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Stability analysis for first-order nonlinear differential equations with three-point boundary conditions

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Abstract. In the present paper, we study a system of nonlinear differential equations with three-point boundary conditions. The given original problem is reduced to the equivalent integral equations using Green function. Several theorems are proved concerning the existence and uniqueness of solutions to the boundary value problems for the first order nonlinear system of ordinary differential equations with three-point boundary conditions. The uniqueness theorem is proved by Banach fixed point principle, and the existence theorem is based on Schafer's theorem. Then, we describe different types of Ulam stability: Ulam-Hyers stability, generalized Ulam-Hyers stability. We discuss the stability results providing suitable example.

Keywords. Three-point boundary conditions, Ulam-Hyers stability, existence and uniqueness, fixed point theorems.

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1 Introduction

Lately, stability theory for functional, differential, integral and integro-differential equations has been intensively studied. The most frequent types of citations are shown in the following examples [1, 2, 4]. Stability theory was first introduced from a well-known question raised by S. M. Ulam at the Mathematics Club of the University of Wisconsin in 1940: "When a solution of an equation differing slightly from a given one must be somehow near to the solution of the given equation?" (see e.g., [7]). The first partial answer to Ulam's question in the case of Cauchy's equation in Banach spaces was given by D.H.Hyers in 1941, and which is obtained property is nowadays called the Ulam-Hyers stability (see for more detail [8]). Afterwards, various generalizations of Ulam-Hyers stability are obtained. We show this with the following examples: Ulam-Hyers-Rassias stability, generalized Ulam–Hyers stability, generalized Ulam–Hyers-Rassias stability [3, 5, 6, 9, 11-16].

In the past few years, the study of differential equations with nonlocal boundary conditions has been an important field of mathematics, that has recently received much attention of researchers; the reader is referred to [10, 17-31]. Stability problems for non-local boundary conditions have been newly studied by several authors (see e.g., [32-35] and the references therein).

Here for the first time we investigate Ulam-Hyers stability for three-point boundary value problems. The rest of this paper is organized as follows. In Section 2, notations, problem statement, general definitions, remark and auxiliary lemmas are given, which will be used in the proof of main results. In Section 3, we discuss existence results for investigated boundary value problems. In Section 4, Ulam stability analysis results are obtained. In Section 5, an example illustrates the application for the Ulam-Hyers stability .

2 Problem statement and preliminaries

In this section, we give problem statement, general definitions, remark and lemmas which are used throughout this paper. We denote by $C([0,T],R^n)$ the Banach space of all continuous functions from $[0,T]$ into R^n with the norm

$$\|x\| = \max\{|x(t)| : t \in [0,T]\},$$

where $|\cdot|$ is the norm in space R^n.

We concerned the existence, uniqueness and stability of the system of nonlinear differential equations of the type

$$\dot{x}(t) = f(t, x(t)), t \in [0,T], \tag{1}$$

subject to three point boundary conditions

$$Ax(0) + Bx(t_1) + Cx(T) = d, \tag{2}$$

where A, B, C are constant square matrices of order n such that $\det N \neq 0$, $N = A + B + C, f : [0,T] \times R^n \to R^n$ is a given function, $d \in R^n$ is a given vector and t_1 satisfies the condition $0 < t_1 < T$.

Definition 2.1. For every $\varepsilon > 0$, the function satisfies $y \in C^1([0,T],R^n)$

$$|\dot{y}(t) - f(t, y(t))| \leq \varepsilon, t \in [0,T], \tag{3}$$

where the function f is defined in (1). Let $x \in C([0,T],R^n)$ be a solution of the problem (1)-(2). If there exists a nonzero positive constant k such that

$$|y(t) - x(t)| \leq k\varepsilon, t \in [0,T],$$

then the problem (1)-(2) is said to be Ulam-Hyers stable.

