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Popoviciu's type inequalities for h-MN-convex functions

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Abstract. In this work, Popoviciu type inequalities for h-MN-convex functions are proved. Some direct examples are pointed out.

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1 Introduction

We recall that a function $\mathrm{M} : (0,\infty) \to (0,\infty)$ is called a mean function if it has

(i) Symmetry: $\mathrm{M}(x,y) = \mathrm{M}(y,x)$;

(ii) Reflexivity: $\mathrm{M}(x,x) = x$;

(iii) Monotonicity: $\min\{x,y\} \leq \mathrm{M}(x,y) \leq \max\{x,y\}$;

(iv) Homogeneity: $\mathrm{M}(\lambda x, \lambda y) = \lambda \mathrm{M}(x,y)$, for any positive scalar λ.

The most famous and old known mathematical means are listed as follows:

(i) The arithmetic mean :

$$A := A(\alpha,\beta) = \frac{\alpha+\beta}{2}, \quad \alpha,\beta \in \mathbb{R}_+.$$

(ii) The geometric mean :

$$G := G(\alpha,\beta) = \sqrt{\alpha\beta}, \quad \alpha,\beta \in \mathbb{R}_+$$

(iii) The harmonic mean :

$$H := H(\alpha,\beta) = \frac{2}{\frac{1}{\alpha}+\frac{1}{\beta}}, \quad \alpha,\beta \in \mathbb{R}_+ - \{0\}.$$

In particular, we have the famous inequality $H \leq G \leq A$.

In 2007, Anderson et.al. in [2] developed a systematic study to the classical theory of continuous and midconvex functions, by replacing a given mean instead of the arithmetic mean.

Definition 1.1. Let $f : I \to (0, \infty)$ be a continuous function where $I \subseteq (0, \infty)$. Let M and N be any two mean functions. We say f is MN-convex (concave) if

$$f\left(\mathrm{M}\left(x, y\right)\right) \leq (\geq) \, \mathrm{N}\left(f(x), f(y)\right), \tag{1}$$

for all $x, y \in I$ and $t \in [0, 1]$.

In fact, the authors in [2] discussed the midconvexity of positive continuous real functions according to some means. Hence, the usual midconvexity is a special case when both mean values are arithmetic means. Also, they studied the dependence of MN-convexity on M and N and give sufficient conditions for MN-convexity of functions defined by Maclaurin series. For other works regarding MN-convexity, see [15, 16].

The class of h-convex functions, which generalizes convex, s-convex (denoted by K_s^2, [4–6, 11]), Godunova-Levin functions (denoted by $Q(I)$, [10]) and P-functions (denoted by $P(I)$, [18]), was introduced by Varošanec in [26]. Namely, the h-convex function is defined as a nonnegative function $f : I \to \mathbb{R}$ which satisfies

$$f\left(t\alpha + (1-t)\,\beta\right) \leq h\left(t\right) f\left(\alpha\right) + h\left(1-t\right) f\left(\beta\right),$$

where h is a nonnegative function, $t \in (0, 1) \subseteq J$ and $x, y \in I$, where I and J are real intervals such that $(0, 1) \subseteq J$. Accordingly, some properties of h-convex functions were discussed in the same work of Varošanec.

Let $h : J \to (0, \infty)$ be a nonnegative function. Define the function $\mathrm{M} : [0, 1] \to [a, b]$ given by $\mathrm{M}\left(t\right) = \mathrm{M}\left(t; a, b\right)$; where by $\mathrm{M}\left(t; a, b\right)$ we mean one of the following functions:

(i) $A_h\left(a, b\right) := h\left(1-t\right) a + h\left(t\right) b$, the generalized arithmetic mean;

(ii) $G_h\left(a, b\right) = a^{h(1-t)} b^{h(t)}$, the generalized geometric mean;

(iii) $H_h\left(a, b\right) := \frac{ab}{h(t)a + h(1-t)b} = \frac{1}{A_h\left(\frac{1}{a}, \frac{1}{b}\right)}$, the generalized harmonic mean.

Note that $\mathrm{M}(h(0);a,b)=a$ and $\mathrm{M}(h(1);a,b)=b$. Clearly, for $h(t)=t$ with $t=\frac{1}{2}$, the means $A_{\frac{1}{2}}$, $G_{\frac{1}{2}}$ and $H_{\frac{1}{2}}$, respectively; represents the midpoint of the A_t, G_t and H_t, respectively; which was discussed in [2] in viewing of Definition 1.1.

For $h(t)=t$, we note that the above means are related with celebrated AM-GM-HM inequality

$$H_t(a,b)\le G_t(a,b)\le A_t(a,b)\,,\qquad \forall\ t\in[0,1].$$

Indeed, one can easily prove more general form of the above inequality; that is if h is positive increasing on $[0,1]$ then the generalized AM-GM-HM inequality is given by

$$H_h(a,b)\le G_h(a,b)\le A_h(a,b)\,,\qquad \forall\ t\in[0,1]\ \text{ and }\ a,b>0. \tag{2}$$

The Definition 1.1 can be extended according to the defined mean $\mathrm{M}(t;a,b)$, as follows: Let $f:I\to(0,\infty)$ be any function. Let M and N be any two mean functions. We say f is MN-convex (concave) if

$$f(\mathrm{M}(t;x,y))\le(\ge)\,\mathrm{N}(t;f(x),f(y))\,,$$

for all $x,y\in I$ and $t\in[0,1]$.

More generally, we introduce the class of $\mathrm{M_tN_h}$-convex functions by generalizing the concept of $\mathrm{M_tN_t}$-convexity and combining it with h-convexity [1].

Definition 1.2. Let $h:J\to(0,\infty)$ be a nonnegative function. Let $f:I\to(0,\infty)$ be any function. Let $\mathrm{M}:[0,1]\to[a,b]$ and $\mathrm{N}:(0,\infty)\to(0,\infty)$ be any two mean functions. We say f is h-MN-convex (-concave) or that f belongs to the class $\overline{\mathcal{MN}}(h,I)$ ($\underline{\mathcal{MN}}(h,I)$) if

$$f(\mathrm{M}(t;x,y))\le(\ge)\,\mathrm{N}(h(t);f(x),f(y))\,, \tag{3}$$

for all $x,y\in I$ and $t\in[0,1]$.

Clearly, if $\mathrm{M}(t;x,y)=A_t(x,y)=\mathrm{N}(t;x,y)$, then Definition 1.2 reduces to the original concept of h-convexity. Also, if we assume f is continuous, $h(t)=t$ and $t=\frac{1}{2}$ in (3), then the Definition 1.2 reduces to the Definition 1.1.

The cases of h-MN-convexity are given with respect to a certain mean, as follows:

(i) f is $\mathrm{A_tG_h}$-convex iff

$$f(t\alpha+(1-t)\beta)\le[f(\alpha)]^{h(t)}[f(\beta)]^{h(1-t)}\,,\qquad 0\le t\le 1; \tag{4}$$

(ii) f is $\mathrm{A_tH_h}$-convex iff

$$f\left(t\alpha+(1-t)\beta\right)\leq\frac{f(\alpha)f(\beta)}{h(1-t)f(\alpha)+h(t)f(\beta)},\qquad 0\leq t\leq 1;\quad (5)$$

(iii) f is $\mathrm{G_tA_h}$-convex iff

$$f\left(\alpha^t\beta^{1-t}\right)\leq h(t)f(\alpha)+h(1-t)f(\beta),\qquad 0\leq t\leq 1;\quad (6)$$

(iv) f is $\mathrm{G_tG_h}$-convex iff

$$f\left(\alpha^t\beta^{1-t}\right)\leq[f(\alpha)]^{h(t)}[f(\beta)]^{h(1-t)},\qquad 0\leq t\leq 1;\quad (7)$$

(v) f is $\mathrm{G_tH_h}$-convex iff

$$f\left(\alpha^t\beta^{1-t}\right)\leq\frac{f(\alpha)f(\beta)}{h(1-t)f(\alpha)+h(t)f(\beta)},\qquad 0\leq t\leq 1;\quad (8)$$

(vi) f is $\mathrm{H_tA_h}$-convex iff

$$f\left(\frac{\alpha\beta}{t\alpha+(1-t)\beta}\right)\leq h(1-t)f(\alpha)+h(t)f(\beta),\qquad 0\leq t\leq 1;\quad (9)$$

(vii) f is $\mathrm{H_tG_h}$-convex iff

$$f\left(\frac{\alpha\beta}{t\alpha+(1-t)\beta}\right)\leq[f(\alpha)]^{h(1-t)}[f(\beta)]^{h(t)},\qquad 0\leq t\leq 1;\quad (10)$$

(viii) f is $\mathrm{H_tH_h}$-convex iff

$$f\left(\frac{\alpha\beta}{t\alpha+(1-t)\beta}\right)\leq\frac{f(\alpha)f(\beta)}{h(t)f(\alpha)+h(1-t)f(\beta)},\quad 0\leq t\leq 1.\quad (11)$$

Remark 1.3. In all previous cases, $h(t)$ and $h(1-t)$ are not equal to zero at the same time. Therefore, if $h(0)=0$ and $h(1)=1$, then a mean function N satisfying the conditions $\mathrm{N}(h(0),f(x),f(y))=f(x)$ and $\mathrm{N}(h(1),f(x),f(y))=f(y)$.

Remark 1.4. According to the Definition 1.2, we may extend the classes $Q(I)$, $P(I)$ and K_s^2 by replacing the arithmetic mean by another given one. Let $\mathrm{M}:[0,1]\to[a,b]$ and $\mathrm{N}:(0,\infty)\to(0,\infty)$ be any two mean functions.

