e-Journal of Analysis and Applied Mathematics **2021** (2021), 1 – 11
DOI 10.2478/ejaam-2021-0001

On the non existence of periodic orbits for a class of two dimensional Kolmogorov systems

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Communicated by Sandra Pinelas

Abstract. In this paper we characterize the integrability and the non-existence of limit cycles of Kolmogorov systems of the form

$$x' = x\left(B_1(x,y)\,ln\left|\frac{A_3(x,y)}{A_4(x,y)}\right| + B_3(x,y)ln\left|\frac{A_1(x,y)}{A_2(x,y)}\right|\right),$$

$$y' = y\left(B_2(x,y)ln\left|\frac{A_5(x,y)}{A_6(x,y)}\right| + B_3(x,y)ln\left|\frac{A_1(x,y)}{A_2(x,y)}\right|\right)$$

where $A_1(x,y), A_2(x,y), A_3(x,y), A_4(x,y), A_5(x,y), A_6(x,y), B_1(x,y), B_2(x,y), B_3(x,y)$ are homogeneous polynomials of degree a, a, b, b, c, c, n, n, m respectively. Concrete example exhibiting the applicability of our result is introduced.

Keywords. Kolmogorov system, first integral, periodic orbits, limit cycle.

2020 Mathematics Subject Classification. 34C05, 34C07, 37C27, 37K10.

1 Introduction

The autonomous differential systems on the plane given by

$$\begin{cases} x' = \dfrac{dx}{dt} = xF(x,y), \\ \\ y' = \dfrac{dy}{dt} = yG(x,y), \end{cases} \tag{1}$$

is known as Kolmogorov systems, the derivatives are performed with respect to the time variable, and F , G are two functions in the variables x and y. Is frequently used to model the iteration of two species occupying the same ecological niche [10, 15, 17]. There are many natural phenomena which can be modeled by the Kolmogorov systems such as mathematical ecology and population dynamics [12, 18, 19] chemical reactions, plasma physics [14], hydrodynamics [5], economics, etc. In the classical Lotka- Volterra-Gause model, F and G are linear and it is well known that there are no limit cycles. There can, of course, only be one critical point in the interior of the positive quadrant $(x > 0, y > 0)$ in this case, but this can be a center; however, there are no isolated periodic solutions. We remind that in the phase plane, a limit cycle of system (1) is an isolated periodic orbit in the set of all periodic orbits of system (1). In the qualitative theory of planar dynamic systems $[4, 7, 8, 9, 16]$, one of the most important topics is related to the second part of the unsolved Hilbert 16th problem $[13]$. There is a huge literature about limit cycles, most of them deal essentially with their detection, their number and their stability and rare are papers concerned by giving them explicitly $[1, 2, 3, 11, 21]$.

System (1) is integrable on an open set Ω of $\mathbb{R}^2$ if there exists a non constant C^1 function $H : \Omega \to \mathbb{R}$, called a first integral of the system on Ω , which is constant on the trajectories of the system (1) contained in Ω, i.e., if

$$\frac{dH\left(x,y\right)}{dt}=\frac{\partial H\left(x,y\right)}{\partial x}xF\left(x,y\right)+\frac{\partial H\left(x,y\right)}{\partial y}yG\left(x,y\right)\equiv 0 \text{ in the points of } \Omega.$$

Moreover, $H = h$ is the general solution of this equation, where h is an arbitrary constant. It is well known that for differential systems defined on the plane $\mathbb{R}^2$ the existence of a first integral determines their phase portrait [6].

In this paper we are interested in studying the integrability and the periodic orbits of the 2-dimensional Kolmogorov systems of the form

$$\begin{cases} x' = x\left(B_1\left(x,y\right)\ln\left|\frac{A_3(x,y)}{A_4(x,y)}\right| + B_3\left(x,y\right)\ln\left|\frac{A_1(x,y)}{A_2(x,y)}\right|\right), \\ \\ y' = y\left(B_2\left(x,y\right)\ln\left|\frac{A_5(x,y)}{A_6(x,y)}\right| + B_3\left(x,y\right)\ln\left|\frac{A_1(x,y)}{A_2(x,y)}\right|\right), \end{cases} \tag{2}$$

where $A_1\left(x,y\right), A_2\left(x,y\right), A_3\left(x,y\right), A_4\left(x,y\right), A_5\left(x,y\right), A_6\left(x,y\right), B_1\left(x,y\right)$, $B_2\left(x,y\right), B_3\left(x,y\right)$ are homogeneous polynomials of degree a, a, b, b, c, c, n, n, m respectively.

