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New solvability condition of 2-d nonlocal boundary value problem for Poisson's operator on rectangle

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Abstract. Differential and difference interpretations of a nonlocal boundary value problem for Poisson's equation in open rectangular domain are studied. New solvability conditions are obtained in respect of existence, uniqueness and a priori estimate of the classical solution. Second order of accuracy difference scheme is presented.

Keywords. Poisson's operator, nonlocal boundary value problem, nonlocal boundary value condition, rectangular domain, difference scheme.

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1 Introduction

First of all, we note that a detailed overview on the nonlocal boundary value problem (NLBVP) that we consider in this paper is enclosed in [16, p. 38-39].

Let Π designates an open rectangle, i.e., $\Pi = (0 < x < 1) \times (0 < y < \pi)$. Our present paper deals with Poisson's equation $\Delta u(x, y) = f(x, y)$ in the rectangular domain Π where nonlocal boundary value condition (NLBVC) is represented by a linear combination of unknown solution values

$$u(1, y) = \alpha_1 u(\xi_1, y) + \alpha_2 u(\xi_2, y) + ... + \alpha_m u(\xi_m, y)$$

for $y \in [0, \pi]$, $\xi_k \in (0, 1)$, $k = 1, ..., m$ and $u(x, y)|_{\partial\Pi\setminus\{x=1\}} = 0$ is given on three sides of the rectangle boundary $\partial\Pi$. Actually, herein the coefficients α_k, $k = 1, ..., m$ have an arbitrary sign. This kind of NLBVP was considered in [3] where the existence and uniqueness of classical solution were proved against the requirement

$$\sum_{k=1}^{m} \frac{1}{2}(\alpha_k + |\alpha_k|) \leq 1,$$

but a priori estimate

$$||u||_{W_2^2(\Pi)} \leq C||f||_{L_2(\Pi)}$$

was established for the same sign coefficients which satisfy the condition

$$-\infty < \sum_{k=1}^{m} \alpha_k \leq 1.$$

In addition, the second order of accuracy finite-difference scheme was offered on a uniform grid. In [5], the existence and uniqueness of classical solution were proved for a similar NLBVP in a rectangular domain when

$$\sum_{k=1}^{m} |\alpha_k| \leq |B_1|^{-1}$$

for $0 < |B_1| < 1$, where the value $|B_1|^{-1}$ could be an unboundedly large if $\xi_m \to 0$, so that the unboundedness for $\sum\limits_{k=1}^{m} |\alpha_k|$ was revealed.

In [16], the differential and difference variants of NLBVP formulated in [3] were researched for the case when NLBVC encloses positive and negative coefficients together without failing. The condition of paper [3] on the coefficients in respect of NLBVC was improved, the well-posedness of the differential problem was established, a second order of accuracy approximation for the suggested difference scheme was proved.

In our present paper, we obtain a new condition that ensures the existence, uniqueness and a priori estimate of classical solution for the class of NLBVPs which was considered in [16]. Our new well-posedness condition for the differential problem reveals the unboundedness effect for the coefficients of NLBVC. In addition, herein, we improve the condition of [16] in respect of the difference problem and obtain a second order of accuracy for the difference scheme.

Before finishing this introduction, we note that for the NLBVP which we consider in our present paper, the most relevant references [1–15] from [16, p. 51-52] are included in the bibliography.

2 Differential problem

We consider NLBVP

$$\begin{cases} \Delta u(x,y) = f(x,y),\ (x,y) \in \Pi, \\ u(x,0) = u(x,\pi) = 0,\ 0 \leq x \leq 1,\ \ u(0,y) = 0,\ 0 \leq y \leq \pi, \\ \ell[u](y) = 0,\ 0 \leq y \leq \pi, \end{cases} \tag{1}$$

where

$$\ell[u](y) \equiv u(1,y) - \sum_{r=1}^{n} \alpha_r u(\zeta_r, y) + \sum_{s=1}^{m} \beta_s u(\eta_s, y),$$

$0 < \zeta_1 < ... < \zeta_n < 1$, $0 < \eta_1 < ... < \eta_m < 1$, $\zeta_r \neq \eta_s$, $\alpha_r > 0$, $\beta_s > 0$, $r = 1, ..., n$, $s = 1, ..., m$. Further in this article, $\mathcal{A}$ denotes following conditions:

$$-\infty < \sum_{r=1}^{n} \alpha_r - \sum_{s=1}^{m} \beta_s < \frac{\sinh 1}{\sinh \zeta_n} \quad \text{when} \quad \zeta_n < \eta_1;$$

$$\sum_{r=1}^{n} \alpha_r < \frac{\sinh 1}{\sinh \zeta_n} \quad \text{when} \quad \zeta_n > \eta_1.$$

Naturally, the classical solution of NLBVP (1) is the function $u(x, y)$ that belongs to $C^2(\Pi) \cap C(\overline{\Pi})$, satisfies the equation and all conditions of (1).