(i) Let $s \in (0,1]$, a function $f : I \to (0,\infty)$ is $\mathrm{M_t N_{t^s}}$-convex function or that f belongs to the class $K_s^2\left(I;\mathrm{M_t},\mathrm{N_{t^s}}\right)$ if for all $x,y \in I$ and $t \in [0,1]$ we have
$$f\left(\mathrm{M}\left(t;x,y\right)\right) \le \mathrm{N}\left(t^s;f(x),f(y)\right). \tag{12}$$

(ii) We say that $f : I \to (0,\infty)$ is an extended Godunova-Levin function or that f belongs to the class $Q\left(I;\mathrm{M_t},\mathrm{N_{1/t}}\right)$ if for all $x,y \in I$ and $t \in (0,1)$ we have
$$f\left(\mathrm{M}\left(t;x,y\right)\right) \le \mathrm{N}\left(\frac{1}{t};f(x),f(y)\right). \tag{13}$$

(iii) We say that $f : I \to (0,\infty)$ is P-$\mathrm{M_t N_{t=1}}$-function or that f belongs to the class $P\left(I;\mathrm{M_t},\mathrm{N_1}\right)$ if for all $x,y \in I$ and $t \in [0,1]$ we have
$$f\left(\mathrm{M}\left(t;x,y\right)\right) \le \mathrm{N}\left(1;f(x),f(y)\right). \tag{14}$$

In (12)–(14), setting $\mathrm{M}\left(t;x,y\right) = \mathrm{A_t}\left(x,y\right) = \mathrm{N}\left(t;x,y\right)$, we then refer to the original definitions of these class of convexities.

Remark 1.5. Let h be a nonnegative function such that $h\left(t\right) \ge t$ for $t \in (0,1)$. For instance $h_r\left(t\right) = t^r$, $t \in (0,1)$ has that property. In particular, for $r \le 1$, if f is a nonnegative $\mathrm{M_t N_t}$-convex function on I, then for $x,y \in I$, $t \in (0,1)$ we have
$$f\left(\mathrm{M}\left(t;x,y\right)\right) \le \mathrm{N}\left(t;f(x),f(y)\right) \le \mathrm{N}\left(t^r;f(x),f(y)\right) = \mathrm{N}\left(h\left(t\right);f(x),f(y)\right),$$
for all $r \le 1$ and $t \in (0,1)$. So that f is $\mathrm{M_t N_h}$-convex. Similarly, if the function satisfies the property $h\left(t\right) \le t$ for $t \in (0,1)$, then f is a nonnegative $\mathrm{M_t N_h}$-concave. In particular, for $r \ge 1$, the function $h_r(t)$ has that property for $t \in (0,1)$. So that if f is a nonnegative $\mathrm{M_t N_t}$-concave function on I, then for $x,y \in I$, $t \in (0,1)$ we have
$$f\left(\mathrm{M}\left(t;x,y\right)\right) \ge \mathrm{N}\left(t;f(x),f(y)\right) \ge \mathrm{N}\left(t^r;f(x),f(y)\right) = \mathrm{N}\left(h\left(t\right);f(x),f(y)\right),$$
for all $r \ge 1$ and $t \in (0,1)$, which means that f is $\mathrm{M_t N_h}$-concave.

As known, it is not easy to determine whether a given function is convex or not. Because of that, Jensen in [12] proved his famous characterization of convex functions. Simply, for a continuous functions f defined on a real interval I, f is convex if and only if
$$f\left(\frac{x+y}{2}\right) \le \frac{f\left(x\right)+f\left(y\right)}{2},$$
for all $x,y \in I$.

In 1965, another characterization was presented by Popoviciu [20], where he proved the following theorem.

Theorem 1.6. *Let $f : I \to \mathbb{R}$ be continuous. Then, f is convex if and only if*

$$\frac{2}{3}\left[f\left(\frac{x+z}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{x+y}{2}\right)\right] \leq f\left(\frac{x+y+z}{3}\right)+\frac{f(x)+f(y)+f(z)}{3}, \quad (15)$$

for all $x, y, z \in I$, and the equality occurred by $f(x) = x$, $x \in I$.

The corresponding version of Popoviciu inequality for G_tG_t-convex (concave) function was presented in [15], where he proved that for all $x, y, z \in I$ the inequality

$$f^2\left(\sqrt{xz}\right) f^2\left(\sqrt{yz}\right) f^2\left(\sqrt{xy}\right) \leq (\geq) f^3\left(\sqrt[3]{xyz}\right) f(x) f(y) f(z), \quad (16)$$

holds.

One of the most applicable benefits of Popoviciu's inequality is to maximize and/or minimize a given function (or certain real quantities) without using derivatives, so that such type of inequalities plays an important role in optimizations and approximations. Another serious usefulness is to generalize some old famous inequalities, e.g., the Popoviciu's inequality can be considered as an elegant generalization of Hlawka's inequality using convexity as a simple tool of geometry. For any real numbers x, y, z, the Hlawka's inequality reads:

$$|x|+|y|+|z|+|x+y+z| \geq |x+z|+|z+y|+|x+y|. \quad (17)$$

D. Smiley & M. Smiley [28] (see also [23], p. 756), interpreted Hlawka's inequality geometrically by saying that: "the total length over all sums of pairs from three vectors is not greater than the perimeter of the quadrilateral defined by the three vectors." For recent comprehensive history regarding Hlawka's inequality see [8]. It's convenient to note that, a normed linear space for which inequality (17) holds for all x, y, z is called a Hlawka space or quadrilateral space, see [24, 25] (also [23]). For instance, each inner product space is a Hlawka space [14].

The extended version of Popoviciu's inequality to several variables was not possible without the help of Hlawka's inequality, as it inspired the authors of [3] to develop a higher dimensional analogue of Popoviciu's inequality based on his characterization. Interesting generalizations and counterparts of Popoviciu inequality with some ramified consequences can be found in [9, 27].

Therefore, as Popoviciu's inequality one of the most popular generalization of Hlawka's inequality, and due to its important usefulness, in this work we establish the corresponding Popoviciu type inequalities according to a given mean used instead of the arithmetic mean. Namely, for h-AN-convex functions several inequalities of Popoviciu type are proved. In this way, we extend Hlawka's inequality based on the geometric structure used under an h-AN-convex mappings.

2 Popoviciu type inequalities for h-AN-convex functions

After focus consideration we find that, there is neither nonnegative $\frac{1}{t}$-$\mathrm{M_tA_t}$-concave nor $\frac{1}{t}$-$\mathrm{M_tH_t}$–convex functions, where $M_t = A_t, G_t, H_t$. The same observation holds for $h(t) = t^k$, $k \leq -1$, $t \in (0,1)$.

To see how this holds, suppose on the contrary that there is a nonnegative function f which is $\mathrm{M_tA_{1/t}}$-concave on I. Thus, for means $\mathrm{M_t}$ and $\mathrm{A_t}$, the reverse inequality of (13) holds for all $x, y \in I$ and $t \in (0,1)$.

$$f(M(t;x,y)) \geq \frac{1}{1-t} f(x) + \frac{1}{t} f(y).$$

Since $M_t(x,x) = x$, so by setting $x = y$ we have

$$f(x) \geq \frac{1}{1-t} f(x) + \frac{1}{t} f(x) = \frac{1}{t(1-t)} f(x),$$

which is equivalent to write $\left(t - t^2 - 1\right) f(x) \geq 0$, $\forall t \in (0,1)$. But since f is nonnegative we must have $t - t^2 - 1 \geq 0$, $0 < t < 1$ which is impossible and thus we got a contradiction. Hence, we must have $f(x) \leq 0$.

In case when f is nonnegative $\mathrm{M_tH_{1/t}}$–convex function, then

$$f(M(t;x,y)) \leq \frac{t(1-t) f(x) f(y)}{t f(x) + (1-t) f(y)},$$

and setting $x = y$ we have

$$f(x) \leq t(1-t) f(x),$$

and this is equivalent to write $(t(1-t) - 1) f(x) \geq 0$, since f is nonnegative we must have $t(1-t) - 1 \geq 0$ which impossible for $t \in (0,1)$, which contradicts the nonnegativity assumption of f. Hence, $f \leq 0$.

Remark 2.1. There are neither nonnegative $\mathrm{M_tA_1}$-concave nor $\mathrm{M_tH_1}$-convex functions, where $M_t = A_t, G_t, H_t$. The proof is simpler than that ones given above.

According to the previous discussion, we need to extend classes $Q\left(I; \mathrm{M_t}, \mathrm{A_{1/t}}\right)$, $Q\left(I; \mathrm{M_t}, \mathrm{H_{1/t}}\right)$, $P(I; \mathrm{M_t}, \mathrm{A_1})$, and $P(I; \mathrm{M_t}, \mathrm{H_1})$. Consequently, we say that a function $f : I \to \mathbb{R}$

(i) is $\mathrm{M_tA_{1/t}}$-concave, if $-f \in Q\left(I; \mathrm{M_t}, \mathrm{A_{1/t}}\right)$, i.e.,

$$f\left(\mathrm{M}\left(t; x, y\right)\right) \geq \frac{1}{1-t} f\left(x\right) + \frac{1}{t} f\left(y\right),$$

for all $x, y \in I$ and $t \in (0,1)$,

(ii) is $\mathrm{M_tH_{1/t}}$-convex, if $-f \in Q\left(I; \mathrm{M_t}, \mathrm{H_{1/t}}\right)$, i.e.,

$$f\left(\mathrm{M}\left(t; x, y\right)\right) \geq \frac{t\left(1-t\right) f\left(x\right) f\left(y\right)}{tf\left(x\right) + \left(1-t\right) f\left(y\right)},$$

for all $x, y \in I$ and $t \in (0,1)$,

(iii) is $\mathrm{M_tA_1}$-concave, if $-f \in P\left(I; \mathrm{M_t}, \mathrm{A_1}\right)$, i.e.,

$$f\left(\mathrm{M}\left(t; x, y\right)\right) \geq f\left(x\right) + f\left(y\right)$$

for all $x, y \in I$ and $t \in (0,1)$,

(iv) is $\mathrm{M_tH_1}$-concave, if $-f \in P\left(I; \mathrm{M_t}, \mathrm{H_1}\right)$, i.e.,

$$f\left(\mathrm{M}\left(t; x, y\right)\right) \geq \frac{f\left(x\right) f\left(y\right)}{f\left(x\right) + f\left(y\right)},$$

for all $x, y \in I$ and $t \in (0,1)$.

