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Common fixed point theorems in complex valued non-negative extended b-metric space

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Abstract. We introduce complex valued non-negative extended b-metric spaces and establish new fixed point results for mappings under some rational contractions. Our idea improves and extends corresponding fixed point theorems in the setting of b-metric, extended b-metric and classical metric spaces. Nontrivial examples are provided to support the hypotheses and usefulness of the main result obtained herein.

Keywords. Complex valued metric space, complex valued non-negative extended b-metric space, fixed point, integral equation.

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1 Introduction

Fixed point theory is one of the highly embraced fields in mathematics. In this area, a huge involvement has been made by Banach [6], who gave the notion of contraction mapping due to a complete metric space to locate fixed point of the specified function. In 1969, Kannan [15] gave an alternate sort of contractive condition that demonstrated fixed point theorem. The distinction in Banach theorem and that of mapping in Kannan is that continuity is necessary for contraction of Banach maps but Kannan maps are not necessarily continuous. Additionally, Chaterjea [7] gave similar kind of contraction. The classical fixed point theorem due to Banach [6] has been generalized by many researchers in various ways (see, for example, [3, 4, 7, 11]) and the references therein. One may also consult Rhoades [17] for different definitions of contractive type mappings. Moreover, all the generalizations of Banach fixed point theorem is further classified in two directions-either the contractive condition is replaced with a more generalized one or the axioms characterizing the ground set is enlarged or weakened. In the second case, some of these metric-like spaces are called semimetric, quasimetric, pseudodometric, b-metric, K-metric, to mention but a few. Along this line, by replacing the set of real numbers as the usual co-domain of a metric, Huang and Zhang [12] established the concept of cone metric as a generalization of metric spaces, thereby, establishing some fixed point theorems for contractive mappings on cone metric spaces. Thereafter, many authors have come up with various important fixed point results in the setting of cone metric spaces (see, for example, [13, 19]). The interested researcher may also want to go deeply into a comprehensive new survey on cone metric spaces by Aleksić et al. [2].

It is well-known that fixed point theorems involving rational contractions cannot be extended or meaningless in cone metric spaces. To circumvent this problem, Azam et al. [3] introduced the notion of complex valued metric spaces and obtained sufficient conditions for the existence of common fixed points of a pair of mappings satisfying contractive type inequalities involving rational expressions. It is interesting to note that complex valued metric space is a special class of cone metric spaces. But, the definition of a cone metric is based on the underlying Banach space which is not a division ring; thus, many results of analysis regarding divisions cannot be generalized to cone metric spaces. On this development, the study of fixed point theorems concerning rational inequalities in complex valued metric spaces have been growing at a geometric rate (see, for example, [9, 16, 20]). Along the line, the idea of b-metric space was presented by Czerwik [8] in 1993. Branciari [21] invented the concept of rectangular metric space by modifying the triangular inequality. Rao [18] introduced the notion of fixed point results on complex valued b-metric spaces, which is broader than complex valued metric spaces. However, every complex valued b-metric space is a cone b-metric space over Banach algebra $\mathbb{C}$ in which the cone is normal with the coefficient of normality $K = 1$, and where the cone has non-empty interior (that is, solid cone). Following [18], various authors have demonstrated fixed point results for different mappings fulfilling rational inequalities with regards to complex valued b-metric spaces (see, for instance, [1, 5]).

Motivated by the ideas presented in [3, 8, 21], we introduce the concept of complex valued non-negative extended b-metric spaces and establish new fixed point theorems for mappings under some rational contractive inequalities. Our idea improves and extends the above mentioned articles and a few others in the corresponding literature. Nontrivial examples are provided to indicate the usefulness and validity of the main result obtained herein.

2 Preliminaries

In this section, we recall some specific concepts which are necessary for the presentation of our main results.