Definition 2.2. Let $y \in C^1([0,T],R^n)$ satisfies the inequality in (3) and $x \in C([0,T],R^n)$ is a solution of (1)-(2). If there is a function $\varphi_f \in C(R^+,R^+)$ with $\varphi_f(0)=0$ satisfying

$$|y(t)-x(t)| \le \varphi_f(\varepsilon), t \in [0,T],$$

then the problem (1)-(2) is said to be generalized Ulam-Hyers stable.

Remark 2.3. A function $y \in C^1([0,T],R^n)$ is said to be a solution to (3) if and only if we can find a function $\varphi \in C([0,T],R^n)$ (dependent on y) such that

(i) $|\varphi(t)| \le \varepsilon$ for all $t \in [0,T]$,

(ii) $\dot{y}(t) = f(t,y(t)) + \varphi(t), \;\; t \in [0,T]$.

For simplicity, we can look at the following problem:

Lemma 2.4. *Suppose* $\mu \in C([0,T],R^n)$ *and* $\det N \neq 0$. *Then the unique solution of the following problem*

$$\dot{x}(t) = \mu(t), \;\; t \in [0,T], \tag{4}$$

with three-point boundary conditions

$$Ax(0) + Bx(t_1) + Cx(T) = d, \tag{5}$$

is given by

$$x(t) = D + \int_0^T G(t,\tau)\mu(\tau)d\tau, \tag{6}$$

where

$$D = N^{-1}d, \;\; G(t,\tau) = \begin{cases} G_1(t,\tau), & t \in [0,t_1], \\ G_2(t,\tau), & t \in (t_1,T], \end{cases}$$

such that

$$G_1(t,\tau) = \begin{cases} N^{-1}A, & 0 \le \tau \le t, \\ -N^{-1}(B+C), & t < \tau \le t_1, \\ -N^{-1}C, & t_1 < \tau \le T, \end{cases}$$

and

$$G_2(t,\tau) = \begin{cases} N^{-1}A, & 0 \le \tau \le t_1, \\ N^{-1}(A+B), & t_1 < \tau \le t, \\ -N^{-1}C, & t < \tau \le T. \end{cases}$$

Proof. If function $x = x(\cdot)$ is a solution of the differential equation (1), then for $t \in (0, T)$

$$x(t) = x_0 + \int_0^t \mu(\tau) d\tau, \tag{7}$$

where x_0 is an arbitrary constant vector. Now we define x_0 so that, the function in equality (7) satisfies condition (5)

$$x_0 = N^{-1}d - N^{-1}B \int_0^{t_1} \mu(t)dt - N^{-1}C \int_0^T \mu(t)dt. \tag{8}$$

Now in (7) we take into account the value x_0 determined from the equality (8) and yield

$$x(t) = N^{-1}d - N^{-1}B \int_0^{t_1} \mu(t)dt - N^{-1}C \int_0^T \mu(t)dt + \int_0^t \mu(\tau)d\tau. \tag{9}$$

Suppose that, $t \in [0, t_1]$. Then we can write the equality (9) as follows:

$$x(t) = N^{-1}d - N^{-1}B \left(\int_0^t \mu(\tau)d\tau + \int_t^{t_1} \mu(\tau)d\tau \right)$$
$$-N^{-1}C \left(\int_0^t \mu(\tau)d\tau + \int_t^{t_1} \mu(\tau)d\tau \right) - N^{-1}C \int_{t_1}^T \mu(t)dt + \int_0^t \mu(\tau)d\tau.$$

We group similar terms and then simplify:

$$\begin{aligned} x(t) &= N^{-1}d + \left(E - N^{-1}B - N^{-1}C\right) \int_0^t \mu(\tau)d\tau \\ &\quad - \left(N^{-1}B + N^{-1}C\right) \int_t^{t_1} \mu(\tau)d\tau, \\ -N^{-1}C \int_{t_1}^T \mu(t)dt &= N^{-1}d + N^{-1}A \int_0^t \mu(\tau)d\tau \\ &\quad -N^{-1}(B + C) \int_t^{t_1} \mu(\tau)d\tau - N^{-1}C \int_{t_1}^T y(t)dt, \end{aligned} \tag{10}$$

where E is an identity matrix. Let us define new function as follows:

$$G_1(t,\tau) = \begin{cases} N^{-1}A, & 0 \le \tau \le t, \\ -N^{-1}(B+C), & t < \tau \le t_1, \\ -N^{-1}C, & t_1 < \tau \le T. \end{cases}$$