In the same way, there is no $\mathrm{M_tG_{1/t}}$-concave function satisfies $f\left(x\right) > 1$. To support this assertion, assume there exists $\mathrm{M_tG_{1/t}}$-concave function, so that for means $\mathrm{M_t}$ and $\mathrm{G_t}$, the reverse inequality of (13) holds for all $x, y \in I$ and $t \in (0,1)$.

$$f\left(M\left(t; x, y\right)\right) \geq \left[f\left(x\right)\right]^{\frac{1}{1-t}} \left[f\left(y\right)\right]^{\frac{1}{t}},$$

since $M_t\left(x, x\right) = x$, so by setting $x = y$ we have

$$f\left(x\right) \geq \left[f\left(x\right)\right]^{\frac{1}{1-t} + \frac{1}{t}},$$

since $f\left(x\right) > 1$ and $t \in (0,1)$ then we must have $\frac{1}{1-t} + \frac{1}{t} \leq 1$ which is equivalent to write $1 \leq t\left(1-t\right)$ for all $t \in (0,1)$ and this is impossible, thus we have a contradiction. Hence, we must have $0 \leq f\left(x\right) \leq 1$.

Remark 2.2. There is no 1-$\mathrm{M_tG_t}$-concave function satisfies $f(x) > 1$. The proof is simpler than that ones given above.

A function $h : I \to \mathbb{R}$ is said to be

(i) additive if $h(s+t) = h(s) + h(t)$,

(ii) subadditive if $h(s+t) \le h(s) + h(t)$,

(iii) superadditive if $h(s+t) \ge h(s) + h(t)$,

for all $s, t \in I$. For example, let $h : I \to (0, \infty)$ given by $h(x) = x^k$, $x > 0$. Then h is

(i) additive if $k = 1$,

(ii) subadditive if $k \in (-\infty, -1] \cup [0, 1)$,

(iii) superadditive if $k \in (-1, 0) \cup (1, \infty)$.

We note here, in all next results and for the classes $\mathrm{M_tA_{1/t}}$-concave, $\mathrm{M_tG_{1/t}}$-concave, $\mathrm{M_tH_{1/t}}$-convex , $\mathrm{M_tA_1}$-concave, and $\mathrm{M_tH_1}$-convex functions, f is defined to be $f : I \to \mathbb{R}$, $I \subseteq (0, \infty)$.

2.1 The case when f is h-AA-convex

Now, we are ready to state our first main result.

Theorem 2.3. *Let $h : I \to (0, \infty)$ be a nonnegative super(sub)additive function. If $f : I \to (0, \infty)$ be an $\mathrm{A_tA_h}$-convex (concave) function, then*

$$\begin{aligned} &f\left(\frac{x+z}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{x+y}{2}\right) \\ &\qquad \le (\ge)\, h(3/2)\, f\left(\frac{x+y+z}{3}\right) + h(1/2)\left[f(x) + f(y) + f(z)\right], \quad (18) \end{aligned}$$

for all $x, y, z \in I$.

Proof. f is $\mathrm{A_tA_h}$-convex iff the inequality

$$f(t\alpha + (1-t)\beta) \le h(t) f(\alpha) + h(1-t) f(\beta), \qquad 0 \le t \le 1$$

holds for all $\alpha, \beta \in I$. Assume that $x \le y \le z$. If $y \le \frac{x+y+z}{3}$, then

$$\frac{x+y+z}{3} \le \frac{x+z}{2} \le z \text{ and } \frac{x+y+z}{3} \le \frac{y+z}{2} \le z,$$

so that there exist two numbers $s, t \in [0,1]$ satisfying

$$\frac{x+z}{2} = s\left(\frac{x+y+z}{3}\right) + (1-s)\,z,$$

and

$$\frac{y+z}{2} = t\left(\frac{x+y+z}{3}\right) + (1-t)\,z.$$

Summing up, we get $(x+y-2z)\left(s+t-\frac{3}{2}\right) = 0$. If $x+y-2z = 0$, then $x = y = z$, and Popoviciu's inequality holds.

If $s+t = \frac{3}{2}$, then since f is $\mathrm{A_tA_h}$-convex, we have

$$\begin{aligned}
f\left(\frac{x+z}{2}\right) &= f\left[s\left(\frac{x+y+z}{3}\right) + (1-s)\,z\right] \\
&\leq h(s)\,f\left(\frac{x+y+z}{3}\right) + h(1-s)\,f(z)\,, \\
f\left(\frac{y+z}{2}\right) &= f\left[t\left(\frac{x+y+z}{3}\right) + (1-t)\,z\right] \\
&\leq h(t)\,f\left(\frac{x+y+z}{3}\right) + h(1-t)\,f(z)\,,
\end{aligned}$$

and

$$f\left(\frac{x+y}{2}\right) \leq h(1/2)\left[f(x) + f(y)\right].$$

Summing up these inequalities taking into account that h is superadditive we get

$$\begin{aligned}
&f\left(\frac{x+z}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{x+y}{2}\right) \\
&\leq h(s)\,f\left(\frac{x+y+z}{3}\right) + h(1-s)\,f(z) + h(t)\,f\left(\frac{x+y+z}{3}\right) \\
&\qquad + h(1-t)\,f(z) + h(1/2)\left[f(x) + f(y)\right] \\
&= \left[h(s) + h(t)\right] f\left(\frac{x+y+z}{3}\right) + \left[h(1-s) + h(1-t)\right] f(z) \\
&\qquad + h(1/2)\left[f(x) + f(y)\right] \\
&\leq h(s+t)\,f\left(\frac{x+y+z}{3}\right) + h(2-s-t)\,f(z) + h(1/2)\left[f(x) + f(y)\right]
\end{aligned}$$

$$= h(3/2) f\left(\frac{x+y+z}{3}\right) + h(1/2) f(z) + h(1/2)[f(x)+f(y)]$$
$$= h(3/2) f\left(\frac{x+y+z}{3}\right) + h(1/2)[f(x)+f(y)+f(z)],$$

as desired. □

Remark 2.4. In (18), setting $z = y$, we have

$$2f\left(\frac{x+y}{2}\right) + f(y) \le (\ge) h(3/2) f\left(\frac{x+2y}{3}\right) + h(1/2)[f(x)+2f(y)].$$

for all $x, y \in I$.

Remark 2.5. In (18), setting $z = y$, we get

$$2f\left(\frac{x+y}{2}\right) + f(y) \le (\ge) h(3/2) f\left(\frac{x+2y}{3}\right) + h(1/2)[f(x)+2f(y)],$$

for all $x, y \in I$.

Corollary 2.6. *Let $h : I \to (0,\infty)$ be a nonnegative super(sub)additive function. If $f : I \to (0,\infty)$ be an* $\mathrm{A_tA_t}$*-convex (concave) function, then*

$$\frac{2}{3}\left[f\left(\frac{x+z}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{x+y}{2}\right)\right] \le (\ge) f\left(\frac{x+y+z}{3}\right) + \frac{f(x)+f(y)+f(z)}{3},$$

for all $x, y, z \in I$. The equality holds when f is affine.

Example 2.7. (i) Let $f(x) = x^p$, $p \ge 1$ then f is $\mathrm{A_tA_t}$-convex for all $x > 0$. Applying Corollary 2.6, we get

$$\frac{2}{3}\left[\left(\frac{x+z}{2}\right)^p + \left(\frac{y+z}{2}\right)^p + \left(\frac{x+y}{2}\right)^p\right] \le \left(\frac{x+y+z}{3}\right)^p + \frac{x^p+y^p+z^p}{3},$$

for all $x, y, z > 0$.

(ii) Let $f(x) = -\log x$, then f is $\mathrm{A_tA_t}$-convex for all $0 < x < 1$. Applying Corollary 2.6, we get

$$(x+z)^2(y+z)^2(x+y)^2 \geq \frac{64}{27}(x+y+z)^3(xyz),$$

for all $1 > x, y, z > 0$.

Corollary 2.8. *If $f : I \to \mathbb{R}$ be an $\mathrm{A_tA_{1/t}}$-concave function, then*

$$\frac{3}{2}\left[f\left(\frac{x+z}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{x+y}{2}\right)\right] \\ \leq (\geq)\, f\left(\frac{x+y+z}{3}\right)+3\left[f(x)+f(y)+f(z)\right],$$

for all $x, y, z \in I$.

Example 2.9. Let $f(x) = \log x$, then f is an $\mathrm{A_tA_{1/t}}$-concave for $0 < x < 1$. Applying Corollary 2.8, we get

$$(x+z)^3(y+z)^3(x+y)^3 \geq \frac{512}{9}(x+y+z)^2(xyz)^6,$$

for all $0 < x, y, z < 1$.

Corollary 2.10. *If $f : I \to \mathbb{R}$ be an $\mathrm{A_tA_1}$-concave function, then*

$$f\left(\frac{x+z}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{x+y}{2}\right) \\ \leq (\geq)\, f\left(\frac{x+y+z}{3}\right)+f(x)+f(y)+f(z),$$

for all $x, y, z \in I$.

Example 2.11. Let $f(x) = \log x$, which is a nonnegative $\mathrm{A_tA_1}$-concave for all $0 < x < 1$. Applying Corollary 2.10, we get

$$(x+z)(y+z)(x+y) \geq \frac{8}{3}(x+y+z)(xyz),$$

for all $0 < x, y, z < 1$.

Corollary 2.12. *In Theorem* 2.3.