Definition 2.1. [3] Let $\mathbb{C}$ be the set of all complex numbers and $z_1, z_2 \in \mathbb{C}$. The

partial order on $\mathbb{C}$ is defined as:
$z_1 \preceq z_2$, if and only if $Re(z_1) \le Re(z_2)$ and $Im(z_1) \le Im(z_2)$. This implies that $z_1 \preceq z_2$ if one of the following conditions holds:

(i) $Re(z_1) = Re(z_2),\ Im(z_1) < Im(z_2)$,

(ii) $Re(z_1) < Re(z_2),\ Im(z_1) = Im(x_2)$,

(iii) $Re(z_1) < Re(z_2),\ Im(z_1) < Im(z_2)$,

(iv) $Re(z_1) = Re(z_2),\ Im(z_1) = Im(z_2)$.

Definition 2.2. [3] Let X be a non-empty set. If the mapping $d : X \times X \to \mathbb{C}$ satisfies the following conditions :
(i) $0 \preceq d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, y) \preceq d(x, z) + d(z, y)$, for all $x, z, y \in X$,
then d is known as a complex valued metric on X, and the pair (X, d) is said to be a complex valued metric space.

Example 2.3. Let $X = X_1 \cup X_2$, where

$$X_1 = \{z \in \mathbb{C} : Re(z) \ge 0 \text{ and } Im(z) = 0\}$$

and

$$X_2 = \{z \in \mathbb{C} : Re(z) = 0 \text{ and } Im(z) \ge 0\}.$$

Define $d : X \times X \longrightarrow \mathbb{C}$ as follows:

$$d(z_1, z_2) = \begin{cases} \frac{2}{3}\left|x_1 - x_2\right| + \frac{i}{2}\left|x_1 - x_2\right|, & \text{if } z_1, z_2 \in X_1; \\ \frac{1}{2}\left|y_1 - y_2\right| + \frac{i}{2}\left|y_1 - y_2\right|, & \text{if } z_1, z_2 \in X_2; \\ \left(\frac{1}{2}y_1 + \frac{2}{3}x_2\right) + i\left(\frac{1}{3}y_1 + \frac{1}{2}x_2\right), & \text{if } z_1 \in X_1, z_2 \in X_2, \end{cases}$$

where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then, (X, d) is a complex valued metric space.

Definition 2.4. [3] Let X be a non-empty set. The mapping $d : X \times X \to \mathbb{C}$ satisfies the following conditions :
(i) $0 \preceq d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, y) \preceq \tau[d(x, z) + d(z, y)]$, for all $x, z, y \in X$, where $\tau \ge 1$.
Then d is known as a complex valued b-metric on X, and the pair (X, d) is said to be a complex valued b-metric space.

Example 2.5. [18] Let $X = [0, 1]$. Define a mapping $d : X \times X \to \mathbb{C}$ by

$$d(x, y) = |x - y|^2 + i|x - y|^2$$

for all $x, y \in X$. Then (X, d) is a complex valued b-metric space with $\tau = 2$.

Definition 2.6. [3] Let X be a non-empty set, $\theta : X \times X \to [1, \infty)$ be a function and the mapping $d : X \times X \to \mathbb{C}$ satisfies the following conditions :
(i) $0 \preceq d(x, y)$ and $d(x, y) = 0 \iff x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, y) \preceq \theta(x, y)[d(x, z) + d(z, y)], \forall\, x, z, y \in X$,
then d is known as a complex valued extended b-metric on X, and the pair (X, d) is said to be a complex valued extended b-metric space.

Example 2.7. [3] Let $X = C([a, b], \mathbb{R})$ be the space of all continuous real valued functions defined on $[a, b]$ and a function $\theta : X \times X \to [1, \infty)$ be given by

$$\theta(x, y) = |x(t)| + |y(t)| + 2.$$

Also, define $d : X \times X \to \mathbb{C}$ by

$$d(x, y) = \max_{t \in [a,b]} \left| x(t) - y(t) \right|^2.$$

Then, (X, d) is complex valued extended b-metric space.

Definition 2.8. Let X be a non-empty set, $\theta_0, \theta : X \times X \to [0, \infty)$ be defined by

$$\theta(x, y) = \theta_0(x, y) + \tau,$$

for all $x,\ y \in X$, and $\tau \geq 1$. Also, define a function $d : X \times X \to \mathbb{C}$, if for all $x, y, z \in X$, the following statements hold.
(i) $0 \preceq d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, y) \preceq \theta(x, y)[d(x, z) + d(z, y)]$.
Then d is called a complex valued non-negative extended b-metric on X and the pair (X, d) is called a complex valued non-negative extended b-metric space.