Lemma 2.1. *For $x \in (0, 1)$ and $t > 1$ the following inequalities hold*

$$1 > \frac{\sinh x}{\sinh 1} > \frac{\sinh tx}{\sinh t}.$$

Proof. Left side of inequality is obvious. Let we show that the other one holds. Let

$$g(t) = \frac{\sinh tx}{\sinh t}$$

for specified $x \in (0, 1)$, then

$$g'(t) = \Big(\frac{\sinh tx}{\sinh t}\Big)' = \frac{x \cosh tx \sinh t - \sinh tx \cosh t}{(\sinh t)^2}.$$

Since

$$\int \sinh at \sinh bt \; dt = \frac{1}{a^2 - b^2}\big(a \sinh bt \cosh at - b \sinh at \cosh bt\big)$$

for $a^2 \neq b^2$,

$$g'(t) = \frac{x \cosh tx \sinh t - \sinh tx \cosh t}{(\sinh t)^2} = \frac{x^2 - 1}{(\sinh t)^2} \int_0^t \sinh x\tau \sinh \tau \; d\tau.$$

Since $g'(t) < 0$ for $t > 0$, $g(t)$ strictly decreases, and therefore, for $t > 1$

$$\frac{\sinh x}{\sinh 1} > \frac{\sinh tx}{\sinh t}.$$

Lemma 2.1 is proved. □

Theorem 2.2. *Let* $f \in C(\overline{\Pi})$. *If* $\mathcal{A}$ *holds, then classical solution of (1) exists, is unique and holds a priori estimate*

$$||u||_{W_2^2(\Pi)} \leq C||f||_{L_2(\Pi)}. \tag{2}$$

Proof. First, we prove a priori estimate (2). We assume that classical solution exists. For $k \in \boldsymbol{N}$ let us denote

$$U_k(x) = \sqrt{2/\pi} \int_0^\pi u(x,y) \sin(ky)\, dy, \tag{3}$$

$$f_k(x) = \sqrt{2/\pi} \int_0^\pi f(x,y) \sin(ky)\, dy, \tag{4}$$

so that using the equation $\Delta u(x,y) = f(x,y)$ and conditions

$$u(0,y) = 0, \;\; u(1,y) = \sum_{r=1}^{n} \alpha_r u(\zeta_r, y) - \sum_{s=1}^{m} \beta_s u(\eta_s, y),$$

we see that $U_k(x)$ satisfies the multipoint problem

$$\begin{cases} L[U_k](x) = f_k(x), \quad 0 < x < 1, \\ U_k(0) = 0, \; \ell[U_k] = 0, \end{cases} \tag{5}$$

where

$$L[U_k](x) \equiv U_k''(x) - k^2 U_k(x), \tag{6}$$

$$\ell[U_k] \equiv U_k(1) - \Big(\sum_{r=1}^{n} \alpha_r U_k(\zeta_r) - \sum_{s=1}^{m} \beta_s U_k(\eta_s) \Big). \tag{7}$$

Letting $U_k(x) = V_k(x) + W_k(x)$, where $V_k(x)$ is the solution of

$$\begin{cases} L[V_k(x)] = f_k(x), \quad 0 < x < 1, \\ V_k(0) = 0, \; V_k(1) = 0, \end{cases} \tag{8}$$

while $W_k(x)$ is the solution of

$$\begin{cases} L[W_k(x)] = 0, \quad 0 < x < 1, \\ W_k(0) = 0, \; \ell[W_k] = -\ell[V_k]. \end{cases} \tag{9}$$

In view of [3, p. 143], the solution of (8) holds the estimates

$$||V_k||_{L_2[0,1]} \ \leq \frac{1}{k^2}\ ||f_k||_{L_2[0,1]}, \tag{10}$$

$$||V_k'||_{L_2[0,1]} \ \leq \frac{1}{k}\ ||f_k||_{L_2[0,1]}, \tag{11}$$

$$||V_k''||_{L_2[0,1]} \ \leq ||f_k||_{L_2[0,1]}. \tag{12}$$

Since $V_k(1)=0$, by virtue of Cauchy-Bunyakovskii inequality

$$\Big|\int_{\zeta_r}^{1}([V_k(x)]^2)'dx\Big| = 2\ \Big|\int_{\zeta_r}^{1} V_k(x)V_k'(x)dx\Big| \leq 2\ ||V_k||_{L_2[0,1]}||V_k'||_{L_2[0,1]}, \tag{13}$$

$$\Big|\int_{\eta_s}^{1}([V_k(x)]^2)'dx\Big| = 2\ \Big|\int_{\eta_s}^{1} V_k(x)V_k'(x)dx\Big| \leq 2\ ||V_k||_{L_2[0,1]}||V_k'||_{L_2[0,1]}. \tag{14}$$

Since for $\xi \in (0,1)$

$$[V_k(\xi)]^2 = \Big|\int_{\xi}^{1}([V_k(x)]^2)'dx\Big|,$$

from (13)-(14), in view of (10)-(11), we get estimates

$$|V_k(\zeta_r)| \leq \frac{\sqrt{2}}{k^{3/2}}||f_k(x)||_{L_2[0,1]}, \qquad |V_k(\eta_s)| \leq \frac{\sqrt{2}}{k^{3/2}}||f_k(x)||_{L_2[0,1]}. \tag{15}$$

Hence,

$$\big|\ell[V_k]\big| \leq \Big(\sum_{r=1}^{n}\alpha_r + \sum_{s=1}^{m}\beta_s\Big)\frac{\sqrt{2}}{k^{3/2}}||f_k(x)||_{L_2[0,1]}. \tag{16}$$