We define the trigonometric functions

$$\begin{aligned} f_1\left(\theta\right) =& B_1\left(\cos\theta,\sin\theta\right)\left(\cos^2\theta\right)\ln\left|\frac{A_3\left(\cos\theta,\sin\theta\right)}{A_4\left(\cos\theta,\sin\theta\right)}\right| \\ &+ B_2\left(\cos\theta,\sin\theta\right)\left(\sin^2\theta\right)\ln\left|\frac{A_5\left(\cos\theta,\sin\theta\right)}{A_6\left(\cos\theta,\sin\theta\right)}\right|, \end{aligned}$$

$$f_2\left(\theta\right) = B_3\left(\cos\theta,\sin\theta\right)\ln\left|\frac{A_1\left(\cos\theta,\sin\theta\right)}{A_2\left(\cos\theta,\sin\theta\right)}\right|,$$

$$f_3(\theta) = (\cos\theta\sin\theta)\left(B_2(\cos\theta,\sin\theta)\ln\left|\frac{A_5(\cos\theta,\sin\theta)}{A_6(\cos\theta,\sin\theta)}\right|\right.$$
$$\left. -B_1(\cos\theta,\sin\theta)\ln\left|\frac{A_3(\cos\theta,\sin\theta)}{A_4(\cos\theta,\sin\theta)}\right|\right).$$

2 Main result

Our main result on the integrability and the periodic orbits of the 2-dimensional Kolmogorov system (2) is the following.

Theorem 2.1. *Consider the Kolmogorov system* (2)*, then the following statements hold.*

(1) *If* $f_3(\theta) \neq 0$, $A_i(\cos\theta,\sin\theta) \neq 0$ *for* $i = 1,2,3,4,5,6$ *and* $n \neq m$*, then system* (2) *has the first integral*

$$\begin{aligned} H(x,y) &= \left(x^2+y^2\right)^{\frac{n-m}{2}} \exp\left((m-n)\int_0^{\arctan\frac{y}{x}} M(\omega)\,d\omega\right) \\ &\quad -(n-m)\int_0^{\arctan\frac{y}{x}} \exp\left((m-n)\int_0^{w} M(\omega)\,d\omega\right) N(w)\,dw, \end{aligned}$$

where $M(\theta) = \frac{f_1(\theta)}{f_3(\theta)}$, $N(\theta) = \frac{f_2(\theta)}{f_3(\theta)}$*, and the curves which are formed by the trajectories of the differential system* (2)*, in Cartesian coordinates are written as*

$$x^2+y^2 = \begin{pmatrix} h\exp\left((n-m)\int_0^{\arctan\frac{y}{x}} M(\omega)\,d\omega\right) \\ +(n-m)\exp\left((n-m)\int_0^{\arctan\frac{y}{x}} M(\omega)\,d\omega\right) \\ \int_0^{\arctan\frac{y}{x}} \exp\left((m-n)\int_0^{w} M(\omega)\,d\omega\right) N(w)\,dw \end{pmatrix}^{\frac{2}{n-m}},$$

where $h \in \mathbb{R}$*. Moreover, the system* (2) *has no limit cycle.*

(2) *If* $f_3(\theta) \neq 0$, $A_i(\cos\theta,\sin\theta) \neq 0$ *for* $i = 1,2,3,4,5,6$ *and* $n - m \neq 1$*, then system* (2) *has the first integral*

$$\begin{aligned} H(x,y) &= \left(x^2+y^2\right)^{\frac{n-m}{2}} \exp\left((m-n)\int_0^{\arctan\frac{y}{x}} M(\omega)\,d\omega\right) \\ &\quad -(n-m)\int_0^{\arctan\frac{y}{x}} \exp\left((m-n)\int_0^{w} M(\omega)\,d\omega\right) N(w)\,dw, \end{aligned}$$

where $M(\theta) = \frac{f_1(\theta)}{f_3(\theta)}$, $N(\theta) = \frac{f_2(\theta)}{f_3(\theta)}$, *and the curves which are formed by the trajectories of the differential system* (2), *in Cartesian coordinates are written as*