Derivation of complex valued extended b-metric spaces
If we put $\theta_0(x, y) = x + y$ and $\tau = 1$, then we get $\theta(x, y) = 1 + x + y = \theta^*(x, y)$. Property (iii) of Definition 2.8 will be replaced by

$$d(x, y) \preceq \theta^*(x, y)[d(x, z) + d(z, y)].$$

Thus, the Definition 2.8 with the replaced property becomes Complex valued extended b-metric space.

Derivation of complex valued b-metric spaces

If we put $\theta_0(x,y)=0$, then $\theta(x,y)=0+\tau$, where $\tau \geq 1$, then Property (iii) of Definition 2.8 is

$$d(x,y) \preceq \tau[d(x,z)+d(z,y)].$$

Then, Definition oct202 with this property is the definition of Complex valued b-metric spaces.

Derivation of complex valued metric space

By the similar way, if we put $\tau = 1$, in the above definition then it defines the complex valued metric space.

Derivation of ordinary metric space

In the above, if we replace ground space $\mathbb{C}$ by $\mathbb{R}$, we get ordinary metric space.

Example 2.9. [18] Let $X=[0,1]$. Define a mapping $d: X \times X \to \mathbb{C}$ by

$$d(x,y)=|x-y|^2+i|x-y|^2$$

for all $x, y \in X$. Then (X,d) is a complex valued non-negative extended b-metric space with $\tau = 2$ and $\theta_0(x,y)=0$.

Example 2.10. [20] Let X be a non-empty set and $\theta_0, \theta : X \times X \to [0,\infty)$ be defined as:

$$\theta(x,y)=1+x+y\ ,\ \theta_0(x,y)=x+y,\ \tau=1.$$

Further, let

(i) $d(x,y)=\frac{i}{xy}$, for all $x,y \in (0,1]$;

(ii) $d(x,y)=0 \Leftrightarrow x=y$ for all $x,y \in [0,1]$;

(iii) $d(x,0)=d(0,x)=\frac{i}{x}$ for all $x \in (0,1]$.

Then the pair (X,d) is a complex valued non-negative extended b-metric space.

Example 2.11. [20] Let $X=[0,\infty)$. $\theta : X \times X \to [0,\infty)$ be a function defined by $\theta(x,y)=1+x+y$ and $d: X \times X \longrightarrow \mathbb{C}$ be given as

$$d(x,y)=\begin{cases} 0, & \text{if} \quad x=y \\ i, & \text{if} \quad x \neq y. \end{cases}$$

Then (X,d) is a complex valued non-negative extended b-metric space.

Definition 2.12. Let S and T be self mappings of a non-empty set X.

(i) A point $x \in X$ is said to be a fixed point of T if $Tx = x$.

(ii) A point $x \in X$ is said to be a coincidence point of S and T if $Tx = Sx$ and called $w = Tx = Sx$ a point of coincidence of S and T.

(iii) A point $x \in X$ is said to be common fixed of T and S if $x = Sx = Tx$.

Lemma 2.13. [3] *Let (X, d) be a complex valued rectangular extended b- metric space and $\{x_p\}$ be a sequence in X. Then $\{x_p\}$ converges to $x \in X$ if $|d(x_p, x)| \to 0$ as $p \to \infty$.*

Lemma 2.14. [3] *Let (X, d) be a complex valued non-negative extended b- metric space and $\{x_p\}$ be a sequence in X. Then $\{x_p\}$ is a Cauchy sequence if and only if $|d(x_p, x_q)| \to 0$ as $p, q \to \infty$.*

3 Main results

Our main result runs as follows.

Theorem 3.1. *Let (X, d) be a complete complex valued non-negative extended b-metric space, $\theta, \theta_i : X \times X \to [0, \infty)$ ($i = 0, 1, 2$ and $\theta = \theta_0 + \tau,\ \tau \geq 1$) and $S, T : X \to X$ be mappings satisfying the following conditions:*

$$(i)\ \lim_{n\to\infty} \theta(x_{n+1}, x_m)\theta_1(x_{n+1}, x_{n+2}) + \theta_2(x_{n+1}, x_{n+2}) < 1;$$

$$(ii)\ d(Sx, Ty) \preceq \theta_1(x, y)d(x, y) + \theta_2(x, y)\frac{d(x, Sx)d(y, Ty)}{1 + d(x, y)}.$$

Then S, T have a unique common fixed point in X.