(i) *If* $h : J \to (0,\infty)$ *is a nonnegative superadditive and* $f : I \to (0,\infty)$ *is an* $\mathrm{A_tA_h}$*-convex and subadditive, then*

$$\begin{aligned} f(x+y+z) &\le f\left(\frac{x+z}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{x+y}{2}\right) \\ &\le h(3/2)\, f\left(\frac{x+y+z}{3}\right) + h(1/2)\left[f(x)+f(y)+f(z)\right] \\ &\le h(3/2)\left[f\left(\frac{x}{3}\right) + f\left(\frac{y}{3}\right) + f\left(\frac{z}{3}\right)\right] + h(1/2)\left[f(x)+f(y)+f(z)\right], \end{aligned}$$

for all $x, y, z \in I$*. If* h *is nonnegative subadditive on* J *and* f *is an* $\mathrm{A_tA_h}$*-concave and superadditive, then the inequality is reversed.*

(ii) *If* $h : J \to (0,\infty)$ *is a nonnegative superadditive and* $f : I \to (0,\infty)$ *is an* $\mathrm{A_tA_h}$*-convex and superadditive, then*

$$\begin{aligned} & f\left(\frac{x+z}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{x+y}{2}\right) \\ &\le h(3/2)\, f\left(\frac{x+y+z}{3}\right) + h(1/2)\left[f(x)+f(y)+f(z)\right] \\ &\le h(3/2)\, f\left(\frac{x+y+z}{3}\right) + h(1/2)\, f(x+y+z), \end{aligned}$$

for all $x, y, z \in I$*. If* h *is a nonnegative subadditive and* f *is an* $\mathrm{A_tA_h}$*-concave and subadditive, then the inequality is reversed.*

2.2 The case when f is h-$\mathrm{A_tG_t}$-convex

Theorem 2.13. *Let* $h : I \to (0,\infty)$ *be a nonnegative super(sub)additive function. If* $f : I \to (0,\infty)$ *be an* $\mathrm{A_tG_h}$*-convex (concave) function, then*

$$\begin{aligned} & f\left(\frac{x+z}{2}\right) f\left(\frac{y+z}{2}\right) f\left(\frac{x+y}{2}\right) \\ &\qquad \le (\ge) \left[f\left(\frac{x+y+z}{3}\right)\right]^{h(3/2)} \left[f(x)\, f(y)\, f(z)\right]^{h(1/2)}, \end{aligned} \tag{19}$$

for all $x, y, z \in I$*.*

Proof. f is $\mathrm{A_tG_h}$-convex iff the inequality

$$f(t\alpha + (1-t)\beta) \le [f(\alpha)]^{h(t)} [f(\beta)]^{h(1-t)}, \qquad 0 \le t \le 1$$

holds for all $\alpha, \beta \in I$. As in the proof of Theorem 2.3, we have

$$(x+y-2z)\left(s+t-\frac{3}{2}\right)=0.$$

If $x+y-2z=0$, then $x=y=z$, and Popoviciu's inequality holds.
If $s+t=\frac{3}{2}$, then since f is $\mathrm{A_tG_t}$-convex, we have

$$\begin{aligned} f\left(\frac{x+z}{2}\right) &= f\left[s\left(\frac{x+y+z}{3}\right)+(1-s)\,z\right] \\ &\le \left[f\left(\frac{x+y+z}{3}\right)\right]^{h(s)}[f\,(z)]^{h(1-s)}, \end{aligned}$$

$$\begin{aligned} f\left(\frac{y+z}{2}\right) &= f\left[t\left(\frac{x+y+z}{3}\right)+(1-t)\,z\right] \\ &\le \left[f\left(\frac{x+y+z}{3}\right)\right]^{h(t)}[f\,(z)]^{h(1-t)}, \end{aligned}$$

and

$$f\left(\frac{x+y}{2}\right)\le [f\,(x)\,f\,(y)]^{h(1/2)}.$$

Multiplying these inequalities we get

$$\begin{aligned} & f\left(\frac{x+z}{2}\right)f\left(\frac{y+z}{2}\right)f\left(\frac{x+y}{2}\right) \\ &\le \left[f\left(\frac{x+y+z}{3}\right)\right]^{h(s)}[f\,(z)]^{h(1-s)}\left[f\left(\frac{x+y+z}{3}\right)\right]^{h(t)} \\ &\qquad\times [f\,(z)]^{h(1-t)}[f\,(x)\,f\,(y)]^{h(1/2)} \\ &= \left[f\left(\frac{x+y+z}{3}\right)\right]^{h(s)+h(t)}[f\,(z)]^{h(1-s)+h(1-t)}[f\,(x)\,f\,(y)]^{h(1/2)} \\ &\le \left[f\left(\frac{x+y+z}{3}\right)\right]^{h(s+t)}[f\,(z)]^{h(2-s-t)}[f\,(x)\,f\,(y)]^{\frac{1}{2}} \\ &= \left[f\left(\frac{x+y+z}{3}\right)\right]^{h(3/2)}[f\,(x)\,f\,(y)\,f\,(z)]^{h(1/2)}, \end{aligned}$$

as desired. □

Remark 2.14. In (19), setting $z = y$ we have

$$f^2\left(\frac{x+y}{2}\right) f(y) \leq (\geq) \left[f\left(\frac{x+2y}{3}\right)\right]^{h(3/2)} \left[f(x) f^2(y)\right]^{h(1/2)},$$

for all $x, y \in I$.

Corollary 2.15. *If $f : I \to (0, \infty)$ be an $\mathrm{A_t G_t}$-convex function, then*

$$f^2\left(\frac{x+z}{2}\right) f^2\left(\frac{y+z}{2}\right) f^2\left(\frac{x+y}{2}\right) \leq f^3\left(\frac{x+y+z}{3}\right) f(x) f(y) f(z),$$

for all $x, y, z \in I$. The equality occurred for $f(x) = \mathrm{e}^x$, $x > 0$.

Example 2.16. $f(x) = \cosh(x)$, $x \in \mathbb{R}$ is $\mathrm{A_t G_t}$-convex function. Applying Corollary 2.15, we get

$$\begin{aligned}&\cosh^2\left(\frac{x+z}{2}\right)\cosh^2\left(\frac{y+z}{2}\right)\cosh^2\left(\frac{x+y}{2}\right)\\ &\leq \cosh^3\left(\frac{x+y+z}{3}\right)\cosh(x)\cosh(y)\cosh(z)\end{aligned}$$

Corollary 2.17. *If $f : I \to (0, \infty)$ be an $\mathrm{A_t G_{1/t}}$-concave function, then*

$$f^3\left(\frac{x+z}{2}\right) f^3\left(\frac{y+z}{2}\right) f^3\left(\frac{x+y}{2}\right) \geq f^2\left(\frac{x+y+z}{3}\right) f^6(x) f^6(y) f^6(z),$$

for all $x, y, z \in I$.

Example 2.18. $f(x) = \arcsin(x)$, is $\frac{1}{t}$-$\mathrm{A_t G_t}$-concave for $x \in [0, 1]$. Applying Corollary 2.17, we get

$$\begin{aligned}&\arcsin^3\left(\frac{x+z}{2}\right)\arcsin^3\left(\frac{y+z}{2}\right)\arcsin^3\left(\frac{x+y}{2}\right)\\ &\geq \arcsin^2\left(\frac{x+y+z}{3}\right)\arcsin^6(x)\arcsin^6(y)\arcsin^6(z),\end{aligned}$$

for all $0 \leq x, y, z \leq 1$.

Corollary 2.19. *If $f : I \to (0, \infty)$ be an 1-$\mathrm{A_t G_t}$-concave function, then*

$$f\left(\frac{x+z}{2}\right) f\left(\frac{y+z}{2}\right) f\left(\frac{x+y}{2}\right) \leq (\geq) f\left(\frac{x+y+z}{3}\right) f(x) f(y) f(z),$$

for all $x, y, z \in I$.

Example 2.20. Let $f(x) = \arcsin(x)$, is $\mathrm{A_tG_1}$-concave for $x \in [0,1]$. Applying Corollary 2.19, we get

$$\arcsin\left(\frac{x+z}{2}\right)\arcsin\left(\frac{y+z}{2}\right)\arcsin\left(\frac{x+y}{2}\right)$$
$$\geq \arcsin\left(\frac{x+y+z}{3}\right)\arcsin(x)\arcsin(y)\arcsin(z),$$

for all $0 \leq x, y, z \leq 1$.

Corollary 2.21. *In Theorem* 2.13*,*

(i) *if* $f : I \to (0,\infty)$ *is an* $\mathrm{A_tG_h}$*-convex and submultiplicative,*

$$f\left(\frac{(x+z)(y+z)(x+y)}{8}\right) \leq f\left(\frac{x+z}{2}\right) f\left(\frac{y+z}{2}\right) f\left(\frac{x+y}{2}\right)$$
$$\leq \left[f\left(\frac{x+y+z}{3}\right)\right]^{h(3/2)} [f(x) f(y) f(z)]^{h(1/2)},$$

for all $x, y, z \in I$*. If* f *is an* h-$\mathrm{A_tG_t}$*-concave and supermultiplicative, then the inequality is reversed;*

(ii) *if* $f : I \to (0,\infty)$ *is an* $\mathrm{A_tG_h}$*-convex and supermultiplicative, then*

$$f\left(\frac{x+z}{2}\right) f\left(\frac{y+z}{2}\right) f\left(\frac{x+y}{2}\right)$$
$$\leq \left[f\left(\frac{x+y+z}{3}\right)\right]^{h(3/2)} [f(x) f(y) f(z)]^{h(1/2)}$$
$$\leq \left[f\left(\frac{x+y+z}{3}\right)\right]^{h(3/2)} [f(xyz)]^{h(1/2)},$$

for all $x, y, z \in I$*. If* f *is an* $\mathrm{A_tA_h}$*-concave and submultiplicative, then the inequality is reversed.*

Corollary 2.22. *In Theorem* 2.13*,*

(i) *if* $f : I \to (0,\infty)$ *is an* $\mathrm{A_tG_h}$*-convex and superadditive,*

$$\left[f\left(\frac{x}{2}\right) + f\left(\frac{z}{2}\right)\right]\left[f\left(\frac{y}{2}\right) + f\left(\frac{z}{2}\right)\right]\left[f\left(\frac{x}{2}\right) + f\left(\frac{y}{2}\right)\right]$$
$$\leq f\left(\frac{x+z}{2}\right) f\left(\frac{y+z}{2}\right) f\left(\frac{x+y}{2}\right)$$
$$\leq \left[f\left(\frac{x+y+z}{3}\right)\right]^{h(3/2)} [f(x) f(y) f(z)]^{h(1/2)},$$

for all $x, y, z \in I$. If f is an $\mathrm{A_tG_h}$*-concave and subadditive, then the inequality is reversed;*

