^d_{k,m}$ are defined by (5), (6), (7), (8) and (15),(16), (17), (18), respectively.

4 Approximation of S

In this section, we present quadrature formula for singular integral S. Applying spline approximation of ρ in the domain $D_{k,m}$, we have

$$S(\widehat{\rho}|z) = -\frac{1}{\pi}\sum_{k=1}^{N-1}\sum_{m=0}^{M_k-1}\Big(a_{k,m}S^a_{k,m}(z) + b_{k,m}S^b_{k,m}(z) + c_{k,m}S^c_{k,m}(z) + d_{k,m}S^d_{k,m}(z)\Big),$$

where

$$S^a_{k,m}(z) = -\frac{1}{\pi}\iint_{D_{k,m}}\frac{\zeta d\zeta}{(\zeta-z)^2},\ S^b_{k,m}(z) = -\frac{1}{\pi}\iint_{D_{k,m}}\frac{\overline{\zeta}d\zeta}{(\zeta-z)^2},$$
$$S^c_{k,m}(z) = -\frac{1}{\pi}\iint_{D_{k,m}}\frac{\left(\zeta^2-\overline{\zeta}^2\right)d\zeta}{(\zeta-z)^2},\ S^d_{k,m}(z) = -\frac{1}{\pi}\iint_{D_{k,m}}\frac{d\zeta}{(\zeta-z)^2}.$$

From (1), it follows that

$$S^a_{k,m}(z) = \frac{\partial T^a_{k,m}(z)}{\partial z}, S^b_{k,m}(z) = \frac{\partial T^b_{k,m}(z)}{\partial z},$$
$$S^c_{k,m}(z) = \frac{\partial T^c_{k,m}(z)}{\partial z}, S^d_{k,m}(z) = \frac{\partial T^d_{k,m}(z)}{\partial z}.$$

Applying (15), (16), (17) and (18), we get

$$\begin{aligned} S^a_{k,m}(z) &= \frac{\overline{z^{k,m}_{0,0}}}{z^{k,m}_{0,0}} \\ &\times\left[z^{k,m}_{1,0} - z^{k,m}_{0,0} + 2z\ln\left(\frac{z^{k,m}_{1,0}-z}{z^{k,m}_{0,0}-z}\right) - z^2\left(\frac{1}{z^{k,m}_{1,0}-z} - \frac{1}{z^{k,m}_{0,0}-z}\right)\right] \\ &+z^{k,m}_{1,0}\overline{z^{k,m}_{1,0}}\left[\frac{1}{z^{k,m}_{1,0}-z} - \frac{1}{z^{k,m}_{1,1}-z}\right] + z^{k,m}_{0,0}\overline{z^{k,m}_{0,0}}\left[\frac{1}{z^{k,m}_{0,1}-z} - \frac{1}{z^{k,m}_{0,0}-z}\right] \\ &+\frac{\overline{z^{k,m}_{1,1}}}{z^{k,m}_{1,1}}\left[z^{k,m}_{0,1} - z^{k,m}_{1,1} + 2z\ln\left(\frac{z^{k,m}_{0,1}-z}{z^{k,m}_{1,1}-z}\right) - z^2\left(\frac{1}{z^{k,m}_{0,1}-z} - \frac{1}{z^{k,m}_{1,1}-z}\right)\right], \end{aligned} \tag{19}$$