Proof. Led $x_0 \in X$ be arbitrary point in X. We construct a sequence $\{x_n\}$ such that

$$x_{2n+1} = Sx_{2n},\ \ x_{2n+2} = Tx_{2n+1} \tag{1}$$

for all $n \geq 0$. From the hypothesis and 1, we get

$$\begin{aligned}
&d(x_{2n+1}, x_{2n+2})\\
&= d(Sx_{2n}, Tx_{2n+1})\\
&\preceq \theta_1(x_{2n}, x_{2n+1})d(x_{2n}, x_{2n+1}) + \frac{\theta_2(x_{2n}, x_{2n+1})d(x_{2n}, Sx_{2n})d(x_{2n+1}, Tx_{2n+1})}{1 + d(x_{2n}, x_{2n+1})}\\
&\preceq \theta_1(x_{2n}, x_{2n+1})d(x_{2n}, x_{2n+1}) + \theta_2(x_{2n}, x_{2n+1})d(x_{2n+1}, Tx_{2n+1})\frac{d(x_{2n}, x_{2n+1})}{1 + d(x_{2n}, x_{2n+1})}\\
&\preceq \theta_1(x_{2n}, x_{2n+1})d(x_{2n}, x_{2n+1}) + \theta_2(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2}).
\end{aligned}$$

This implies that

$$(1 - \theta_2(x_{2n}, x_{2n+1}))d(x_{2n+1}, x_{2n+2}) \preceq \theta_1(x_{2n}, x_{2n+1})d(x_{2n}, x_{2n+1}).$$

That is,

$$\begin{aligned}
&d(x_{2n+1}, x_{2n+2})\\
&\preceq \frac{\theta_1(x_{2n}, x_{2n+1})}{1-\theta_2(x_{2n}, x_{2n+1})} d(x_{2n}, x_{2n+1})\\
&\preceq \frac{\theta_1(x_{2n}, x_{2n+1})}{1-\theta_2(x_{2n}, x_{2n+1})} \frac{\theta_1(x_{2n-1}, x_{2n})}{1-\theta_2(x_{2n-1}, x_{2n})} d(x_{2n-1}, x_{2n})\\
&\preceq \frac{\theta_1(x_{2n}, x_{2n+1})}{1-\theta_2(x_{2n}, x_{2n+1})} \frac{\theta_1(x_{2n-1}, x_{2n})}{1-\theta_2(x_{2n-1}, x_{2n})} \frac{\theta_1(x_{2n-2}, x_{2n-1})}{1-\theta_2(x_{2n-2}, x_{2n-1})} d(x_{2n-2}, x_{2n-1}).
\end{aligned}$$

By continuing in this way, we get

$$\begin{aligned}
d(x_{2n+1}, x_{2n+2}) \quad &\preceq \quad \frac{\theta_1(x_{2n}, x_{2n+1})}{1-\theta_2(x_{2n}, x_{2n+1})} \frac{\theta_1(x_{2n-1}, x_{2n})}{1-\theta_2(x_{2n-1}, x_{2n})} \frac{\theta_1(x_{2n-2}, x_{2n-1})}{1-\theta_2(x_{2n-2}, x_{2n-1})}\\
&\vdots\\
&\quad \times \frac{\theta_1(x_0, x_1)}{1-\theta_2(x_0, x_1)} d(x_0, x_1).
\end{aligned}$$