Problem (9) has the solution

$$W_k(x) = \mathcal{W}_k\frac{\sinh kx}{\sinh k}, \tag{17}$$

where

$$\mathcal{W}_k = \frac{-\ell[V_k(x)]}{1-(\sinh k)^{-1}\big(\sum\limits_{r=1}^{n}\alpha_r \sinh k\zeta_r - \sum\limits_{s=1}^{m}\beta_s \sinh k\eta_s\big)} \tag{18}$$

and since the denominator of the fraction in (18) is nonzero, moreover,

$$1-(\sinh k)^{-1}\Big(\sum_{r=1}^{n}\alpha_r \sinh k\zeta_r - \sum_{s=1}^{m}\beta_s \sinh k\eta_s\Big) > 0. \tag{19}$$

Indeed,

$$1-\sum_{r=1}^{n}\alpha_r\frac{\sinh k\zeta_r}{\sinh k}+\sum_{s=1}^{m}\beta_s\frac{\sinh k\eta_s}{\sinh k}\geq 1-\frac{\sinh k\zeta_n}{\sinh k}\sum_{r=1}^{n}\alpha_r+\frac{\sinh k\eta_1}{\sinh k}\sum_{s=1}^{m}\beta_s\geq S_k$$

for

$$S_k=\begin{cases}1, \quad \text{if} \quad -\infty<\sum\limits_{r=1}^{n}\alpha_r-\sum\limits_{s=1}^{m}\beta_s\leq 0, \ \ \zeta_n<\eta_1;\\[2ex] 1-(\sum\limits_{r=1}^{n}\alpha_r-\sum\limits_{s=1}^{m}\beta_s)\frac{\sinh k\zeta_n}{\sinh k}, \quad \text{if} \quad 0<\sum\limits_{r=1}^{n}\alpha_r-\sum\limits_{s=1}^{m}\beta_s, \ \ \zeta_n<\eta_1;\\[2ex] 1-(\sum\limits_{r=1}^{n}\alpha_r)\frac{\sinh k\zeta_n}{\sinh k}, \quad \text{if} \quad 0<\sum\limits_{r=1}^{n}\alpha_r, \ \ \zeta_n>\eta_1.\end{cases}$$

By virtue of Lemma 1,

$$1>\frac{\sinh\zeta_n}{\sinh 1}>\frac{\sinh k\zeta_n}{\sinh k},$$

then, in view of $\mathcal{A}$, we get that $S_k\geq S_0>0$ for

$$S_0=\begin{cases}1, \quad \text{when} \quad -\infty<\sum\limits_{r=1}^{n}\alpha_r-\sum\limits_{s=1}^{m}\beta_s\leq 0, \ \ \zeta_n<\eta_1,\\[2ex] 1-(\sum\limits_{r=1}^{n}\alpha_r-\sum\limits_{s=1}^{m}\beta_s)\frac{\sinh\zeta_n}{\sinh 1}, \quad \text{when} \quad 0<\sum\limits_{r=1}^{n}\alpha_r-\sum\limits_{s=1}^{m}\beta_s, \ \ \zeta_n<\eta_1,\\[2ex] 1-(\sum\limits_{r=1}^{n}\alpha_r)\frac{\sinh\zeta_n}{\sinh 1}, \quad \text{when} \quad 0<\sum\limits_{r=1}^{n}\alpha_r, \ \zeta_n>\eta_1.\end{cases}$$

Therefore,

$$1-(\sinh k)^{-1}\Big(\sum_{r=1}^{n}\alpha_r\sinh k\zeta_r-\sum_{s=1}^{m}\beta_s\sinh k\eta_s\Big)\geq S_0>0. \tag{20}$$

Hence, in view of (16)-(20),

$$|W_k(1)|\leq C_0\,\frac{\sqrt{2}}{k^{3/2}}||\,f_k(x)||_{L_2[0,1]} \tag{21}$$

for

$$C_0=\frac{1}{S_0}\Big(\sum_{r=1}^{n}\alpha_r+\sum_{s=1}^{m}\beta_s\Big).$$

Since, in view of (17),

$$W_k(x) = W_k(1)\frac{\sinh kx}{\sinh k}$$

is the explicit solution of (9), then

$$||W_k||_{L_2[0,1]} \le |\ W_k(1)\ | \left(\frac{\int_0^1 \sinh^2(kx)dx}{\sinh^2 k}\right)^{1/2}, \tag{22}$$

$$||W_k'||_{L_2[0,1]} \le k\ |\ W_k(1)\ | \left(\frac{\int_0^1 \cosh^2(kx)dx}{\sinh^2 k}\right)^{1/2}, \tag{23}$$

$$||W_k''||_{L_2[0,1]} \le k^2\ |\ W_k(1)\ | \left(\frac{\int_0^1 \sinh^2(kx)dx}{\sinh^2 k}\right)^{1/2}. \tag{24}$$

Because

$$\frac{\int_0^1 \sinh^2(kx)dx}{\sinh^2 k} \le \frac{1}{k}, \quad \frac{\int_0^1 \cosh^2(kx)dx}{\sinh^2 k} \le \frac{5}{2k},$$

then, in view of (21), the inequalities (22), (23) and (24) result in

$$||W_k||_{L_2[0,1]} \le C_0\sqrt{2}\frac{1}{k^2}||f_k||_{L_2[0,1]}, \tag{25}$$

$$||W_k'||_{L_2[0,1]} \le C_0\sqrt{5}\frac{1}{k}||f_k||_{L_2[0,1]}, \tag{26}$$

$$||W_k''||_{L_2[0,1]} \le C_0\sqrt{2}\ ||f_k||_{L_2[0,1]}. \tag{27}$$

Hence, in view of (10)-(12),

$$||U_k||_{L_2[0,1]}\ \le C_1\ \frac{1}{k^2}\ ||f_k||_{L_2[0,1]}, \tag{28}$$

$$||\ U_k'||_{L_2[0,1]}\ \le C_2\ \frac{1}{k}\ ||f_k||_{L_2[0,1]}, \tag{29}$$

$$||\ U_k''||_{L_2[0,1]}\ \le C_3\ ||f_k||_{L_2[0,1]}, \tag{30}$$

where $C_1 = C_3 = 1 + C_0\sqrt{2}$, $C_2 = 1 + C_0\sqrt{5}$. Therefore, in view of [3, p. 142-143], we have