$$
\begin{aligned}
&S_{k,m}^{b}(z)=\frac{1}{2}\left(\frac{\overline{z_{0,0}^{k,m}}}{z_{0,0}^{k,m}}\right)^{2} \\
&\times\left[z_{1,0}^{k,m}-z_{0,0}^{k,m}+2z\ \ln\left(\frac{z_{1,0}^{k,m}-z}{z_{0,0}^{k,m}-z}\right)+z^{2}\ \left(\frac{1}{z_{0,0}^{k,m}-z}-\frac{1}{z_{1,0}^{k,m}-z}\right)\right] \\
&+\frac{1}{2}\left(\overline{z_{1,1}^{k,m}}z_{1,1}^{k,m}\right)^{2}\left[-\frac{2}{z^{3}}\ \ln\frac{\left(z_{1,1}^{k,m}-z\right)z_{1,0}^{k,m}}{\left(z_{1,0}^{k,m}-z\right)z_{1,1}^{k,m}}+\frac{1}{z^{2}}\left(\frac{1}{z_{1,0}^{k,m}-z}-\frac{1}{z_{1,1}^{k,m}-z}\right)\right. \\
&\left.-\frac{1}{z^{2}}\ \left(\frac{1}{z_{1,1}^{k,m}}-\frac{1}{z_{1,0}^{k,m}}\right)\ \right]+\frac{1}{2}\left(\frac{\overline{z_{0,1}^{k,m}}}{z_{0,1}^{k,m}}\right)^{2} \\
&\times\left[z_{0,1}^{k,m}-z_{1,1}^{k,m}+2z\ \ln\left(\frac{z_{0,1}^{k,m}-z}{z_{1,1}^{k,m}-z}\right)+z^{2}\ \left(\frac{1}{z_{1,1}^{k,m}-z}-\frac{1}{z_{0,1}^{k,m}-z}\right)\right] \\
&+\frac{1}{2}\left(\overline{z_{0,0}^{k,m}}z_{0,0}^{k,m}\right)^{2}\left[-\frac{2}{z^{3}}\ \ln\frac{\left(z_{0,0}^{k,m}-z\right)z_{0,1}^{k,m}}{\left(z_{0,1}^{k,m}-z\right)z_{0,0}^{k,m}}+\frac{1}{z^{2}}\left(\frac{1}{z_{0,1}^{k,m}-z}-\frac{1}{z_{0,0}^{k,m}-z}\right)\right. \\
&\left.-\frac{1}{z^{2}}\ \left(\frac{1}{z_{0,0}^{k,m}}-\frac{1}{z_{0,1}^{k,m}}\right)\right],
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
&S_{k,m}^{c}(z)=\left(\left(\frac{\overline{\mathrm{z}_{0,0}^{k,m}}}{\mathrm{z}_{0,0}^{k,m}}\right)-\frac{1}{3}\left(\frac{\overline{\mathrm{z}_{0,0}^{k,m}}}{\mathrm{z}_{0,0}^{k,m}}\right)^{3}\right)\left[\frac{\left(z_{1,0}^{k,m}\right)^{2}}{2}-\frac{\left(z_{0,0}^{k,m}\right)^{2}}{2}\right. \\
&\left.+2z\left(z_{1,0}^{k,m}-z_{0,0}^{k,m}\right)+\ 3z^{2}\ln\frac{z_{1,0}^{k,m}-z}{z_{0,0}^{k,m}-z}+z^{3}\left(\frac{1}{z_{0,0}^{k,m}-z}-\frac{1}{z_{1,0}^{k,m}-z}\right)\right] \\
&+\left(z_{1,0}^{k,m}\overline{z_{1,0}^{k,m}}\right)\left\{\ln\left(\frac{z_{1,1}^{k,m}-z}{z_{1,0}^{k,m}-z}\right)+z\left(\frac{1}{z_{1,0}^{k,m}-z}-\frac{1}{z_{1,1}^{k,m}-z}\right)\right\}-\frac{\left(z_{1,0}^{k,m}\overline{z_{1,0}^{k,m}}\right)^{3}}{3} \\
&\times\left\{\frac{3}{z^{4}}\cdot\ln\left(\frac{z_{1,1}^{k,m}}{z_{1,0}^{k,m}}\right)-\frac{2}{z^{3}}\cdot\left(\frac{1}{z_{1,1}^{k,m}}-\frac{1}{z_{1,0}^{k,m}}\right)-\frac{1}{2z^{2}}\cdot\left(\frac{1}{\left(z_{1,1}^{k,m}\right)^{2}}-\frac{1}{\left(z_{1,0}^{k,m}\right)^{2}}\right)\right. \\
&\left.-\frac{3}{z^{4}}\cdot\ln\left(\frac{z_{1,1}^{k,m}-z}{z_{1,0}^{k,m}-z}\right)+\frac{1}{z^{3}}\cdot\left(\frac{1}{z_{1,0}^{k,m}-z}-\frac{1}{z_{1,1}^{k,m}-z}\right)\right\} \\
&+\left(\left(\frac{\overline{\mathrm{z}_{0,1}^{k,m}}}{\mathrm{z}_{0,1}^{k,m}}\right)-\frac{1}{3}\left(\frac{\overline{\mathrm{z}_{0,1}^{k,m}}}{\mathrm{z}_{0,1}^{k,m}}\right)^{3}\right)\left[\frac{\left(z_{0,1}^{k,m}\right)^{2}}{2}-\frac{\left(z_{1,1}^{k,m}\right)^{2}}{2}\ +2z\left(z_{0,1}^{k,m}-z_{1,1}^{k,m}\right)\right. \\
&\left.+3z^{2}\ln\left(\frac{z_{0,1}^{k,m}-z}{z_{1,1}^{k,m}-z}\right)+z^{3}\left(\frac{1}{z_{1,1}^{k,m}-z}-\frac{1}{z_{0,1}^{k,m}-z}\right)\right] \\
&+\left(z_{0,0}^{k,m}\overline{z_{0,0}^{k,m}}\right)\left\{\ln\left(\frac{z_{0,0}^{k,m}-z}{z_{0,1}^{k,m}-z}\right)+z\left(\frac{1}{z_{0,1}^{k,m}-z}-\frac{1}{z_{0,0}^{k,m}-z}\right)\right\}-\frac{\left(z_{0,0}^{k,m}\overline{z_{0,0}^{k,m}}\right)^{3}}{3} \\
&\times\left\{\frac{3}{z^{4}}\cdot\ln\left(\frac{z_{0,0}^{k,m}}{z_{0,1}^{k,m}}\right)-\frac{2}{z^{3}}\cdot\left(\frac{1}{z_{0,0}^{k,m}}-\frac{1}{z_{0,1}^{k,m}}\right)-\frac{1}{2z^{2}}\cdot\left(\frac{1}{\left(z_{0,0}^{k,m}\right)^{2}}-\frac{1}{\left(z_{0,1}^{k,m}\right)^{2}}\right)\right. \\
&\qquad\left.-\frac{3}{z^{4}}\cdot\ln\left(\frac{z_{0,0}^{k,m}-z}{z_{0,1}^{k,m}-z}\right)+\frac{1}{z^{3}}\cdot\left(\frac{1}{z_{0,1}^{k,m}-z}-\frac{1}{z_{0,0}^{k,m}-z}\right)\right\},
\end{aligned}
\tag{21}
$$