For $m > n$, we have

$$\begin{aligned}
d(x_n, x_m) &\preceq \theta(x_n, x_m) d(x_n, x_{n+1}) + \theta(x_n, x_m)\theta(x_{n+1}, x_m) d(x_{n+1}, x_{n+2}) + \cdots\\
&+ \theta(x_n, x_m)\theta(x_{n+1}, x_m)\theta(x_{n+2}, x_m)\cdots\theta(x_{m-1}, x_m) d(x_{m-1}, x_m),\\
&\preceq \theta(x_0, x_m)\theta(x_1, x_m)\theta(x_2, x_m)\cdots\theta(x_{n-1}, x_m)\theta(x_n, x_m)\\
&\frac{\theta_1(x_0, x_1)}{1-\theta_2(x_0, x_1)} \frac{\theta_1(x_1, x_2)}{1-\theta_2(x_1, x_2)} \cdots \frac{\theta_1(x_{n-1}, x_n)}{1-\theta_2(x_{n-1}, x_n)} d(x_0, x_1)\\
&+ \theta(x_0, x_m)\theta(x_1, x_m)\theta(x_2, x_m)\cdots\theta(x_n, x_m)\theta(x_{n+1}, x_m)\\
&\frac{\theta_1(x_0, x_1)}{1-\theta_2(x_0, x_1)} \frac{\theta_1(x_1, x_2)}{1-\theta_2(x_1, x_2)} \cdots \frac{\theta_1(x_{n+1}, x_{n+2)}}{1-\theta_2(x_{n+1}, x_{n+2})} d(x_0, x_1)\\
&\vdots\\
&+ \theta(x_0, x_m)\theta(x_1, x_m)\theta(x_2, x_m)\cdots\theta(x_{m-2}, x_m)\theta(x_{m-1}, x_m)\\
&\frac{\theta_1(x_0, x_1)}{1-\theta_2(x_0, x_1)} \frac{\theta_1(x_1, x_2)}{1-\theta_2(x_1, x_2)} \cdots \frac{\theta_1(x_{m-2}, x_{m-1)}}{1-\theta_2(x_{m-2}, x_{m-1})} d(x_0, x_1).
\end{aligned}$$

This implies

$$\begin{aligned} d(x_n,x_m) &\preceq \theta(x_n,x_m)d(x_n,x_{n+1}) + \theta(x_n,x_m)\theta(x_{n+1},x_m)d(x_{n+1},x_{n+2}) + \cdots \\ &+ \theta(x_n,x_m)\theta(x_{n+1},x_m)\theta(x_{n+2},x_m)\cdots\theta(x_{m-1},x_m)d(x_{m-1},x_m), \\ &\preceq d(x_0,x_1)\Bigg[\theta(x_0,x_m)\theta(x_1,x_m)\theta(x_2,x_m)\cdots\theta(x_{n-1},x_m)\theta(x_n,x_m) \\ &\frac{\theta_1(x_0,x_1)}{1-\theta_2(x_0,x_1)}\frac{\theta_1(x_1,x_2)}{1-\theta_2(x_1,x_2)}\cdots\frac{\theta_1(x_{n-1},x_n)}{1-\theta_2(x_{n-1},x_n)} \\ &+ \theta(x_0,x_m)\theta(x_1,x_m)\theta(x_2,x_m)\cdots\theta(x_n,x_m)\theta(x_{n+1},x_m) \\ &\frac{\theta_1(x_0,x_1)}{1-\theta_2(x_0,x_1)}\frac{\theta_1(x_1,x_2)}{1-\theta_2(x_1,x_2)}\cdots\frac{\theta_1(x_{n+1},x_{n+2)}}{1-\theta_2(x_{n+1},x_{n+2})} \\ &\vdots \\ &+ \theta(x_0,x_m)\theta(x_1,x_m)\theta(x_2,x_m)\cdots\theta(x_{m-2},x_m)\theta(x_{m-1},x_m) \\ &\frac{\theta_1(x_0,x_1)}{1-\theta_2(x_0,x_1)}\frac{\theta_1(x_1,x_2)}{1-\theta_2(x_1,x_2)}\cdots\frac{\theta_1(x_{m-2},x_{m-1)}}{1-\theta_2(x_{m-2},x_{m-1})}\Bigg]. \end{aligned}$$