(ii) *if $f : I \to (0, \infty)$ is an* $\mathrm{A_tG_h}$*-convex and subadditive, then*

$$\begin{aligned} &f\left(\frac{x+z}{2}\right) f\left(\frac{y+z}{2}\right) f\left(\frac{x+y}{2}\right) \\ &\le \left[f\left(\frac{x+y+z}{3}\right)\right]^{h(3/2)} \left[f(x) f(y) f(z)\right]^{h(1/2)} \\ &\le \left[f\left(\frac{x}{3}\right) + f\left(\frac{y}{3}\right) + f\left(\frac{z}{3}\right)\right]^{h(3/2)} \left[f(x) f(y) f(z)\right]^{h(1/2)}, \end{aligned}$$

for all $x, y, z \in I$. If f is an $\mathrm{A_tG_h}$*-concave and submultiplicative, then the inequality is reversed.*

2.3 The case when f is $\mathrm{A_tH_h}$-convex

Theorem 2.23. *Let $h : I \to (0, \infty)$ be a nonnegative super(sub)additive function. If $f : I \to (0, \infty)$ is an* $\mathrm{A_tH_h}$*-concave (convex), then*

$$\begin{aligned} &\frac{1}{f\left(\frac{x+z}{2}\right)} + \frac{1}{f\left(\frac{y+z}{2}\right)} + \frac{1}{f\left(\frac{x+y}{2}\right)} \\ &\le (\ge)\, h(1/2) \left[\frac{1}{f(y)} + \frac{1}{f(x)} + \frac{1}{f(z)}\right] + \frac{h(3/2)}{f\left(\frac{x+y+z}{3}\right)}, \end{aligned} \tag{20}$$

for all $x, y, z \in I$.

Proof. f is $\mathrm{A_tH_h}$-convex iff the inequality

$$f(t\alpha + (1-t)\beta) \le \frac{f(\alpha) f(\beta)}{h(1-t) f(\alpha) + h(1-t) f(\beta)}, \qquad 0 \le t \le 1$$

holds for all $\alpha, \beta \in I$. As in the proof of Theorem 2.3, we have

$$(x + y - 2z)\left(s + t - \frac{3}{2}\right) = 0.$$

If $x + y - 2z = 0$, then $x = y = z$, and Popoviciu's inequality holds.
If $s + t = \frac{3}{2}$, then since f is $\mathrm{A_tH_h}$-convex, we have

$$\begin{aligned} f\left(\frac{x+z}{2}\right) &= f\left[s\left(\frac{x+y+z}{3}\right) + (1-s) z\right] \\ &\ge \frac{f\left(\frac{x+y+z}{3}\right) f(z)}{h(1-s) f\left(\frac{x+y+z}{3}\right) + h(s) f(z)}, \end{aligned}$$

and this equivalent to write

$$\frac{1}{f\left(\frac{x+z}{2}\right)} \leq \frac{h\left(1-s\right) f\left(\frac{x+y+z}{3}\right) + h\left(s\right) f\left(z\right)}{f\left(\frac{x+y+z}{3}\right) f\left(z\right)}, \tag{21}$$

similarly,

$$\begin{aligned} f\left(\frac{y+z}{2}\right) &= f\left[t\left(\frac{x+y+z}{3}\right) + \left(1-t\right) z\right] \\ &\geq \frac{f\left(\frac{x+y+z}{3}\right) f\left(z\right)}{h\left(1-t\right) f\left(\frac{x+y+z}{3}\right) + h\left(t\right) f\left(z\right)}, \end{aligned}$$

which equivalent to write

$$\frac{1}{f\left(\frac{y+z}{2}\right)} \leq \frac{h\left(1-t\right) f\left(\frac{x+y+z}{3}\right) + h\left(t\right) f\left(z\right)}{f\left(\frac{x+y+z}{3}\right) f\left(z\right)}, \tag{22}$$

and

$$\begin{aligned} f\left(\frac{x+y}{2}\right) &\geq \frac{f\left(x\right) f\left(y\right)}{h\left(1/2\right)\left(f\left(x\right) + f\left(y\right)\right)} \\ \Longleftrightarrow \frac{1}{f\left(\frac{x+y}{2}\right)} &\leq \frac{h\left(1/2\right)\left(f\left(x\right) + f\left(y\right)\right)}{f\left(x\right) f\left(y\right)}, \end{aligned} \tag{23}$$

Summing the inequalities (21)–(23), we get

$$\begin{aligned} &\frac{1}{f\left(\frac{x+z}{2}\right)} + \frac{1}{f\left(\frac{y+z}{2}\right)} + \frac{1}{f\left(\frac{x+y}{2}\right)} \\ &\leq \frac{h\left(1-s\right) f\left(\frac{x+y+z}{3}\right) + h\left(s\right) f\left(z\right)}{f\left(\frac{x+y+z}{3}\right) f\left(z\right)} + \frac{h\left(1-t\right) f\left(\frac{x+y+z}{3}\right) + h\left(t\right) f\left(z\right)}{f\left(\frac{x+y+z}{3}\right) f\left(z\right)} \\ &\quad + \frac{h\left(1/2\right)\left(f\left(x\right) + f\left(y\right)\right)}{f\left(x\right) f\left(y\right)} \\ &= \frac{\left[h\left(1-s\right) + h\left(1-t\right)\right] f\left(\frac{x+y+z}{3}\right) + \left[h\left(s\right) + h\left(t\right)\right] f\left(z\right)}{f\left(\frac{x+y+z}{3}\right) f\left(z\right)} \\ &\quad + \frac{h\left(1/2\right)\left(f\left(x\right) + f\left(y\right)\right)}{f\left(x\right) f\left(y\right)} \\ &\leq \frac{h\left(2-s-t\right) f\left(\frac{x+y+z}{3}\right) + h\left(s+t\right) f\left(z\right)}{f\left(\frac{x+y+z}{3}\right) f\left(z\right)} + \frac{h\left(1/2\right)\left(f\left(x\right) + f\left(y\right)\right)}{f\left(x\right) f\left(y\right)} \\ &= \frac{h\left(1/2\right) f\left(\frac{x+y+z}{3}\right) + h\left(3/2\right) f\left(z\right)}{f\left(\frac{x+y+z}{3}\right) f\left(z\right)} + \frac{h\left(1/2\right)\left(f\left(x\right) + f\left(y\right)\right)}{f\left(x\right) f\left(y\right)} \end{aligned}$$

$$= h\left(1/2\right)\left[\frac{1}{f\left(y\right)}+\frac{1}{f\left(x\right)}+\frac{1}{f\left(z\right)}\right]+\frac{h\left(3/2\right)}{f\left(\frac{x+y+z}{3}\right)},$$

as desired. □

Remark 2.24. In (20), setting $z=y$, we have

$$\frac{2}{f\left(\frac{x+y}{2}\right)}+\frac{1}{f\left(\frac{y+z}{2}\right)} \le (\ge)\, h\left(1/2\right)\left[\frac{2}{f\left(y\right)}+\frac{1}{f\left(x\right)}\right]+\frac{h\left(3/2\right)}{f\left(\frac{x+2y}{3}\right)},$$

for all $x,y\in I$.

Corollary 2.25. *If $f:I\to(0,\infty)$ is an* $\mathrm{A_tH_t}$*-concave (convex), then*

$$\begin{aligned}&\frac{2}{3}\left[\frac{1}{f\left(\frac{x+z}{2}\right)}+\frac{1}{f\left(\frac{y+z}{2}\right)}+\frac{1}{f\left(\frac{x+y}{2}\right)}\right]\\ &\le (\ge)\,\frac{1}{3}\left[\frac{1}{f\left(y\right)}+\frac{1}{f\left(x\right)}+\frac{1}{f\left(z\right)}\right]+\frac{1}{f\left(\frac{x+y+z}{3}\right)},\end{aligned}$$

for all $x,y,z\in I$. The equality holds with $f(x)=\frac{1}{x}$, $x>0$.

Example 2.26. Let $f(x)=x^p$, $p\ge 1$. Then, $\mathrm{A_tH_t}$-concave for $x\ge 1$. Applying Corollary 2.25, we get

$$\begin{aligned}&\frac{2}{3}\left[\left(\frac{x+z}{2}\right)^{-p}+\left(\frac{y+z}{2}\right)^{-p}+\left(\frac{x+y}{2}\right)^{-p}\right]\\ &\le \frac{x^{-p}+y^{-p}+z^{-p}}{3}+\left(\frac{x+y+z}{3}\right)^{-p}\end{aligned}$$

for all $x,y,z\ge 1$.

Corollary 2.27. *If $f:I\to(0,\infty)$ is an* $\mathrm{A_tH_{1/t}}$*-convex, then*

$$\begin{aligned}&\frac{3}{2}\left[\frac{1}{f\left(\frac{x+z}{2}\right)}+\frac{1}{f\left(\frac{y+z}{2}\right)}+\frac{1}{f\left(\frac{x+y}{2}\right)}\right]\\ &\le 3\left[\frac{1}{f\left(y\right)}+\frac{1}{f\left(x\right)}+\frac{1}{f\left(z\right)}\right]+\frac{1}{f\left(\frac{x+y+z}{3}\right)},\end{aligned}$$

for all $x,y,z\in I$.

Example 2.28. Let $f(x) = -\log(x)$, $x \gneqq 1$. Then, f is $\mathrm{A_tH_{1/t}}$-convex for $x \gneqq 1$. Applying Corollary 2.27, we get

$$\frac{3}{2}\left[\frac{1}{\log\left(\frac{x+z}{2}\right)} + \frac{1}{\log\left(\frac{y+z}{2}\right)} + \frac{1}{\log\left(\frac{x+y}{2}\right)}\right] \\ \leq 3\left(\frac{1}{\log x} + \frac{1}{\log y} + \frac{1}{\log z}\right) + \log(xyz)^{\frac{1}{3}},$$

for all $x, y, z \gneqq 1$.