$$x^2+y^2 = \left(\begin{array}{c} h\exp\left((n-m)\int_0^{\arctan\frac{y}{x}} M(\omega)\,d\omega\right) \\ +(n-m)\exp\left((n-m)\int_0^{\arctan\frac{y}{x}} M(\omega)\,d\omega\right) \\ \int_0^{\arctan\frac{y}{x}} \exp\left((m-n)\int_0^{w} M(\omega)\,d\omega\right) N(w)\,dw \end{array}\right)^{\frac{2}{n-m}},$$

where $h \in \mathbb{R}$. *Moreover, the system* (2) *has no limit cycle.*

(3) *If* $f_3(\theta) \neq 0$, $A_i(\cos\theta, \sin\theta) \neq 0$ *for* $i = 1,2,3,4,5,6$ *and* $n = m$, *then system* (2) *has the first integral*

$$H(x,y) = \left(x^2+y^2\right)^{\frac{1}{2}} \exp\left(-\int_0^{\arctan\frac{y}{x}} \left(M(\omega)+N(\omega)\right) d\omega\right),$$

and the curves which are formed by the trajectories of the differential system (2), *in cartesian coordinates are written as*

$$\left(x^2+y^2\right)^{\frac{1}{2}} - h\exp\left(\int_0^{\arctan\frac{y}{x}} \left(M(\omega)+N(\omega)\right) d\omega\right) = 0,$$

where $h \in \mathbb{R}$. *Moreover, the system* (2) *has no limit cycle.*

(4) *If* $f_3(\theta) = 0$ *for all* $\theta \in \mathbb{R}$, *then system* (2) *has the first integral* $H = \frac{y}{x}$, *and the curves which are formed by the trajectories of the differential system* (2), *in cartesian coordinates are written as* $y - hx = 0$, *where* $h \in \mathbb{R}$. *Moreover, the system* (2) *has no limit cycle.*

Proof. In order to prove our results we write the differential system (2) in polar coordinates (r,θ), defined by $x = r\cos\theta$ and $y = r\sin\theta$, then system (2) becomes

$$\begin{cases} r' = \dfrac{dr}{dt} = f_1(\theta)\, r^{n+1} + f_2(\theta)\, r^{m+1}, \\ \\ \theta' = \dfrac{d\theta}{dt} = f_3(\theta)\, r^{n}, \end{cases} \tag{3}$$

where the trigonometric functions $f_1(\theta)$, $f_2(\theta)$, $f_3(\theta)$ are given in introduction.

Suppose that $f_3(\theta) \neq 0$, $A_i(\cos\theta, \sin\theta) \neq 0$ for $i = 1,2,3,4,5,6$ and $n \neq m$.

Taking as independent variable the coordinate θ, this differential system (3) writes

$$\frac{dr}{d\theta} = M(\theta)\, r + N(\theta)\, r^{1+m-n}, \tag{4}$$

where $M(\theta) = \dfrac{f_1(\theta)}{f_3(\theta)}$ and $N(\theta) = \dfrac{f_2(\theta)}{f_3(\theta)}$, which is a Bernoulli equation. By introducing the standard change of variables $\rho = r^{n-m}$, we obtain the linear equation

$$\frac{d\rho}{d\theta} = (n-m)\left(M(\theta)\rho + N(\theta)\right). \tag{5}$$

The general solution of linear equation (5) is

$$\begin{aligned}\rho(\theta) &= \exp\left((n-m)\int_0^\theta M(\omega)\,d\omega\right)\\ &\quad \left(\mu + (n-m)\int_0^\theta \exp\left((m-n)\int_0^w M(\omega)\,d\omega\right)N(w)\,dw\right),\end{aligned}$$

where $\mu \in \mathbb{R}$, which has the first integral

$$\begin{aligned}H(x,y) &= \left(x^2+y^2\right)^{\frac{n-m}{2}}\exp\left((m-n)\int_0^{\arctan\frac{y}{x}} M(\omega)\,d\omega\right)\\ &\quad + (m-n)\int_0^{\arctan\frac{y}{x}}\exp\left((m-n)\int_0^w M(\omega)\,d\omega\right)N(w)\,dw.\end{aligned}$$

The curves $H = h$ with $h \in \mathbb{R}$, which are formed by trajectories of the differential system (2), in cartesian coordinates are written as

$$x^2+y^2 = \left(\begin{array}{c} h\exp\left((n-m)\int_0^{\arctan\frac{y}{x}} M(\omega)\,d\omega\right)\\ +(n-m)\exp\left((n-m)\int_0^{\arctan\frac{y}{x}} M(\omega)\,d\omega\right)\\ \int_0^{\arctan\frac{y}{x}}\exp\left((m-n)\int_0^w M(\omega)\,d\omega\right)N(w)\,dw \end{array}\right)^{\frac{2}{n-m}},$$

where $h \in \mathbb{R}$.