Since, $\lim_{n\to\infty}\theta(x_{n+1},x_m)\theta_1(x_{n+1},x_{n+2})+\theta_2(x_{n+1},x_{n+2})<1$, so the series $\sum_{n=0}^{\infty}\prod_{i=0}^{n}\frac{\theta(x_i,x_m)\theta_1(x_i,x_{i+1})}{1-\theta_1(x_i,x_{i+1})}$ converges by ratio test. For each $m\in\mathbb{N}$, let

$$S_{m-1}=\sum_{j=0}^{m-1}\prod_{i=0}^{j}\frac{\theta(x_i,x_m)\theta_1(x_i,x_{i+1})}{1-\theta_1(x_i,x_{i+1})},\quad S_n=\sum_{j=0}^{n}\prod_{i=0}^{j}\frac{\theta(x_i,x_m)\theta_1(x_i,x_{i+1})}{1-\theta_1(x_i,x_{i+1})}.$$

Thus, for $m>n$, the above inequality implies that

$$d(x_n,x_m)\preceq d(x_0,x_1)\left[S_{m-1}-S_n\right].$$

That is,

$$|d(x_n,x_m)|\leq|d(x_0,x_1)|\left[S_{m-1}-S_n\right]. \tag{2}$$

Letting $n\longrightarrow\infty$ in (2), we conclude that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $u\in X$ such that $x_n\longrightarrow u$ $(n\longrightarrow\infty)$.
To see that $Su=u$, consider

$$\begin{aligned} &d(u,Su) \\ &\preceq d(u,x_{2n+2})+d(x_{2n+2},Su) \\ &= d(u,x_{2n+2})+d(Tx_{2n+1},Su) \\ &= d(u,x_{2n+2})+d(Su,Tx_{2n+1}) \\ &\preceq d(u,x_{2n+2})+\theta_1(u,x_{2n+1})d(u,x_{2n+1})+\frac{\theta_2(u,x_{2n+1})d(u,Su)d(x_{2n+1},Tx_{2n+1})}{1+d(u,x_{2n+1})}. \end{aligned}$$

The above inequality implies that

$$\begin{aligned}&\Big|d(u,Su)\Big|\\&\le\Big|d(u,x_{2n+2})\Big|+\Big|\theta_1(u,x_{2n+1})d(u,x_{2n+1})\Big|+\Big|\frac{\theta_2(u,x_{2n+1})d(u,Su)d(x_{2n+1},Tx_{2n+1})}{1+d(u,x_{2n+1})}\Big|.\end{aligned}$$

By letting $n\longrightarrow\infty$ in the above expression, $|d(u,Su)|\le 0$. Thus, we proved that $u=Su$. Similarly, we can prove that $u=Tu$. Hence, u is the common fixed point of S and T. Finally, to show that u is the unique fixed point of S and T, assume that u^* is another fixed point of S and T with $u\neq u^*$. Then,

$$d(u,u^*)=d(Su,Tu^*)\preceq\theta_1(u,u^*)d(u,u^*)+\frac{\theta_2(u,u^*)d(u,Su)d(u^*,Tu^*)}{1+d(u,u^*)},$$

which gives

$$\Big|d(u,u^*)\Big|\le\theta_1(u,u^*)\Big|d(u,u^*)\Big|+\Big|\frac{\theta_2(u,u^*)d(u,Su)d(u^*,Tu^*)}{1+d(u,u^*)}\Big|$$

and

$$\Big(1-\theta_1(u,u^*)\Big)d(u,u^*)\le 0,$$

which implies that $u=u^*$. Hence u is the unique common fixed point of S and T. □

Corollary 3.2. *Let (X,d) be a complete complex valued non-negative extended b-metric space, $\theta,\theta_i:X\times X\to[0,\infty)$($i=0,1,2$ and $\theta=\theta_0+\tau,\ \tau\ge 1$) and $T:X\to X$ be a mapping satisfying the following conditions:*

$$(i)\ \lim_{n\to\infty}\theta(x_{n+1},x_m)\theta_1(x_{n+1},x_{n+2})+\theta_2(x_{n+1},x_{n+2})<1;$$

$$(ii)\ d(Tx,Ty)\preceq\theta_1(x,y)d(x,y)+\theta_2(x,y)\frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}.$$

Then T has a unique fixed point.