$$\sum_{k=1}^{\infty}\int_0^1 U_k^2(x)dx \le C_1^2\ ||f||^2_{L_2(\Pi)},$$

$$\sum_{k=1}^{\infty}\int_0^1 \left(U_k'(x)\right)^2 dx \le \frac{1}{k^2}\ C_2^2\ ||f||^2_{L_2(\Pi)},$$

$$\sum_{k=1}^{\infty}\int_0^1 \left(U_k''(x)\right)^2 dx \le C_3^2\ ||f||^2_{L_2(\Pi)},$$

so that (28)-(30) result [3, p. 142-143] in

$$||u||_{W_2^2(\Pi)} \le C_1||f||_{L_2(\Pi)}, \tag{31}$$

$$||u_{xx}||_{W_2^2(\Pi)} \le C_2||f||_{L_2(\Pi)}, \tag{32}$$

$$||u_{xy}||_{W_2^2(\Pi)} \le C_3||f||_{L_2(\Pi)}. \tag{33}$$

In view of (32), from the equation $\Delta u(x,y) = f(x,y)$ we get

$$||u_{yy}||_{W_2^2(\Pi)} \le C_4||f||_{L_2(\Pi)}. \tag{34}$$

Finally, a priori estimate (2) results from (31)-(34). Since, the uniqueness of classical solution follows from (2), then the existence results from Fredholm's property [2] which is inherent to the problem (1). Theorem 2.2 is proved. □

Corollary 2.3. *Let* $f \in C(\overline{\Pi})$, $n = m$ *and* $\zeta_r < \eta_r$, $r = 1, ..., n$. *If*

$$\sum_{r=1}^{n}\frac{(\alpha_r - \beta_r) + |\alpha_r - \beta_r|}{2} = 0,$$

or if

$$0 < \sum_{r=1}^{n}\frac{(\alpha_r - \beta_r) + |\alpha_r - \beta_r|}{2} < \frac{\sinh 1}{\sinh \zeta_p} \tag{35}$$

for $p \le n$, *so that* $\frac{(\alpha_p-\beta_p)+|\alpha_p-\beta_p|}{2} > 0$, *but* $\frac{(\alpha_{p+i}-\beta_{p+i})+|\alpha_{p+i}-\beta_{p+i}|}{2} = 0$ *for* $1 < i \le n-p$ *(if such* i *does not exists we put* $p = n$*), then classical solution of (1) exists, is a unique and holds a priori estimate (2).*

Proof. In view of (3)-(7), we find that $U_k(x)$ satisfies the multipoint problem (5)

$$\begin{cases} L[U_k(x)] = f_k(x), \quad 0 < x < 1, \\ U_k(0) = 0,\ \ell[U_k] = 0, \end{cases}$$

where

$$\ell[U_k] \equiv U_k(1) - \sum_{r=1}^{n}(\alpha_r U_k(\zeta_r) - \beta_r U_k(\eta_r)). \tag{36}$$

Put $U_k(x) = V_k(x) + W_k(x)$, where $V_k(x)$ is the solution of (8), $W_k(x)$ is the solution of (9). Similar to the proof of Theorem 2.2, estimates (10)-(12) hold, then estimates (13)-(15) hold for $r = s$. Hence, in view of (15),

$$\big|\ell[V_k]\big| \leq \Big(\sum_{r=1}^{n}(\alpha_r + \beta_r)\Big)\frac{\sqrt{2}}{k^{3/2}}||f_k(x)||_{L_2[0,1]}. \tag{37}$$

In view of (17)-(18),

$$\mathcal{W}_k = \frac{-\ell[V_k]}{1-(\sinh k)^{-1}\sum\limits_{r=1}^{n}(\alpha_r \sinh k\zeta_r - \beta_r \sinh k\eta_r)}. \tag{38}$$

Noting that the denominator of the fraction $\mathcal{W}_k$ is nonzero, we have

$$1-\frac{\sum\limits_{r=1}^{n}(\alpha_r \sinh k\zeta_r - \beta_r \sinh k\eta_r)}{\sinh k} \geq 1-\frac{\sum\limits_{r=1}^{n}(\alpha_r - \beta_r)\sinh k\zeta_r}{\sinh k} \geq S_k$$

for

$$S_k = \begin{cases} 1, \quad \text{if} \ \sum\limits_{r=1}^{n}\frac{(\alpha_r-\beta_r)+|\alpha_r-\beta_r|}{2} = 0, \\ 1-\big(\sum\limits_{r=1}^{n}\frac{(\alpha_r-\beta_r)+|\alpha_r-\beta_r|}{2}\big)\frac{\sinh k\zeta_p}{\sinh k}, \quad \text{if} \ \sum\limits_{r=1}^{n}\frac{(\alpha_r-\beta_r)+|\alpha_r-\beta_r|}{2} > 0. \end{cases}$$