Corollary 2.29. *If $f : I \to (0, \infty)$ is an $\mathrm{A_tH_1}$-convex, then*

$$\frac{1}{f\left(\frac{x+z}{2}\right)} + \frac{1}{f\left(\frac{y+z}{2}\right)} + \frac{1}{f\left(\frac{x+y}{2}\right)} \leq \left[\frac{1}{f(y)} + \frac{1}{f(x)} + \frac{1}{f(z)}\right] + \frac{1}{f\left(\frac{x+y+z}{3}\right)},$$

for all $x, y, z \in I$.

Example 2.30. Let $f(x) = -\log(x)$, $x \gneqq 1$. Then, f is $\mathrm{A_tH_1}$-convex on $x \gneqq 1$. Applying Corollary 2.29, we get

$$\frac{1}{\log\left(\frac{x+z}{2}\right)} + \frac{1}{\log\left(\frac{y+z}{2}\right)} + \frac{1}{\log\left(\frac{x+y}{2}\right)} \leq \frac{1}{\log x} + \frac{1}{\log y} + \frac{1}{\log z} + \log(xyz)^{\frac{1}{3}},$$

for all $x, y, z \gneqq 1$.

3 Popoviciu inequalities for h-GN-convex functions

3.1 The case when f is $\mathrm{G_tA_h}$-convex

Theorem 3.1. *Let $h : I \to (0, \infty)$ be a nonnegative super(sub)additive function. If $f : I \to (0, \infty)$ is $\mathrm{G_tA_h}$-convex function, then*

$$f\left(\sqrt{xz}\right) + f\left(\sqrt{yz}\right) + f\left(\sqrt{xy}\right) \\ \leq (\geq)\, h(3/2)\, f\left(\sqrt[3]{xyz}\right) + h(1/2)\left[f(x) + f(y) + f(z)\right], \quad (24)$$

for all $x, y, z \in I$.

Proof. f is $\mathrm{G_tA_h}$-convex iff the inequality

$$f\left(\alpha^t \beta^{1-t}\right) \leq h(t)\, f(\alpha) + h(1-t)\, f(\beta), \qquad 0 \leq t \leq 1$$

holds for all $\alpha, \beta \in I$. Assume that $x \leq y \leq z$. If $y \leq (xyz)^{1/3}$, then

$$(xyz)^{1/3} \leq (xz)^{1/2} \leq z \text{ and } (xyz)^{1/3} \leq (yz)^{1/2} \leq z,$$

so that there exist two numbers $s, t \in [0, 1]$ satisfying

$$(xz)^{1/2} = (xyz)^{s/3} z^{1-s}$$

and

$$(yz)^{1/2} = (xyz)^{t/3} z^{1-t}.$$

Multiplying the above equations, we get

$$(xyz)^{1/2} z^{1/2} = (xyz)^{(s+t)/3} z^{2-(s+t)}$$

or

$$(xyz)^{\frac{(s+t)}{3}-\frac{1}{2}} z^{2-(s+t)-\frac{1}{2}} = 1.$$

If $xyz^2 = 1$, then $x = y = z$, and Popoviciu's inequality holds.

If $s + t = \frac{3}{2}$, then since f is $\mathrm{G_tA_h}$-convex, we have

$$\begin{aligned}
f\left(\sqrt{xz}\right) &= f\left[(xyz)^{s/3} z^{1-s}\right] \leq h(s)\left[f\left(\sqrt[3]{xyz}\right)\right] + h(1-s)\left[f(z)\right] \\
f\left(\sqrt{yz}\right) &= f\left[(xyz)^{t/3} z^{1-t}\right] \leq h(t)\left[f\left(\sqrt[3]{xyz}\right)\right] + h(1-t)\left[f(z)\right] \\
f\left(\sqrt{xy}\right) &\leq h\left(\frac{1}{2}\right)\left[f(x) + f(y)\right]
\end{aligned}$$

Summing up these inequalities, we get

$$\begin{aligned}
& f\left(\sqrt{xz}\right) + f\left(\sqrt{yz}\right) + f\left(\sqrt{xy}\right) \\
& \leq h(s) f\left(\sqrt[3]{xyz}\right) + h(1-s) f(z) + h(t) f\left(\sqrt[3]{xyz}\right) + h(1-t) f(z) \\
& \qquad + h(1/2)\left[f(x) + f(y)\right] \\
& = \left[h(s) + h(t)\right] f\left(\sqrt[3]{xyz}\right) + \left[h(1-s) + h(1-t)\right] f(z) \\
& + h(1/2)\left[f(x) + f(y)\right] \\
& \leq h(s+t) f\left(\sqrt[3]{xyz}\right) + h(2-s-t) f(z) + h(1/2)\left[f(x) + f(y)\right] \\
& = h(3/2) f\left(\sqrt[3]{xyz}\right) + h(1/2) f(z) + h(1/2)\left[f(x) + f(y)\right] \\
& = h(3/2) f\left(\sqrt[3]{xyz}\right) + h(1/2)\left[f(x) + f(y) + f(z)\right],
\end{aligned}$$

which proves the inequality (24). □

Remark 3.2. In (24), setting $z = y$, we get

$$2f\left(\sqrt{xy}\right) + f\left(y\right) \leq\ (\geq)\, h\left(3/2\right) f\left(\sqrt[3]{xy^2}\right) + h\left(1/2\right)\left[f\left(x\right) + 2f\left(y\right)\right],$$

for all $x, y \in I$.

Corollary 3.3. *If $f : I \to (0, \infty)$ is* $\mathrm{G_tA_t}$*-convex function, then*

$$\frac{2}{3}\left[f\left(\sqrt{xz}\right) + f\left(\sqrt{yz}\right) + f\left(\sqrt{xy}\right)\right] \leq f\left(\sqrt[3]{xyz}\right) + \frac{f\left(x\right) + f\left(y\right) + f\left(z\right)}{3},$$

for all $x, y, z \in I$. The equality holds with $f(x) = \log(x)$, $x > 1$.

Example 3.4. Let $f(x) = \cosh(x)$, $x > 0$. Then, f is $\mathrm{G_tA_t}$-convex on $(0, \infty)$. Applying Corollary 3.3, we get

$$\frac{2}{3}\left[\cosh\left(\sqrt{xz}\right) + \cosh\left(\sqrt{yz}\right) + \cosh\left(\sqrt{xy}\right)\right]$$
$$\leq \cosh\left(\sqrt[3]{xyz}\right) + \frac{\cosh\left(x\right) + \cosh\left(y\right) + \cosh\left(z\right)}{3},$$

for all $x, y, z > 0$.

Corollary 3.5. *If $f : I \to (0, \infty)$ is* $\mathrm{G_tA_{1/t}}$*-concave function, then*

$$\frac{3}{2}\left[f\left(\sqrt{xz}\right) + f\left(\sqrt{yz}\right) + f\left(\sqrt{xy}\right)\right] \geq f\left(\sqrt[3]{xyz}\right) + 3\left(f\left(x\right) + f\left(y\right) + f\left(z\right)\right)$$

for all $x, y, z \in I$.

Example 3.6. Let $f(x) = -x^2$, $x > 0$. Then, f is $\mathrm{G_tA_{1/t}}$-concave on $(0, \infty)$. Applying Corollary 3.5 we get

$$\frac{3}{2}\left(xz + yz + xy\right) \leq \left(\sqrt[3]{xyz}\right)^2 + 3\left(x^2 + y^2 + z^2\right)$$

for all $x, y, z > 0$.

Corollary 3.7. *If $f : I \to (0, \infty)$ is* $\mathrm{G_tA_1}$*-concave function, then*

$$f\left(\sqrt{xz}\right) + f\left(\sqrt{yz}\right) + f\left(\sqrt{xy}\right) \geq f\left(\sqrt[3]{xyz}\right) + f\left(x\right) + f\left(y\right) + f\left(z\right),$$

for all $x, y, z \in I$.

Example 3.8. Let $f(x) = -x^2$, $x > 0$. Then, f is $\mathrm{G_tA_1}$-convex on $(0,\infty)$. Applying Corollary 3.7, we get

$$xz + yz + xy \le \left(\sqrt[3]{xyz}\right)^2 + x^2 + y^2 + z^2$$

for all $x, y, z > 0$.

Corollary 3.9. *In Theorem* 3.1*,*

(i) *if* $f : I \to (0,\infty)$ *is an* $\mathrm{G_tA_h}$*-convex and superadditive,*

$$\begin{aligned} &f\left(\sqrt{xz}\right) + f\left(\sqrt{yz}\right) + f\left(\sqrt{xy}\right) \\ &\le h(3/2)\, f\left(\sqrt[3]{xyz}\right) + h(1/2)\left[f(x) + f(y) + f(z)\right] \\ &\le h(3/2)\, f\left(\sqrt[3]{xyz}\right) + h(1/2)\, f(x+y+z), \end{aligned}$$

for all $x, y, z \in I$*. If* f *is an* $\mathrm{G_tA_h}$*-concave and subadditive, then the inequality is reversed;*

(ii) *if* $f : I \to (0,\infty)$ *is an* $\mathrm{G_tA_h}$*-convex and subadditive, then*

$$\begin{aligned} f\left(\sqrt{xz} + \sqrt{yz} + \sqrt{xy}\right) &\le f\left(\sqrt{xz}\right) + f\left(\sqrt{yz}\right) + f\left(\sqrt{xy}\right) \\ &\le h(3/2)\, f\left(\sqrt[3]{xyz}\right) + h(1/2)\left[f(x) + f(y) + f(z)\right], \end{aligned}$$

for all $x, y, z \in I$*. If* f *is an* $\mathrm{G_tA_h}$*-concave and superadditive, then the inequality is reversed.*

Example 3.10. Let $f(x) = \cosh(x)$, which is $\mathrm{G_tA_t}$-convex and superadditive on $(0,\infty)$. Applying Corollary 3.9, we get

$$\begin{aligned} &\frac{2}{3}\left[\cosh\left(\sqrt{xz}\right) + \cosh\left(\sqrt{yz}\right) + \cosh\left(\sqrt{xy}\right)\right] \\ &\le \cosh\left(\sqrt[3]{xyz}\right) + \frac{\cosh(x) + \cosh(y) + \cosh(z)}{3} \\ &\le \cosh\left(\sqrt[3]{xyz}\right) + \frac{1}{3}\cosh(x+y+z), \end{aligned}$$

for all $x, y, z > 0$.