$$\begin{aligned}
S^d_{k,m}(z) &= \frac{\overline{z^{k,m}_{0,0}}}{z^{k,m}_{0,0}}\left[\ln\frac{z^{k,m}_{1,0}-z}{z^{k,m}_{0,0}-z}+z\left(\frac{1}{z^{k,m}_{0,0}-z}-\frac{1}{z^{k,m}_{1,0}-z}\right)\right]\\
&-z^{k,m}_{1,1}\overline{z^{k,m}_{1,1}}\frac{1}{z^2}\ln\frac{\left(z^{k,m}_{1,1}-z\right)z^{k,m}_{1,0}}{\left(z^{k,m}_{1,0}-z\right)z^{k,m}_{1,1}}+z^{k,m}_{1,1}\overline{z^{k,m}_{1,1}}\frac{1}{z}\left(\frac{1}{z^{k,m}_{1,0}-z}-\frac{1}{z^{k,m}_{1,1}-z}\right)\\
&+\frac{\overline{z^{k,m}_{1,1}}}{z^{k,m}_{1,1}}\left[\ln\frac{z^{k,m}_{0,1}-z}{z^{k,m}_{1,1}-z}+z\left(\frac{1}{z^{k,m}_{1,1}-z}-\frac{1}{z^{k,m}_{0,1}-z}\right)\right]\\
&-\frac{z^{k,m}_{0,0}\overline{z^{k,m}_{0,0}}}{z^2}\ln\left(\frac{\left(z^{k,m}_{0,0}-z\right)z^{k,m}_{0,1}}{\left(z^{k,m}_{0,1}-z\right)z^{k,m}_{0,0}}\right)+\frac{z^{k,m}_{0,0}\overline{z^{k,m}_{0,0}}}{z}\left(\frac{1}{z^{k,m}_{0,1}-z}-\frac{1}{z^{k,m}_{0,0}-z}\right).
\end{aligned} \tag{22}$$