By virtue of Lemma 2.1,

$$1 > \frac{\sinh \zeta_p}{\sinh 1} > \frac{\sinh k\zeta_p}{\sinh k},$$

and then $S_k \geq S_0$ for

$$S_0 = \begin{cases} 1, \quad \text{if} \ \sum\limits_{r=1}^{n}\frac{(\alpha_r-\beta_r)+|\alpha_r-\beta_r|}{2} = 0, \\ 1-\big(\sum\limits_{r=1}^{n}\frac{(\alpha_r-\beta_r)+|\alpha_r-\beta_r|}{2}\big)\frac{\sinh \zeta_p}{\sinh 1}, \ \text{if} \ \sum\limits_{r=1}^{n}\frac{(\alpha_r-\beta_r)+|\alpha_r-\beta_r|}{2} > 0. \end{cases} \tag{39}$$

In view of corollary conditions, $S_k \geq S_0 > 0$. Therefore,

$$1-(\sinh k)^{-1}\sum_{r=1}^{n}(\alpha_r \sinh k\zeta_r - \beta_r \sinh k\eta_r) \geq S_0 > 0.$$

Hence, in view of (17) and (36)-(39),

$$|W_k(1)| \leq \frac{\sum\limits_{r=1}^{n}(\alpha_r+\beta_r)}{S_0}\frac{\sqrt{2}}{k^{3/2}}\ ||f_k(x)||_{L_2[0,1]},$$

i.e., (21) holds for $C_0 = S_0^{-1} \sum_{r=1}^{n} (\alpha_r + \beta_r)$. Then (22)-(34) hold similarly as in Theorem 2.2. Finally, a priori estimate (2) results from (31)-(34). Since the uniqueness of classical solution follows from (2), then the existence results from Fredholm's property [2] which is inherent to the problem (1). Corollary 2.3 is proved. □

Note 2.1. To prove Theorem 2.2 and Corollary 2.3, the fulfillment of condition $\mathcal{A}$ and (35) is required correspondingly. Obviously, these conditions cover the condition $S \leq 1$, where

$$S = \begin{cases} \sum_{r=1}^{n} \alpha_r - \sum_{s=1}^{m} \beta_s \text{ if } \zeta_n < \eta_1, \\ \sum_{r=1}^{n} \alpha_r \text{ if } \zeta_n > \eta_1, \\ \sum_{r=1}^{n} \frac{(\alpha_r - \beta_r) + |\alpha_r - \beta_r|}{2}. \end{cases}$$

The condition $S \leq 1$ was required (see [16, p. 39-44]) to prove the well-posedness of NLBVP (1). Obviously, irrespective of ζ_n and ζ_p location, this result also follows from Theorem 2.2 and Corollary 2.3 correspondingly. In addition, for any value $S > 1$, by virtue of Theorem 2.2, we can define an open interval for the location of ζ_n, i.e.,

$$0 < \zeta_n < \operatorname{arsinh}\big(S^{-1} \sinh 1\big),$$

so that the NLBVP (1) remains well-posed. Similarly, by virtue of Corollary 2.3, for any $S > 1$ we can define an interval for ζ_p, i.e.,

$$0 < \zeta_p < \operatorname{arsinh}\big(S^{-1} \sinh 1\big),$$

so that the NLBVP (1) remains well-posed.

Note 2.2. Actually, the requirement $\mathcal{A}$, as well the condition (35), reveals the unboundedness effect, i.e., the corresponding value S could be an arbitrarily large positive real number that depends on $\zeta_n \to 0$, or on $\zeta_p \to 0$, correspondingly, but nevertheless the NLBVP (1) remains well-posed.

Note 2.3. By virtue of Theorem 2.2, we can improve the condition of well-posed solvability for formulated in [3, p. 140] NLBVP (1) and write it as following:

$$\sum_{k=1}^{m} \alpha_k^+ < \frac{\sinh 1}{\sinh \xi_p},$$

where $\alpha_k^+ = 2^{-1}(\alpha_k + |\alpha_k|)$ and p is the largest subindex of ξ_k, $k = 1, ..., m$, so that $\alpha_p > 0$ (we assume that there is at least one α_k, $k = 1, ..., m$ which has positive value), but $\alpha_{p+i} \leq 0$, $1 < i \leq n - p$ ($p = n$ if such i does not exists).

3 Difference problem

We consider difference interpretation of NLBVP (1)

$$\begin{cases} \Lambda Y = Y_{\bar{x}x} + Y_{\bar{y}y} = f(x,y), \quad (x_i, y_j) \in \Pi, \\ Y|_{y=0} = Y|_{y=\pi} = 0, \ x_i \in [0,1), \quad Y|_{x=0} = 0, \quad y_j \in [0,\pi], \\ \mathcal{L}Y = \sum\limits_{r=1}^{n} \alpha_r \Big(Y_{i_{\zeta_r},j} \frac{[(i_{\zeta_r}+1)h_1 - \zeta_r]}{h_1} + Y_{i_{\zeta_r}+1,j} \frac{[\zeta_r - i_{\zeta_r} h_1]}{h_1} \Big) - \\ - \sum\limits_{s=1}^{m} \beta_s \Big(Y_{i_{\eta_s},j} \frac{[(i_{\eta_s}+1)h_1 - \eta_s]}{h_1} + Y_{i_{\eta_s}+1,j} \frac{[\eta_s - i_{\eta_s} h_1]}{h_1} \Big) - Y_{N_1,j} = 0, \\ j = 1, ..., N_2 - 1, \end{cases} \tag{40}$$