Let Γ be a periodic orbit located in the positive quadrant, then periodic orbit Γ is contained in the curve

$$x^2+y^2 = \left(\begin{array}{c} h_\Gamma\exp\left((n-m)\int_0^{\arctan\frac{y}{x}} M(\omega)\,d\omega\right)\\ +(n-m)\exp\left((n-m)\int_0^{\arctan\frac{y}{x}} M(\omega)\,d\omega\right)\\ \int_0^{\arctan\frac{y}{x}}\exp\left((m-n)\int_0^w M(\omega)\,d\omega\right)N(w)\,dw \end{array}\right)^{\frac{2}{n-m}}.$$

The intersection points (x,y) of this curve with straight line $y = \eta x$ for all

$\eta \in \left]0,+\infty\right[$, is given by

$$\begin{cases} y=\eta x, \\ \text{and} \\ x^2+y^2=\left(\begin{array}{c} h_\Gamma \exp\left((n-m)\int_0^{\arctan\frac{y}{x}} M(\omega)\,d\omega\right) \\ +(n-m)\exp\left((n-m)\int_0^{\arctan\frac{y}{x}} M(\omega)\,d\omega\right) \\ \int_0^{\arctan\frac{y}{x}} \exp\left((m-n)\int_0^{w} M(\omega)\,d\omega\right) N(w)\,dw \end{array}\right)^{\frac{2}{n-m}}. \end{cases}$$

Then, the abscissa points of intersection is given by

$$x=\frac{1}{\sqrt{1+\eta^2}}\left(\begin{array}{c} h_\Gamma \exp\left((n-m)\int_0^{\arctan\eta} M(\omega)\,d\omega\right)+ \\ (n-m)\exp\left((n-m)\int_0^{\arctan\eta} M(\omega)\,d\omega\right) \\ \int_0^{\arctan\eta} \exp\left((m-n)\int_0^{w} M(\omega)\,d\omega\right) N(w)\,dw \end{array}\right)^{\frac{2}{n-m}}.$$

From this last formula of x, at most a unique value of x on every half straight OX^+, consequently at most a unique point in positive quadrant $(x>0, y>0)$. So this curve cannot contain the periodic orbit.

Hence statement (1) of Theorem 1 is proved.

Suppose now that $f_3(\theta)\neq 0$, $A_i(\cos\theta,\sin\theta)\neq 0$ for $i=1,2,3,4,5,6$ and $n-m=1$.

Taking as independent variable the coordinate θ, this differential system (3) writes

$$\frac{dr}{d\theta}=M(\theta)\,r+N(\theta), \tag{6}$$

where $M(\theta)=\dfrac{f_1(\theta)}{f_3(\theta)}$ and $N(\theta)=\dfrac{f_2(\theta)}{f_3(\theta)}$, which is a linear equation.

The general solution of linear equation (6) is

$$\begin{aligned}\rho(\theta) &= \exp\left(\int_0^{\theta} M(\omega)\,d\omega\right) \\ &\quad \left(\mu+\int_0^{\theta}\exp\left(-\int_0^{w} M(\omega)\,d\omega\right)N(w)\,dw\right),\end{aligned}$$

where $\mu\in\mathbb{R}$, which has the first integral

$$\begin{aligned}H(x,y)=&\sqrt{x^2+y^2}\exp\left(-\int_0^{\arctan\frac{y}{x}} M(\omega)\,d\omega\right) \\ &-\int_0^{\arctan\frac{y}{x}}\exp\left(-\int_0^{w} M(\omega)\,d\omega\right)N(w)\,dw.\end{aligned}$$

The curves $H = h$ with $h \in \mathbb{R}$, which are formed by trajectories of the differential system (2), in cartesian coordinates are written as

$$x^2 + y^2 = \left(\frac{h \exp\left(\int_0^{\arctan \frac{y}{x}} M(\omega)\, d\omega\right) + \exp\left(\int_0^{\arctan \frac{y}{x}} M(\omega)\, d\omega\right)}{\int_0^{\arctan \frac{y}{x}} \exp\left(-\int_0^{w} M(\omega)\, d\omega\right) N(w)\, dw} \right)^2,$$

where $h \in \mathbb{R}$.