Equality (10) can be rewritten as integral equation (11), this implies that,

$$x(t) = N^{-1}d + \int_0^T G_1(t,\tau)\mu(\tau)d\tau. \tag{11}$$

Now assume that, $t \in (t_1, T]$. Then we can write the equality (9) as follows:

$$x(t) = N^{-1}d - N^{-1}B\int_0^{t_1}\mu(t)dt - N^{-1}C\int_0^{t_1}\mu(t)dt$$

$$-N^{-1}C\left(\int_{t_1}^{t}\mu(\tau)d\tau + \int_t^T\mu(\tau)d\tau\right) + \int_0^{t_1}\mu(t)dt + \int_{t_1}^{t}\mu(\tau)d\tau$$

$$= N^{-1}d + \left(E - N^{-1}B - N^{-1}C\right)\int_0^{t_1}\mu(t)dt + \left(E - N^{-1}C\right)\int_{t_1}^{t}\mu(\tau)d\tau$$

$$-N^{-1}C\int_t^T\mu(\tau)d\tau = N^{-1}d + N^{-1}A\int_0^{t_1}\mu(t)dt$$

$$N^{-1}\left(A+B\right)\int_{t_1}^{t}\mu(\tau)d\tau - N^{-1}C\int_t^T\mu(\tau)d\tau.$$

We introduce a new function as follows:

$$G_2(t,\tau) = \begin{cases} N^{-1}A, & 0 \le \tau \le t_1, \\ N^{-1}(A+B), & t_1 < \tau \le t, \\ -N^{-1}C, & t < \tau \le T. \end{cases}$$

Hence, if $t \in (t_1, T]$, then we can write the equality (9) as follows:

$$x(t) = N^{-1}d + \int_0^T G_2(t,\tau)\mu(\tau)d\tau.$$

Thus, the solution of the boundary value problem (4)-(5) can be shown as follows:

$$x(t) = D + \int_0^T G(t,\tau)\mu(\tau)d\tau.$$

We showed the validity of (6). Proof is completed. □

Lemma 2.5. *Assume that $f \in C([0,T], R^n)$. Then the function $x(t)$ is a solution of the boundary value problem* (1)-(2) *if and only if $x(t)$ is a solution of the integral equation*

$$x(t) = D + \int_0^T G(t,\tau)f(\tau, x(\tau))d\tau. \tag{12}$$

Proof. Let $x(t)$ be a solution of the boundary value problem (1)-(2). This lemma can be derived by a similar argument to Lemma 2.4. By checking directly we make sure that the solution of integral equation (12) satisfies the boundary value problem (1)-(2). Lemma 2.5 is proved. □

3 Existence results

Let P be an operator such that, $P : C([0,T], R^n) \to C([0,T], R^n)$ as

$$Px(t) = D + \int_0^T G(t,\tau)f(\tau, x(\tau))d\tau.$$

Obviously, the problem (1)-(2) is equivalent to the fixed point problem $x = Px$. So, the problem (1)-(2) has a solution if and only if the operator P has a fixed point. In Lemma 2.4, we use the most basic fixed point theorem named the contraction mapping principle and it uses the assumption:

(H1) There exists a continuous function $M(t) \geq 0$ such that

$$|f(t,x) - f(t,y)| \leq M(t)\,|x-y|$$

for each $t \in [0,T]$ and all $x, y \in R^n$.