where same as in the differential problem we require $0 < \zeta_1 < ... < \zeta_n < 1$, $0 < \eta_1 < ... < \eta_m < 1$, $\zeta_r \neq \eta_s$, $\alpha_r > 0$, $\beta_s > 0$, $r = 1, ..., n$, $s = 1, ..., m$, and additionally, we define the numbers i_{ζ_r} and i_{η_s} by corresponding inequalities $i_{\zeta_r} h_1 \leq \zeta_r < (i_{\zeta_r} + 1)h_1$ for $r = 1, ..., n$ and $i_{\eta_s} h_1 \leq \eta_s < (i_{\eta_s} + 1)h_1$ for $s = 1, ..., m$, at least we put $\zeta_0 = \eta_0 = 0$, $\zeta_{n+1} = \eta_{m+1} = 1$, $h_1 = 1/N_1$, $h_2 = \pi/N_2$ and require $h_1 \leq c_0 h_2$, $c_0 = const$ add $h_1 < \theta$, $\theta = \frac{1}{2}\min\{\zeta_{r+1} - \zeta_r,\ r = 0, 1, ..., n;\ \eta_{s+1} - \eta_s,\ s = 0, 1, ..., m;\ |\zeta_r - \eta_s|,\ r = 1, ..., n,\ s = 1, ..., m\}$.
Let $\overline{\mathcal{A}}$ denotes the condition:

$$-\infty < \sum_{r=1}^{n} \alpha_r - \sum_{s=1}^{m} \beta_s < \big(1 + \frac{4}{\pi}\big)^{1-\zeta_n-\theta} \quad \text{when} \quad \zeta_n < \eta_1,$$

$$\sum_{r=1}^{n} \alpha_r < \big(1 + \frac{4}{\pi}\big)^{1-\zeta_n-\theta} \quad \text{when} \quad \zeta_n > \eta_1.$$

Theorem 3.1. *Let $f(x,y)$ so that $u(x,y) \in C^{(4)}(\overline{\Pi})$ is a solution of NLBVP (1) when the condition $\mathcal{A}$ holds. If, additionally, the condition $\overline{\mathcal{A}}$ holds too, then difference solution of (40) approximates $u(x,y)$ by the second order of accuracy in terms of $h = \sqrt{h_1^2 + h_2^2}$, $h_2 \to 0$ in each of the difference metrics C, W_2^2.*

Proof. We denote $z = Y - u$, then z satisfies the difference problem

$$\begin{cases} \Lambda z = f - \Lambda u = F, \quad (ih_1, jh_2) \in \Pi, \\ z|_{x=0} = z|_{y=0} = z|_{y=\pi} = 0, \quad \mathcal{L}z = -\mathcal{L}u. \end{cases} \tag{41}$$

For this problem $F = O(h^2)$ and $\mathcal{L}u = O(h^2)$ [10, p. 81, 229]. Put $z = \tilde{z} + \hat{z}$, where $\tilde{z}$ is the solution of

$$\begin{cases} \Lambda \tilde{z} = 0, \quad (ih_1, jh_2) \in \Pi, \\ \tilde{z}|_{x=0} = \tilde{z}|_{y=0} = \tilde{z}|_{y=\pi} = 0, \quad \mathcal{L}\tilde{z} = -\mathcal{L}u, \end{cases} \tag{42}$$

and $\hat{z}$ is the solution of

$$\begin{cases} \Lambda\hat{z} = F, \ (ih_1, jh_2) \in \Pi, \\ \hat{z}|_{x=0} = \hat{z}|_{y=0} = \hat{z}|_{y=\pi} = 0, \ \ \mathcal{L}\hat{z} = 0. \end{cases} \tag{43}$$

Further, to estimate $\tilde{z}$ we use [10, p. 113] the orthogonal system of mesh functions $\{\sin(ky)\}|_{k=1}^{k=N_2-1}$, so that from the representation

$$\tilde{z} = \sum_{k=1}^{N_2-1} \tilde{z}_k \sin(ky), \quad y = jh_2, \ \ j = 0, 1, ..., N_2$$

it follows, that $\tilde{z}_k, \ k = 1, ..., N_2 - 1$ is the difference solution of the problem

$$\begin{cases} \Lambda_1\tilde{z}_k - \lambda_k\tilde{z}_k = 0, \\ \tilde{z}_k|_{x=0} = 0, \ \ \mathcal{L}\tilde{z}_k = -Q_k \ , \end{cases} \tag{44}$$

where $\Lambda_1\tilde{z} = \tilde{z}_{\bar{x}x}$, $\lambda_k = 4h_2^{-2}\sin^2(kh_2)$, $Q_k = (\mathcal{L}u)_k$ and, in view of [3, p. 142-143],

$$\tilde{z}_k|_{x_i=ih_1} = A_k \sinh(i \ln q_k),$$

$$A_k = -Q_k/\mathcal{L}[\sinh(i \ln q_k)], \ \ i = 0, ..., N_1,$$

$$q_k = 1 + \lambda_k h_1^2/2 + \sqrt{\lambda_k h_1^2 + \lambda_k^2 h_1^4/4}.$$

Denote $\mathcal{D} = \mathcal{L}[\sinh(i \ln q_k)]$. By acting $\mathcal{L}$ on $\sinh(i \ln q_k)$ in the denominator of the fraction for A_k , we get

$$-\mathcal{D} \geq \sinh(N_1 \ln q_k) - \sum_{r=1}^{n} \alpha_r \sinh((i_{\zeta_n} + 1) \ln q_k) + \sum_{s=1}^{m} \beta_s \sinh(i_{\eta_1} \ln q_k). \tag{45}$$