Let Γ be a periodic orbit located in the positive quadrant, then periodic orbit Γ is contained in the curve

$$x^2 + y^2 = \left(\frac{h_\Gamma \exp\left(\int_0^{\arctan \frac{y}{x}} M(\omega)\, d\omega\right) + \exp\left(\int_0^{\arctan \frac{y}{x}} M(\omega)\, d\omega\right)}{\int_0^{\arctan \frac{y}{x}} \exp\left(-\int_0^{w} M(\omega)\, d\omega\right) N(w)\, dw} \right)^2.$$

The intersection points (x, y) of this curve with straight line $y = \eta x$ for all $\eta \in \left]0, +\infty\right[$, is given by

$$\begin{cases} y = \eta x, \\ \text{and} \\ x^2 + y^2 = \left(\dfrac{h_\Gamma \exp\left(\int_0^{\arctan \frac{y}{x}} M(\omega)\, d\omega\right) + \exp\left(\int_0^{\arctan \frac{y}{x}} M(\omega)\, d\omega\right)}{\int_0^{\arctan \frac{y}{x}} \exp\left(-\int_0^{w} M(\omega)\, d\omega\right) N(w)\, dw} \right)^2. \end{cases}$$

Then, the abscissa points of intersection is given by

$$x = \frac{1}{\sqrt{1+\eta^2}} \left(\frac{h_\Gamma \exp\left(\int_0^{\arctan \eta} M(\omega)\, d\omega\right) + \exp\left(\int_0^{\arctan \eta} M(\omega)\, d\omega\right)}{\int_0^{\arctan \eta} \exp\left(-\int_0^{w} M(\omega)\, d\omega\right) N(w)\, dw} \right)^2.$$

From this last formula of x, at most a unique value of x on every half straight OX^+, consequently at most a unique point in positive quadrant $(x > 0, y > 0)$. So this curve cannot contain the periodic orbit.

Hence statement (2) of Theorem 1 is proved.

Suppose now that $f_3(\theta) \neq 0$, $A_i(\cos\theta, \sin\theta) \neq 0$ for $i = 1, 2, 3, 4, 5, 6$ and $n = m$.

Taking as independent variable the coordinate θ, this differential system (3) writes

$$\frac{dr}{d\theta} = (M(\theta) + N(\theta))\, r. \tag{7}$$

The general solution of equation (7) is

$$r(\theta) = \mu \exp\left(\int_0^{\theta} (M(\omega) + N(\omega))\, d\omega\right),$$

where $\mu \in \mathbb{R}$, which has the first integral

$$H(x,y) = \left(x^2+y^2\right)^{\frac{1}{2}} \exp\left(-\int_0^{\arctan\frac{y}{x}} \left(M(\omega)+N(\omega)\right) d\omega\right).$$

The curves $H = h$ with $h \in \mathbb{R}$, which are formed by trajectories of the differential system (2), in cartesian coordinates are written as

$$\left(x^2+y^2\right)^{\frac{1}{2}} - h\exp\left(\int_0^{\arctan\frac{y}{x}} \left(M(\omega)+N(\omega)\right) d\omega\right) = 0,$$

where $h \in \mathbb{R}$.

Let Γ be a periodic orbit located in the positive quadrant, then periodic orbit Γ is contained in the curve

$$\left(x^2+y^2\right)^{\frac{1}{2}} = h_\Gamma \exp\left(\int_0^{\arctan\frac{y}{x}} \left(M(\omega)+N(\omega)\right) d\omega\right).$$

The intersection points (x,y) of this curve with straight line $y = \eta x$ for all $\eta \in \left]0,+\infty\right[$, is given by

$$\begin{cases} y = \eta x, \\ \text{and} \\ \left(x^2+y^2\right)^{\frac{1}{2}} = h_\Gamma \exp\left(\int_0^{\arctan\frac{y}{x}} \left(M(\omega)+N(\omega)\right) d\omega\right). \end{cases}$$

Then, the abscissa points of intersection is given by

$$x = \frac{h_\Gamma}{\sqrt{(1+\eta^2)}} \exp\left(\int_0^{\arctan\eta} \left(M(\omega)+N(\omega)\right) d\omega\right).$$

From this last formula of x, at most a unique value of x on every half straight OX^+, consequently at most a unique point in positive quadrant $(x>0, y>0)$. So this curve cannot contain the periodic orbit.

Hence statement (3) of Theorem 1 is proved.