Therefore, we obtain quadrature formula

$$\begin{aligned}
S\left(\rho|z\right) \approx \sum_{k=1}^{N-1}\sum_{m=0}^{M_k-1}&\Big(a_{k,m}S^a_{k,m}\left(z\right)+\,b_{k,m}S^b_{k,m}\left(z\right)\\
&+c_{k,m}S^c_{k,m}\left(z\right)\,+\,d_{k,m}S^d_{k,m}\left(z\right)\Big),\, z\in\Omega^{(2)},
\end{aligned}$$

where coefficients $a_{k,m}$, $b_{k,m}$, $c_{k,m}$, and $d_{k,m}$ and integrals $S^a_{k,m}$, $S^b_{k,m}$, $S^c_{k,m}$ and $S^d_{k,m}$ are defined by (5), (6), (7), (8) and (19), (20), (21), (22), respectively.

5 Error analysis

In this section, we present numerical results of error for proposed approximate formulas of T and S in test examples by using MATLAB program.

Let us take $l_2(\Omega^{(j)})$ $(j=1,2)$, the spaces of the grid functions $\rho^{\tau,h}$, defined on $\Omega^{(j)}$ equipped with the norm

$$\left\|\rho^{\tau,h}\right\|_{l_2(\Omega^{(j)})}=\sqrt{\pi}\tau\left[\sum_{k=1}^{N-1}\sum_{m=1}^{M_k-1}|\rho(r_ke^{i\theta_{k,m}})|^2\right]^{1/2}, \tag{23}$$

where $\rho_{k,m}=\rho(r_ke^{i\theta_{k,m}}), r_ke^{i\theta_{k,m}}\in\Omega^{(j)}$.

Table 1 displays the value of error of $T(\rho|z)$ defined by

$$ErT=\sqrt{\pi}\tau\left[\sum_{k=1}^{N-1}\sum_{m=1}^{M_k-1}|T(\rho|z^{*k,m}_{0,0})-T(\widehat{\rho}|z^{*k,m}_{0,0})\,|^2\right]^{1/2} \tag{24}$$

for $N=5,10,20,40$.

Table 1. Error analysis for $T(\rho|z)$

	$\rho(z)$	$T(\rho\|z)$	$N=5$	$N=10$	$N=20$	$N=40$
1	$z\overline{z}$	$\frac{z\overline{z}^2}{2}$	0.2450	0.0278	0.0059	0.0014
2	$z\overline{z}^2$	$\frac{z\overline{z}^3}{3}$	0.1803	0.0162	0.0038	9.4×10^{-04}
3	$z\overline{z}^3$	$\frac{z\overline{z}^4}{4}$	0.1611	0.0338	0.0076	0.0019
4	z	$z\overline{z}-1$	0.3018	0.0377	0.0047	5.8×10^{-04}
5	$\overline{z}$	$\frac{\overline{z}^2}{2}$	0.0989	0.0124	0.0015	1.9×10^{-04}
6	z^2	$z^2\overline{z}-z$	0.1902	0.0227	0.0048	0.0012
7	$\overline{z}^2$	$\frac{\overline{z}^3}{3}$	0.1489	0.0206	0.0047	0.0012
8	$\overline{z}^3$	$\frac{\overline{z}^4}{4}$	0.2027	0.0526	0.0133	0.0033
9	$z^2\overline{z}$	$\frac{z^2\overline{z}^2}{2}-\frac{1}{2}$	0.1112	0.0226	0.0059	0.0015
10	$z^2\overline{z}^2$	$\frac{z^2\overline{z}^3}{3}$	0.1417	0.0194	0.0048	0.0012
11	$z^2\overline{z}^3$	$\frac{z^2\overline{z}^4}{4}$	0.1215	0.0212	0.0049	0.0012
12	$z^3\overline{z}$	$\frac{z^3\overline{z}^2}{2}-\frac{z}{2}$	0.1369	0.0279	0.0061	0.0015
13	$z^3\overline{z}^2$	$\frac{z^3\overline{z}^3}{3}-\frac{1}{3}$	0.1144	0.0344	0.0086	0.0022
14	$z^3\overline{z}^3$	$\frac{z^3\overline{z}^4}{4}$	0.0885	0.0181	0.0045	0.0011
15	$z^3\overline{z}^4$	$\frac{z^3\overline{z}^5}{5}$	0.1096	0.0254	0.0057	0.0014
16	z^3	$z^3\overline{z}-z^2$	0.1807	0.0388	0.0096	0.0024
17	$z^2-\overline{z}^2$	$z^2\overline{z}-z-\frac{\overline{z}^3}{3}$	0.1785	0.0112	6.9×10^{-04}	4.3×10^{-05}