Hence,

$$-\mathcal{D} \geq \sinh(N_1 \ln q_k) - S \sinh((i_{\zeta_n} + 1) \ln q_k) \tag{46}$$

for

$$S = \begin{cases} 0, \quad \text{if} \ \ -\infty < \sum\limits_{r=1}^{n} \alpha_r - \sum\limits_{s=1}^{m} \beta_s \leq 0, \quad \zeta_n < \eta_1, \\ \sum\limits_{r=1}^{n} \alpha_r - \sum\limits_{s=1}^{m} \beta_s, \quad \text{if} \ \ 0 < \sum\limits_{r=1}^{n} \alpha_r - \sum\limits_{s=1}^{m} \beta_s, \ \zeta_n < \eta_1, \\ \sum\limits_{r=1}^{n} \alpha_r, \quad \text{if} \quad \ \zeta_n > \eta_1. \end{cases}$$

Then

$$-\mathcal{D} \geq \sinh(N_1 \ln q_k) \Big[1 - S \ \frac{\sinh(i_{\zeta_n} + 1) \ln q_k)}{\sinh(N_1 \ln q_k)} \Big], \tag{47}$$

therefore,

$$-\mathcal{D} \geq \sinh(N_1 \ln q_k)\Big[1 - \ S\ \frac{q_k^{i_{\zeta_n}+1} - q_k^{-(i_{\zeta_n}+1)}}{q_k^{N_1} - q_k^{-N_1}}\Big].$$

Since $q_k \geq 1$, we get

$$\frac{q_k^{i_{\zeta_n}+1} - q_k^{-(i_{\zeta_n}+1)}}{q_k^{N_1} - q_k^{-N_1}} \leq \frac{q_k^{i_{\zeta_n}+1}[1 - q_k^{-2(i_{\zeta_n}+1)}]}{q_k^{N_1}[1 - q_k^{-2N_1}]} \leq \frac{q_k^{i_{\zeta_n}+1}}{q_k^{N_1}}.$$

Since $h_1 < \theta$ for $\theta = \frac{1}{2}\min\{\zeta_{r+1} - \zeta_r,\ r = \overline{0,n},\ \eta_{s+1} - \eta_s,\ s = \overline{0,m}\}$, for specified $\delta = 1 - \zeta_n - \theta$ the inequality $\zeta_n + h_1 \leq 1 - \delta$ holds. Hence, $i_{\zeta_n} + 1 \leq h_1^{-1}(1 - \delta)$. Then

$$\frac{q_k^{i_{\zeta_n}+1} - q_k^{-(i_{\zeta_n}+1)}}{q_k^{N_1} - q_k^{-N_1}} \leq \frac{q_k^{N_1(1-\delta)}}{q_k^{N_1}} \leq \frac{1}{q_k^{N_1\delta}}. \tag{48}$$

Therefore,

$$-\mathcal{D} \geq \Big(1 - S\frac{1}{q_k^{N_1\delta}}\Big)\sinh(N_1 \ln q_k). \tag{49}$$

Since

$$q_k^{N_1} \geq (1 + \sqrt{\lambda_k}h_1)^{N_1} \geq (1 + \sqrt{\lambda_1}h_1)^{N_1} \geq (1 + \sqrt{\lambda_1}) \geq 1 + \frac{4}{\pi}, \tag{50}$$

we have

$$-\mathcal{D} \geq \Big[1 - S\frac{1}{(1 + 4/\pi)^{\delta}}\Big]\sinh(N_1 \ln q_k), \tag{51}$$

so that

$$-\mathcal{D} \geq C\sinh(N_1 \ln q_k) \tag{52}$$

for

$$C = \begin{cases} 1, \quad \text{if} \quad -\infty < \sum\limits_{r=1}^{n}\alpha_r - \sum\limits_{s=1}^{m}\beta_s \leq 0, \quad \zeta_n < \eta_1, \\ 1 - \big(1 + 4/\pi\big)^{-\delta}\big(\sum\limits_{r=1}^{n}\alpha_r - \sum\limits_{s=1}^{m}\beta_s\big), \quad \text{if} \quad 0 < \sum\limits_{r=1}^{n}\alpha_r - \sum\limits_{s=1}^{m}\beta_s, \quad \zeta_n < \eta_1, \\ 1 - \big(1 + 4/\pi\big)^{-\delta}\sum\limits_{r=1}^{n}\alpha_r, \quad \text{if} \quad \zeta_n > \eta_1. \end{cases}$$

In summary, since the condition $\overline{\mathcal{A}}$ holds,

$$-\mathcal{L}[\sinh(i \ln q_k)] \geq C\sinh(N_1 \ln q_k) > 0. \tag{53}$$

Finally, in view of (53), by virtue of [3, 150-151], we obtain the estimates

$$\max_{i,j} |\tilde{z}_{ij}| = O(h^2), \ \ ||\tilde{z}||_{W_2^2} = O(h^2), \ \ \max_{i,j} |\hat{z}_{ij}| = O(h^2), \ \ ||\hat{z}||_{W_2^2} = O(h^2).$$

Therefore, $\max_{i,j} |z_{ij}| = O(h^2), \ \ ||z||_{W_2^2} = O(h^2)$. Theorem 3.1 is proved. □