Assume now that $f_3(\theta) = 0$ for all $\theta \in \mathbb{R}$, then from system (3) it follows that $\theta' = 0$. So the straight lines through the origin of coordinates of the differential system (2) are invariant by the flow of this system. Hence, $\frac{y}{x}$ is a first integral of the system, then curves which are formed by the trajectories of the differential system (2), in cartesian coordinates are written as $y - hx = 0$, where $h \in \mathbb{R}$, since all straight lines through the origin are formed by trajectories, clearly the system has no periodic orbits, consequently no limit cycle.

This completes the proof of statement (4) of Theorem 1. □

3 Example

The following example is given to illustrate our result.

Example 3.1. If we take $A_1(x,y) = 5x^2 + 4y^2$, $A_2(x,y) = x^2 + y^2$, $A_3(x,y) = eA_4(x,y)$, $A_5(x,y) = eA_6(x,y)$, $B_1(x,y) = x^4 + x^3y + 2x^2y^2 + xy^3 + y^4$, $B_2(x,y) = x^4 + 2x^3y + 2x^2y^2 + 2xy^3 + y^4$ and $B_3(x,y) = 3x^2 - xy + 3y^2$, then system (2) reads

$$\begin{cases} x' = x\left(\left(x^4 + x^3y + 2x^2y^2 + xy^3 + y^4\right)\right. \\ \left. + \left(3x^2 - xy + 3y^2\right)\ln\left|\frac{5x^2+4y^2}{x^2+y^2}\right|\right), \\ \\ y' = y\left(\left(x^4 + 2x^3y + 2x^2y^2 + 2xy^3 + y^4\right)\right. \\ \left. + \left(3x^2 - xy + 3y^2\right)\ln\left|\frac{5x^2+4y^2}{x^2+y^2}\right|\right), \end{cases} \tag{8}$$

the 2-dimensional Kolmogorov system (8) in polar coordinates (r,θ) becomes

$$\begin{aligned} r' &= \left(1 + \tfrac{3}{4}\sin 2\theta - \tfrac{1}{8}\sin 4\theta\right) r^5 + (3 - \cos\theta\sin\theta)\ln\left(\tfrac{9}{2} + \tfrac{1}{2}\cos 2\theta\right) r^3, \\ \theta' &= \left(\cos^2\theta\sin^2\theta\right) r^4, \end{aligned}$$

where $f_1(\theta) = 1 + \frac{3}{4}\sin 2\theta - \frac{1}{8}\sin 4\theta$, $f_2(\theta) = (3 - \cos\theta\sin\theta)\ln\left(\frac{9}{2} + \frac{1}{2}\cos 2\theta\right)$ and $f_3(\theta) = \cos^2\theta\sin^2\theta$. In the positive quadrant $(x > 0, y > 0)$ it is the case (a) of the Theorem 1, then the Kolmogorov system (8) has the first integral

$$\begin{aligned} H(x,y) &= \left(x^2 + y^2\right)\exp\left(-2\int_0^{\arctan\frac{y}{x}} M(\omega)\,d\omega\right) \\ &\quad -2\int^{\arctan\frac{y}{x}} \exp\left(-2\int_0^{w} M(\omega)\,d\omega\right) B(w)\,dw, \end{aligned}$$

where

$$M(\omega) = \frac{1 + \frac{3}{4}\sin 2\omega - \frac{1}{8}\sin 4\omega}{\cos^2\omega\sin^2\omega}, N(w) = \frac{(3 - \cos w\sin w)\ln\left(\frac{9}{2} + \frac{1}{2}\cos 2w\right)}{\cos^2 w\sin^2 w}.$$

The curves $H = h$ with $h \in \mathbb{R}$, which are formed by trajectories of the differential system (8), in Cartesian coordinates are written as

$$\begin{aligned} x^2 + y^2 &= h\exp\left(2\int_0^{\arctan\frac{y}{x}} M(\omega)\,d\omega\right) + 2\exp\left(2\int_0^{\arctan\frac{y}{x}} M(\omega)\,d\omega\right) \\ &\quad \int_0^{\arctan\frac{y}{x}} \exp\left(-2\int_0^{w} N(\omega)\,d\omega\right) N(w)\,dw \end{aligned}$$

where $h \in \mathbb{R}$. The system (8) has no periodic orbits, and consequently no limit cycle**.**

Acknowledgement. We acknowledge support of "Direction Générale de la Recherche Scientifique et du Développement Technologique « DGRSDT»". MESRS, Algeria.

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Received January 14, 2021; revised July 5, 2021; accepted August 31, 2021.

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