3.2 The case when f is $\mathrm{G_tG_h}$-convex

Theorem 3.11. *Let* $h : I \to (0,\infty)$ *be a nonnegative super(sub)additive function. If* $f : I \to (0,\infty)$ *is* $\mathrm{G_tG_h}$*-convex function, then*

$$\begin{aligned} &f\left(\sqrt{xz}\right) f\left(\sqrt{yz}\right) f\left(\sqrt{xy}\right) \\ &\le (\ge)\ \left[f\left(\sqrt[3]{xyz}\right)\right]^{h(3/2)} \left[f(x)\, f(y)\, f(z)\right]^{h(1/2)}, \end{aligned} \tag{25}$$

for all $x, y, z \in I$.

Proof. f is $\mathrm{G_tG_h}$-convex iff the inequality

$$f\left(\alpha^t\beta^{1-t}\right) \le [f(\alpha)]^{h(t)}[f(\beta)]^{h(1-t)}, \qquad 0 \le t \le 1$$

holds for all $\alpha, \beta \in I$. As in the proof of Theorem 3.1, if $xyz^2 = 1$, then $x = y = z$, and Popoviciu's inequality holds.

If $s + t = \frac{3}{2}$, then since f is $\mathrm{G_tG_h}$-convex, we have

$$\begin{aligned}
f\left(\sqrt{xz}\right) &= f\left[(xyz)^{s/3} z^{1-s}\right] \le \left[f\left(\sqrt[3]{xyz}\right)\right]^{h(s)}[f(z)]^{h(1-s)},\\
f\left(\sqrt{yz}\right) &= f\left[(xyz)^{t/3} z^{1-t}\right] \le \left[f\left(\sqrt[3]{xyz}\right)\right]^{h(t)}[f(z)]^{h(1-t)},\\
f\left(\sqrt{xy}\right) &\le h\left(\frac{1}{2}\right)[f(x)+f(y)].
\end{aligned}$$

Multiplying these inequalities we get

$$\begin{aligned}
&f\left(\sqrt{xz}\right) f\left(\sqrt{yz}\right) f\left(\sqrt{xy}\right)\\
&\le \left[f\left(\sqrt[3]{xyz}\right)\right]^{h(s)}[f(z)]^{h(1-s)}\left[f\left(\sqrt[3]{xyz}\right)\right]^{h(t)}[f(z)]^{h(1-t)}[f(x)f(y)]^{h(1/2)}\\
&= \left[f\left(\sqrt[3]{xyz}\right)\right]^{h(s)+h(t)}[f(z)]^{h(1-s)+h(1-t)}[f(x)f(y)]^{h(1/2)}\\
&\le \left[f\left(\sqrt[3]{xyz}\right)\right]^{h(s+t)}[f(z)]^{h(2-s-t)}[f(x)f(y)]^{h(1/2)}\\
&= \left[f\left(\sqrt[3]{xyz}\right)\right]^{h(3/2)}[f(z)]^{h(1/2)}[f(x)f(y)]^{h(1/2)}\\
&= \left[f\left(\sqrt[3]{xyz}\right)\right]^{h(3/2)}[f(x)f(y)f(z)]^{h(1/2)},
\end{aligned}$$

as desired. □

Remark 3.12. In (25), setting $z = y$ we get

$$f^2\left(\sqrt{xy}\right) f(y) \le (\ge) \left[f\left(\sqrt[3]{xy^2}\right)\right]^{h(3/2)}\left[f(x) f^2(y)\right]^{h(1/2)},$$

for all $x, y \in I$.

Corollary 3.13. *If $f : I \to (0,\infty)$ is $\mathrm{G_tG_t}$-convex (concave) function, then*

$$f^2\left(\sqrt{xz}\right) f^2\left(\sqrt{yz}\right) f^2\left(\sqrt{xy}\right) \le (\ge) f^3\left(\sqrt[3]{xyz}\right) f(x) f(y) f(z),$$

for all $x, y, z \in I$. The equality holds with $f(x) = \mathrm{e}^x$, $x > 0$.

Example 3.14. Let $f(x) = \cosh(x)$, which is $\mathrm{G_tG_t}$-convex on $(0,\infty)$. Applying Corollary 3.13, we get

$$\cosh^2\left(\sqrt{xz}\right)\cosh^2\left(\sqrt{yz}\right)\cosh^2\left(\sqrt{xy}\right) \le f^3\left(\sqrt[3]{xyz}\right)\cosh(x)\cosh(y)\cosh(z),$$

for all $x, y, z > 0$.

Corollary 3.15. *If $f : I \to (0,\infty)$ is $\mathrm{G_tG_1/t}$-concave function, then*

$$f^3\left(\sqrt{xz}\right)f^3\left(\sqrt{yz}\right)f^3\left(\sqrt{xy}\right) \ge f^2\left(\sqrt[3]{xyz}\right)f^6(x)f^6(y)f^6(z),$$

for all $x, y, z \in I$.

Example 3.16. Let $f(x) = \exp(-x)$ which is $\frac{1}{t}$-$\mathrm{G_tG_t}$-concave on $(0,\infty)$. Applying Corollary 3.15 we get

$$\sqrt{xz} + \sqrt{yz} + \sqrt{xy} \le \frac{2}{3}\sqrt[3]{xyz} + 2x + 2y + 2z,$$

for all $x, y, z > 0$.

Corollary 3.17. *If $f : I \to (0,\infty)$ is $\mathrm{G_tG_1}$-concave function, then*

$$f\left(\sqrt{xz}\right)f\left(\sqrt{yz}\right)f\left(\sqrt{xy}\right) \le f\left(\sqrt[3]{xyz}\right)f(x)f(y)f(z),$$

for all $x, y, z \in I$.

Example 3.18. Let $f(x) = \exp(-x)$, which is $\mathrm{G_tG_1}$-concave on $(0,\infty)$. Applying Corollary 3.17, we get

$$\sqrt{xz} + \sqrt{yz} + \sqrt{xy} \le \sqrt[3]{xyz} + x + y + z,$$

for all $x, y, z > 0$.

Corollary 3.19. *In Theorem* 3.11.

(i) *If $f : I \to (0,\infty)$ is an $\mathrm{G_tG_h}$-convex and supermultiplicative,*

$$\begin{aligned} f\left(\sqrt{xz}\right)f\left(\sqrt{yz}\right)f\left(\sqrt{xy}\right) &\le \left[f\left(\sqrt[3]{xyz}\right)\right]^{h(3/2)}\left[f(x)f(y)f(z)\right]^{h(1/2)} \\ &\le \left[f\left(\sqrt[3]{xyz}\right)\right]^{h(3/2)}\left[f(xyz)\right]^{h(1/2)}, \end{aligned}$$

for all $x, y, z \in I$.

(ii) *If* $f : I \to (0,\infty)$ *is an* $\mathrm{G_tG_h}$*-convex and submultiplicative, then*

$$\begin{aligned} f(xzy) &\le f\left(\sqrt{xz}\right) f\left(\sqrt{yz}\right) f\left(\sqrt{xy}\right) \\ &\le \left[f\left(\sqrt[3]{xyz}\right)\right]^{h(3/2)} \left[f(x) f(y) f(z)\right]^{h(1/2)} \\ &\le \left[f\left(\sqrt[3]{x}\right) f\left(\sqrt[3]{y}\right) f\left(\sqrt[3]{z}\right)\right]^{h(3/2)} \left[f(x) f(y) f(z)\right]^{h(1/2)}, \end{aligned}$$

for all $x, y, z \in I$.

Example 3.20. Let $f(x) = \cosh(x)$, which is $\mathrm{G_tG_t}$-convex and supermultiplicative on $[1,\infty)$. Applying Corollary 3.19, we get

$$\begin{aligned} \cosh^2\left(\sqrt{xz}\right)\cosh^2\left(\sqrt{yz}\right)\cosh^2\left(\sqrt{xy}\right) &\le \cosh^3\left(\sqrt[3]{xyz}\right)\cosh(x)\cosh(y)\cosh(z) \\ &\le \cosh^3\left(\sqrt[3]{xyz}\right)\cosh(xyz) \end{aligned}$$

for all $x, y, z \ge 1$.

3.3 The case when f is $\mathrm{G_tH_h}$-convex

Theorem 3.21. *Let* $h : I \to (0,\infty)$ *be a nonnegative super(sub)additive function. If* $f : I \to (0,\infty)$ *is* $\mathrm{G_tH_h}$*-concave (convex) function, then*

$$\begin{aligned} &\frac{1}{f\left(\sqrt{xz}\right)} + \frac{1}{f\left(\sqrt{yz}\right)} + \frac{1}{f\left(\sqrt{xy}\right)} \\ &\le (\ge)\, h\left(\frac{1}{2}\right)\left[\frac{1}{f(x)} + \frac{1}{f(y)} + \frac{1}{f(z)}\right] + \frac{h(3/2)}{f\left(\sqrt[3]{xyz}\right)}, \end{aligned} \tag{26}$$

for all $x, y, z \in I$.

Proof. f is $\mathrm{G_tH_h}$-convex iff the inequality

$$f\left(\alpha^t\beta^{1-t}\right) \le \frac{f(\alpha) f(\beta)}{h(1-t) f(\alpha) + h(t) f(\beta)}, \qquad 0 \le t \le 1$$

holds for all $\alpha, \beta \in I$. As in the proof of Theorem 3.1, if $xyz^2 = 1$, then $x = y = z$, and Popoviciu's inequality holds.