Proof. Take $S=T$ in Theorem 3.1. □

Corollary 3.3. *(see Azam et al.* [3, Theorem 4]*) Let (X,d) be a complete complex valued metric space and $S,T:X\to X$. If S and T satisfy*

$$d(Sx,Ty)\preceq\lambda d(x,y)+\frac{\mu d(x,Sx)d(y,Ty)}{1+d(x,y)}$$

for all $x,\,y\in X$, where $\lambda,\ \mu$ are non-negative reals with $\lambda+\mu<1$. Then S and T have a common fixed point in X.

Proof. Put $\theta_1(x,y)=\lambda$ and $\theta_2(x,y)=\mu$ in Corollary 3.2. □

Theorem 3.4. *Let (X,d) be a complete complex valued non-negative extended b-metric space, $\theta,\theta_i: X\times X\to[0,\infty)$ ($i=0,1,2$ and $\theta=\theta_0+\tau,\ \tau\geq 1$) and $T:X\to X$ be mappings satisfying the following conditions :*

$$(i)\ \lim_{n\to\infty}\theta(x_{n+1},x_m)\theta_1(x_{n+1},x_{n+2})+\theta_2(x_{n+1},x_{n+2})<1;$$

$$(ii)\ d(T^nx,T^ny)\preceq\theta_1(x,y)d(x,y)+\theta_2(x,y)\frac{d(x,T^nx)d(y,T^ny)}{1+d(x,y)}.$$

Then T has a unique fixed point in X.

Proof. From Corollary 3.2, we get that T^n has unique fixed point u, that is, $T^nu=u$. The result then follows from the fact that

$$\begin{aligned}d(Tu,u)&=d(TT^nu,T^nu)=d(T^nTu,T^nu)\\&\preceq\theta_1(Tu,u)d(Tu,u)+\frac{\theta_2(Tu,u)d(Tu,T^nTu)d(u,T^nu)}{1+d(Tu,u)}\\&\preceq\theta_1(Tu,u)d(Tu,u)+\frac{\theta_2(Tu,u)d(Tu,T^nTu)d(u,u)}{1+d(Tu,u)}\\&=\theta_1(Tu,u)d(Tu,u),\end{aligned}$$

from which we have $\Big(1-\theta_1(Tu,u)\Big)d(Tu,u)\preceq 0$. This implies that T has a unique fixed point. □

Example 3.5. Let $X=C([1,3],\mathbb{R})$, $a>0$ and for every $x,\ y\in X$, take

$$d(x,y)=\max_{t\in[1,3]}\big|x(t)-y(t)\big|\sqrt{1+a^2}e^{itan^{-1}a}.$$

Define $T:X\to X$ by

$$T(x(t))=4+\int_1^t(x(u)+u^2)e^{u-1}du,\ t\in[1,3].$$

Then, for every $x,\ y\in X$,

$$\begin{aligned}d(Tx,Ty)&=\max_{t\in[1,3]}\big|T(x(t))-T(y(t))\big|\sqrt{1+a^2}e^{itan^{-1}a}\\&\preceq\theta_1(x,y)\int_1^3\max_{t\in[1,3]}\big|x(u)-y(u)\big|e^2\sqrt{1+a^2}e^{itan^{-1}a}du\\&\preceq 2\theta_1(x,y)e^2d(x,y).\end{aligned}$$

Similarly,

$$e^{2n}\theta_1^n(x,y)\frac{2^n}{n!} = \begin{cases} 109, & \text{if } n=2 \\ 1987, & \text{if } n=4 \\ 1.31, & \text{if } n=37 \\ 0.53, & \text{if } n=38. \end{cases}$$

By a routine calculation, we get

$$d(T^n x, T^n y) \preceq e^{2n}\theta_1^n(x,y)\frac{2^n}{n!}d(x,y).$$

Thus, for $\theta_1(x,y) = 0.53$, $\theta_2(x,y) = 0$, $n = 38$, all the conditions of Theorem 3.4 are satisfied and so T has a unique fixed point, which is the unique solution of the integral equation:

$$x(t) = 4 + \int_1^t (x(u) + u^2)e^{u-1}du, \ t \in [1,3],$$

or the differential equation:

$$\frac{dx}{dt} = (x + t^2)e^{t-1}, \ t \in [1,3], \ x(1) = 4.$$

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