Table 2 layouts the value of error of $S(\rho|z)$ defined by

$$ErS = \sqrt{\pi}\tau \left[\sum_{k=1}^{N-1} \sum_{m=1}^{M_k-1} |S(\rho|z_{0,0}^{*k,m}) - S(\widehat{\rho}|z_{0,0}^{*k,m})\,|^2 \right]^{1/2} \tag{25}$$

for $N = 5, 10, 20, 40$.

6 Conclusion

In this paper, we construct second-order of accuracy quadrature formulas for the numerical calculation of the Vekua types two-dimensional potential and singular integral operators in the unit disk of complex plane. We propose quadrature formulas for these integrals which based on first-order spline approximation of two-dimensional function. MATLAB program are used for numerical experiments in test examples.

Table 2. Error analysis for $S(\rho|z)$

	$\rho(z)$	$S(\rho\|z)$	$N=5$	$N=10$	$N=20$	$N=40$
1	$z\overline{z}$	$\frac{\overline{z}^2}{2}$	0.0775	0.0138	0.0030	7.1×10^{-04}
2	$z\overline{z}^2$	$\frac{\overline{z}^3}{3}$	0.4025	0.0683	0.0230	0.0083
3	$z\overline{z}^3$	$\frac{\overline{z}^4}{4}$	0.6517	0.1549	0.0567	0.0207
4	z	$\overline{z}$	8.7×10^{-15}	1.5×10^{-13}	2.1×10^{-12}	5.1×10^{-11}
5	$\overline{z}$	0	0.3072	0.0768	0.0192	0.0048
6	z^2	$2z\overline{z}-1$	0.6155	0.1169	0.0324	0.0110
7	$\overline{z}^2$	0	0.6525	0.1174	0.0322	0.0110
8	$\overline{z}^3$	0	0.7794	0.1901	0.0685	0.0249
9	$z^2\overline{z}$	$z\overline{z}^2$	0.4335	0.0724	0.0239	0.0085
10	$z^2\overline{z}^2$	$\frac{2}{3}z\overline{z}^3$	0.0985	0.0261	0.0067	0.0017
11	$z^2\overline{z}^3$	$\frac{1}{2}z\overline{z}^4$	0.3466	0.0906	0.0332	0.0121
12	$z^3\overline{z}$	$\frac{3}{2}z^2\overline{z}^2-\frac{1}{2}$	0.5976	0.1478	0.0557	0.0206
13	$z^3\overline{z}^2$	$z^2\overline{z}^3$	0.3912	0.0981	0.0346	0.0124
14	$z^3\overline{z}^3$	$\frac{3}{4}z^2\overline{z}^4$	0.1748	0.0463	0.0118	0.0030
15	$z^3\overline{z}^4$	$\frac{3}{5}z^2\overline{z}^5$	0.3654	0.1114	0.0407	0.0150
16	z^3	$3z^2\overline{z}-2z$	0.8176	0.1897	0.0682	0.0248
17	$z^2-\overline{z}^2$	$2z\overline{z}-1$	0.7305	0.0913	0.0114	0.0014

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