Corollary 3.2. *Let* $n = m, \ \zeta_r < \eta_r, \ r = 1, ..., n$. *Let* $f(x,y)$ *and so that* $u(x,y) \in C^{(4)}(\overline{\Pi})$ *is a solution of NLBVP (1) when condition (35) holds for* $2^{-1} \sum_{r=1}^{n} \big(\alpha_r - \beta_r + |\alpha_r - \beta_r|\big) > 0$. *If*

$$0 < \sum_{r=1}^{n} \frac{(\alpha_r - \beta_r) + |\alpha_r - \beta_r|}{2} < \big(1 + \frac{4}{\pi}\big)^{1-\zeta_p-\theta} \tag{54}$$

for $1 \le p \le n$, *so that* $\frac{(\alpha_p-\beta_p)+|\alpha_p-\beta_p|}{2} > 0$, *but* $\frac{(\alpha_{p+i}-\beta_{p+i})+|\alpha_{p+i}-\beta_{p+i}|}{2} = 0$ *for all* $1 < i \le n-p$ *(if such* i *does not exist, we put* $p = n$*), then difference solution of (40) approximates* $u(x,y)$ *by the second order of accuracy in terms of* $h = \sqrt{h_1^2 + h_2^2}, \ h_2 \to 0$ *in each of the difference metrics* $C, \ W_2^2$.

Proof. In view of (41)-(45), for $\mathcal{D} = \mathcal{L}[\sinh(i \ln q_k)]$ we obtain the inequality

$$-\mathcal{D} \ge \sinh(N_1 \ln q_k) - \sum_{r=1}^{n} \alpha_r \sinh((i_{\zeta_r} + 1) \ln q_k) + \sum_{r=1}^{n} \beta_r \sinh(i_{\eta_r} \ln q_k).$$

Since $i_{\zeta_r} + 1 < i_{\eta_r}, \ r = \overline{1, n}$, we get

$$-\mathcal{D} \ge \sinh(N_1 \ln q_k) - \sum_{r=1}^{n} (\alpha_r - \beta_r) \ \sinh((i_{\zeta_r} + 1) \ln q_k).$$

Hence,

$$-\mathcal{D} \ge \ \Big[1 - \sum_{r=1}^{n} (\alpha_r - \beta_r) \ \Big(\frac{q_k^{i_{\zeta_r}+1} - q_k^{-(i_{\zeta_r}+1)}}{q_k^{N_1} - q_k^{-N_1}}\Big)\Big] \ \sinh(N_1 \ln q_k).$$

Also,

$$-\mathcal{D} \ge \Big[1 - S \ \frac{q_k^{i_{\zeta_p}+1} - q_k^{-(i_{\zeta_p}+1)}}{q_k^{N_1} - q_k^{-N_1}}\Big] \ \sinh(N_1 \ln q_k) \tag{55}$$

for

$$S = \sum_{r=1}^{n} \frac{(\alpha_r - \beta_r) + |\alpha_r - \beta_r|}{2}.$$

By analogy with (48), for $q_k \geq 1$ and $\delta = 1 - \zeta_p - \theta$, we get

$$\frac{q_k^{i_{\zeta_p}+1} - q_k^{-(i_{\zeta_p}+1)}}{q_k^{N_1} - q_k^{-N_1}} \leq \frac{1}{q_k^{N_1\delta}} \tag{56}$$

since the inequalities $\zeta_p + h_1 \leq 1 - \delta$ and $i_{\zeta_p} + 1 \leq h_1^{-1}(1-\delta)$ hold. In view of (50) and (55)-(56), the analogies of (51)-(53) hold for

$$C = 1 - \big(1 + 4/\pi\big)^{-\delta}\Big(\sum_{r=1}^{n} \frac{(\alpha_r - \beta_r) + |\alpha_r - \beta_r|}{2}\Big).$$

In view of (53), similar to Theorem 3.1, we obtain

$$\max_{i,j} |\tilde{z}_{ij}| = O(h^2), \ \ ||\tilde{z}||_{W_2^2} = O(h^2), \ \ \max_{i,j} |\hat{z}_{ij}| = O(h^2), \ \ ||\hat{z}||_{W_2^2} = O(h^2),$$

and therefore, $\max_{i,j} |z_{ij}| = O(h^2)$, $||z||_{W_2^2} = O(h^2)$. Corollary 3.2 is proved. □

4 Conclusion

In this paper we used an approach which is based on modified methods of papers [3] and [16].

The basic result of our paper demonstrates new conditions on the well-posedness of NLBVP (1) (see Theorem 2.2 and Corollary 2.3). The newness of the condition $\mathcal{A}$ and (35) is shown in Note 2.1. As it is shown in Note 2.2, condition $\mathcal{A}$, as well as the requirement (35), reveals the unboundedness effect for the value S, which is specified by corresponding values of the coefficients in NLBVC of the differential problem (1).

The difference interpretation of NLBVP (1) is proposed by the finite-difference scheme (40). In Theorem 3.1, under the condition $\overline{\mathcal{A}}$, and in Corollary 3.2 under the requirement (54), correspondingly, we proved the second order of accuracy approximation for smooth classical solution of NLBVP (1) on a uniform grid with sufficiently small step. The required new condition $\overline{\mathcal{A}}$ and the inequality (54) covers the condition $S \leq 1$ which was used by the author earlier in the paper [16, p. 45-48] to obtain the second order of accuracy approximation.

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