Theorem 3.1. *Assume that, the assumption (H1) holds, and*

$$L = TSM < 1, \tag{13}$$

then the boundary-value problem (1)-(2) *has a unique solution on* $[0,T]$, *where*

$$M = \max_{[0,T]} M(t),$$

$$S = \max_{[0,T]\times[0,T]} \|G(t,\tau)\|.$$

Proof. Setting $\max\limits_{[0,T]} |f(t,0)| = M_f$ and choosing $r \geq \frac{\|D\|+M_f TS}{1-L}$ we show that $PB_r \subset B_r$ where

$$B_r = \{x \in C([0,T], R^n) : \|x\| \leq r\}.$$

For $x \in B_r$, we have

$$\|Px(t)\| \leq \|D\| + \int_0^T |G(t,\tau)| \left(|f(\tau,x(\tau)) - f(\tau,0)| + |f(\tau,0)|\right) d\tau$$

$$\leq \|D\| + S \int_0^T (M\,|x| + M_f)\, dt \leq \|D\|$$

$$+SMrT + M_f TS \leq \frac{\|D\| + M_f TS}{1-L} \leq r.$$

Now for any $x, y \in B_r$ we have

$$|Px - Py| \leq \int_0^T |G(t,\tau)\left(f(\tau,x(\tau)) - f(\tau,y(\tau))\right)| d\tau$$

$$\leq \int_0^T |G(t,\tau)|\,|f(\tau,x(\tau)) - f(\tau,y(\tau))|\, d\tau$$

$$\leq S \int_0^T M(t)\,|x(t) - y(t)|\, dt \leq SMT \max_{[0,T]} |x(t) - y(t)| \leq SMT\,\|x - y\|$$

or

$$\|Px - Py\| \leq L\,\|x - y\|.$$

It is seen that, P is contraction by condition (13). So, the boundary-value problem (1)-(2) has a unique solution. □

Theorem 3.2. ***(Schafer's fixed point theorem).*** *Let X be a Banach space. Assume that, $G : X \to X$ is a completely continuous operator and the set $\rho = \{x \in X | x = \beta Gx, 0 < \beta < 1\}$ is bounded. Then G has a fixed point in X.*

Now we apply Schafer's fixed point theorem and it uses the following assumption:

(H2) The function $f : [0, T] \times R^n \to R^n$ is continuous.

Theorem 3.3. *Assume that there exists $\rho \in C([0, T], R^+)$ such that $|f(t, x(t))| \leq \rho(t), \forall t \in [0, T], x \in C([0, T], R^n)$ with $\sup\limits_{t\in[0,T]} |\rho(t)| = \|\rho\|$. Then the boundary value problem* (1)-(2) *has at least one solution on* $[0, T]$.

Proof. Since f is continuous, the operator P is continuous. Let $\phi \in C([0, T], R^n)$ be bounded subset. Then $\forall x \in \phi$ together with the given assumption $|f(t, x(t))| \leq \rho(t)$, we get

$$|P(x)(t)| \leq \sup_{t\in[0,T]} \left\{ D + \int_0^T |G(t, \tau)| \, |f(\tau, x(\tau))| \, d\tau \right\}.$$

Hence,

$$|P(x)(t)| \leq \|D\| + ST\rho.$$

Thus,

$$\|P(x)(t)\| \leq \|D\| + ST\rho = l.$$

This shows that, P is bounded. Now for $0 < \tau_1 < \tau_2 < T$, we have

$$|P(x)(\tau_2) - P(x)(\tau_1)| \leq \|\rho\| (\tau_2 - \tau_1)$$

which tends to zero as $\tau_2 \to \tau_1$. We conclude that the mapping $P : C([0, T], R^n) \to C([0, T], R^n)$ is completely continuous by Arzela-Ascoli theorem. We show that a set $\Omega = \{x \in C([0, T], R^n) : x = \lambda P(x), \text{ for some } 0 < \lambda < 1\}$ is bounded. Assume that, $x = \lambda P(x)$ for some $0 < \lambda < 1$. Then for each $t \in [0, T]$, we can write

$$x(t) = \lambda D + \lambda \int_0^T G(t, \tau) f(\tau, x(\tau)) d\tau.$$

From here

$$\|x\| \leq \|D\| + S \|\rho\| T.$$

Therefore, the set Ω is bounded. Since all conditions of Theorem 3.2 are satisfied, P has at least one fixed point. So, there exists at least one solution for the problem (1)-(2) on $[0, T]$. □

4 Ulam stability analysis results

Lemma 4.1. *For every $\varepsilon > 0$, the function $y \in C^1([0,T], R^n)$ satisfies the inequality*

$$|\dot{y}(t) - f(t, y(t))| \le \varepsilon,$$

then y is a solution of the inequality

$$|y(t) - Py(t)| \le ST\varepsilon. \tag{14}$$

Proof. From Remark 2.3 (ii) and Lemma 2.5, we have

$$y(t) = D + \int_0^T G(t,\tau)\left(f(\tau, y(\tau)) + \varphi(\tau)\right) d\tau.$$

Then by Remark 2.1 (i), we obtain

$$y(t) - Py(t) = \int_0^T G(t,\tau)\left(f(\tau, y(\tau)) + \varphi(\tau)\right) d\tau$$

$$- \int_0^T G(t,\tau) f(\tau, y(\tau)) d\tau,$$

$$|y(t) - Py(t)| \le S \int_0^T |\varphi(\tau)|\, d\tau \le ST\varepsilon.$$

Clearly, the inequality (14) is satisfied. This proves our statement. □

Theorem 4.2. *If the condition (H1) is satisfied and* $\det N \neq 0$ *holds, then the problem* (1)-(2) *is Ulam-Hyers stable.*

Proof.

$$|y(t) - x(t)| = \left| y(t) - \int_0^T G(t,\tau) f(\tau, x(\tau)) d\tau \right|$$

$$= |y(t) - Py(t) + Py(t) - Px(t)| \le |y(t) - Py(t)|$$

$$+ \left| \int_0^T G(t,\tau)\left(f(\tau, y(\tau)) - f(\tau, x(\tau))\right) d\tau \right|$$

$$\leq ST\varepsilon + SMT\,\|y(t) - x(t)\|\,,$$

$$(1 - SMT)\,\|y - x\| \leq ST\varepsilon,$$

$$\|y - x\| \leq \frac{ST\varepsilon}{1 - SMT}.$$

Obviously, the problem (1)-(2) is Ulam-Hyers stable. So, by setting $\varphi_f(\varepsilon) = \frac{ST\varepsilon}{1-SMT}$, $\varphi_f(0) = 0$, problem (1)-(2) is generalized Ulam-Hyers stable. The proof is completed. □

5 Example

Consider the following system of differential equation

$$\begin{cases} \dot{x}_1 = \sin \alpha x_2, \\ \dot{x}_2 = \frac{\beta|x_1|}{1+t^2}, \end{cases}$$

subject to

$$x_1(0) + x_2(0) - x_2\left(\frac{1}{2}\right) = 1, \; - x_1\left(\frac{1}{2}\right) + x_1(1) + x_2(1) = 0.$$

Obviously,

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix},$$

$$A + B + C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For $t \in \left[0, \frac{1}{2}\right]$, we obtain

$$G_1(t,\tau) = \begin{cases} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, & 0 \leq \tau \leq t, \\ \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, & t < \tau \leq \frac{1}{2}, \\ \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix}, & \frac{1}{2} < \tau \leq 1 \end{cases}$$

and for $t \in \left(\frac{1}{2}, 1\right]$

$$G_2(t,\tau) = \begin{cases} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, & 0 \le \tau \le \frac{1}{2}, \\ \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, & \frac{1}{2} < \tau \le t, \\ \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix}, & t < \tau \le 1. \end{cases}$$

Here, $\|G\| \le 2$ and $1 \cdot 2 \cdot \max\{|\alpha|, |\beta|\} < 1$. So, $\max\{|\alpha|, |\beta|\} < \frac{1}{2}$. We can easily see that the given system is Ulam-Hyers stable.

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