If $s + t = \frac{3}{2}$, then since f is $\mathrm{G_tH_h}$-convex, we have

$$f\left(\sqrt{xz}\right) = f\left[(xyz)^{s/3} z^{1-s}\right] \ge \frac{f\left(\sqrt[3]{xyz}\right) f(z)}{h(1-s) f\left(\sqrt[3]{xyz}\right) + h(s) f(z)}$$

and this equivalent to write

$$\frac{1}{f\left(\sqrt{xz}\right)} \leq \frac{h\left(1-s\right)f\left(\sqrt[3]{xyz}\right)+h\left(s\right)f\left(z\right)}{f\left(\sqrt[3]{xyz}\right)f\left(z\right)}, \tag{27}$$

similarly,

$$f\left(\sqrt{yz}\right)=f\left[\left(xyz\right)^{t/3}z^{1-t}\right] \geq \frac{f\left(\sqrt[3]{xyz}\right)f\left(z\right)}{h\left(1-t\right)f\left(\sqrt[3]{xyz}\right)+h\left(t\right)f\left(z\right)}$$

which equivalent to write

$$\frac{1}{f\left(\sqrt{yz}\right)} \leq \frac{h\left(1-t\right)f\left(\sqrt[3]{xyz}\right)+h\left(t\right)f\left(z\right)}{f\left(\sqrt[3]{xyz}\right)f\left(z\right)}, \tag{28}$$

and

$$\begin{aligned} f\left(\sqrt{xy}\right) &\geq \frac{f\left(x\right)f\left(y\right)}{h\left(1/2\right)\left(f\left(x\right)+f\left(y\right)\right)} \\ \Longleftrightarrow \frac{1}{f\left(\sqrt{xy}\right)} &\leq \frac{h\left(1/2\right)\left(f\left(x\right)+f\left(y\right)\right)}{f\left(x\right)f\left(y\right)}. \end{aligned} \tag{29}$$

Summing the inequalities (27)–(29), we get

$$\begin{aligned} &\frac{1}{f\left(\sqrt{xz}\right)}+\frac{1}{f\left(\sqrt{yz}\right)}+\frac{1}{f\left(\sqrt{xy}\right)} \\ &\leq \frac{h\left(1-s\right)f\left(\sqrt[3]{xyz}\right)+h\left(s\right)f\left(z\right)}{f\left(\sqrt[3]{xyz}\right)f\left(z\right)}+\frac{h\left(1-t\right)f\left(\sqrt[3]{xyz}\right)+h\left(t\right)f\left(z\right)}{f\left(\sqrt[3]{xyz}\right)f\left(z\right)} \\ &\qquad +\frac{h\left(1/2\right)\left(f\left(x\right)+f\left(y\right)\right)}{f\left(x\right)f\left(y\right)} \\ &= \frac{\left[h\left(1-s\right)+h\left(1-t\right)\right]f\left(\sqrt[3]{xyz}\right)+\left[h\left(s\right)+h\left(t\right)\right]f\left(z\right)}{f\left(\sqrt[3]{xyz}\right)f\left(z\right)} \\ &\qquad +\frac{h\left(1/2\right)\left(f\left(x\right)+f\left(y\right)\right)}{f\left(x\right)f\left(y\right)} \\ &\leq \frac{h\left(2-s-t\right)f\left(\sqrt[3]{xyz}\right)+h\left(s+t\right)f\left(z\right)}{f\left(\sqrt[3]{xyz}\right)f\left(z\right)}+\frac{h\left(1/2\right)\left(f\left(x\right)+f\left(y\right)\right)}{f\left(x\right)f\left(y\right)} \\ &\leq \frac{h\left(1/2\right)f\left(\sqrt[3]{xyz}\right)+h\left(3/2\right)f\left(z\right)}{f\left(\sqrt[3]{xyz}\right)f\left(z\right)}+\frac{h\left(1/2\right)\left(f\left(x\right)+f\left(y\right)\right)}{f\left(x\right)f\left(y\right)} \\ &= h\left(\frac{1}{2}\right)\left[\frac{1}{f\left(x\right)}+\frac{1}{f\left(y\right)}+\frac{1}{f\left(z\right)}\right]+\frac{h\left(3/2\right)}{f\left(\sqrt[3]{xyz}\right)}, \end{aligned}$$

which proves the inequality in (26). □

Remark 3.22. In (26), setting $z = y$ then we get

$$\frac{2}{f\left(\sqrt{xy}\right)} + \frac{1}{f\left(y\right)} \leq (\geq)\, h\left(\frac{1}{2}\right)\left[\frac{1}{f\left(x\right)} + \frac{2}{f\left(y\right)}\right] + \frac{h\left(3/2\right)}{f\left(\sqrt[3]{xy^2}\right)},$$

for all $x, y \in I$.

Corollary 3.23. *If $f : I \to (0, \infty)$ is* $\mathrm{G_tH_t}$*-concave (convex) function, then*

$$\frac{2}{3}\left[\frac{1}{f\left(\sqrt{xz}\right)} + \frac{1}{f\left(\sqrt{yz}\right)} + \frac{1}{f\left(\sqrt{xy}\right)}\right]$$
$$\leq (\geq)\, \frac{1}{3}\left[\frac{1}{f\left(x\right)} + \frac{1}{f\left(y\right)} + \frac{1}{f\left(z\right)}\right] + \frac{1}{f\left(\sqrt[3]{xyz}\right)},$$

for all $x, y, z \in I$. The equality holds with $f\left(x\right) = \frac{1}{\log(x)}$, $x \gneqq 1$.

Example 3.24. Let $f\left(x\right) = \cosh\left(x\right)$, then f is $\mathrm{G_tH_t}$-convex for all $x \geq 1$. Applying Corollary 3.23, we get

$$\frac{2}{3}\left[\frac{1}{\cosh\left(\sqrt{xz}\right)} + \frac{1}{\cosh\left(\sqrt{yz}\right)} + \frac{1}{\cosh\left(\sqrt{xy}\right)}\right]$$
$$\geq \frac{1}{3}\left[\frac{1}{\cosh\left(x\right)} + \frac{1}{\cosh\left(y\right)} + \frac{1}{\cosh\left(z\right)}\right] + \frac{1}{\cosh\left(\sqrt[3]{xyz}\right)},$$

for all $x, y, z \geq 1$.

Corollary 3.25. *If $f : I \to (0, \infty)$ is* $\mathrm{G_tH_{1/t}}$*-convex function, then*

$$\frac{3}{2}\left[\frac{1}{f\left(\sqrt{xz}\right)} + \frac{1}{f\left(\sqrt{yz}\right)} + \frac{1}{f\left(\sqrt{xy}\right)}\right] \geq 3\left[\frac{1}{f\left(x\right)} + \frac{1}{f\left(y\right)} + \frac{1}{f\left(z\right)}\right] + \frac{1}{f\left(\sqrt[3]{xyz}\right)},$$

for all $x, y, z \in I$.

Example 3.26. Let $f\left(x\right) = -\log\left(x\right)$, then f is $\mathrm{G_tH_{1/t}}$-convex for all $x > 1$. Applying Corollary 3.25, then we get

$$\frac{3}{2}\left[\frac{1}{\log\left(\sqrt{xz}\right)} + \frac{1}{\log\left(\sqrt{yz}\right)} + \frac{1}{\log\left(\sqrt{xy}\right)}\right]$$
$$\leq 3\left[\frac{1}{\log\left(x\right)} + \frac{1}{\log\left(y\right)} + \frac{1}{\log\left(z\right)}\right] + \frac{1}{\log\left(\sqrt[3]{xyz}\right)},$$

for all $x, y, z > 1$.

Corollary 3.27. *If* $f : I \to (0,\infty)$ *is* 1-$\mathrm{G_tH_t}$*-convex function, then*

$$\frac{1}{f\left(\sqrt{xz}\right)}+\frac{1}{f\left(\sqrt{yz}\right)}+\frac{1}{f\left(\sqrt{xy}\right)}\geq\left[\frac{1}{f\left(x\right)}+\frac{1}{f\left(y\right)}+\frac{1}{f\left(z\right)}\right]+\frac{1}{f\left(\sqrt[3]{xyz}\right)},$$

for all $x, y, z \in I$.

Example 3.28. Let $f(x) = -\log(x)$, then f is $\mathrm{G_tH_1}$-convex for all $x > 1$. Applying Corollary 3.27, we get

$$\begin{aligned}&\frac{1}{\log\left(\sqrt{xz}\right)}+\frac{1}{\log\left(\sqrt{yz}\right)}+\frac{1}{\log\left(\sqrt{xy}\right)}\\&\leq\left[\frac{1}{\log\left(x\right)}+\frac{1}{\log\left(y\right)}+\frac{1}{\log\left(z\right)}\right]+\frac{1}{\log\left(\sqrt[3]{xyz}\right)},\end{aligned}$$

for all $x, y, z > 1$.

4 Popoviciu inequalities for h-HN-convex functions

4.1 The case when f is $\mathrm{H_tA_h}$-convex

Theorem 4.1. *Let* $h : I \to (0,\infty)$ *be a nonnegative super(sub)additive. If* $f : I \to (0,\infty)$ *is* $\mathrm{H_tA_h}$*-convex (concave) function, then*

$$\begin{aligned}&f\left(\frac{2xz}{x+z}\right)+f\left(\frac{2yz}{y+z}\right)+f\left(\frac{2xy}{x+y}\right)\\&\leq(\geq)\,h\left(3/2\right)f\left(\frac{3xyz}{xy+yz+xz}\right)+h\left(1/2\right)\left[f\left(x\right)+f\left(y\right)+f\left(z\right)\right],\end{aligned}\tag{30}$$

for all $x, y, z \in I$.

Proof. f is $\mathrm{H_tA_h}$-convex iff the inequality

$$f\left(\frac{\alpha\beta}{t\alpha+(1-t)\beta}\right)\leq h\left(1-t\right)f\left(\alpha\right)+h\left(t\right)f\left(\beta\right),\qquad 0\leq t\leq 1,$$

holds for all $\alpha, \beta \in I$. Assume that $x \leq y \leq z$. If $y \leq \frac{3xyz}{xy+yz+xz}$, then

$$\frac{3xyz}{xy+yz+xz}\leq\frac{2xz}{x+z}\leq z\text{ and }\frac{3xyz}{xy+yz+xz}\leq\frac{2yz}{y+z}\leq z,$$

so that there exist two numbers $s, t \in [0,1]$ satisfying

$$\frac{2xz}{x+z} = \frac{\frac{3xyz}{xy+yz+xz} \cdot z}{s\frac{3xyz}{xy+yz+xz} + (1-s)\,z},$$

and

$$\frac{2yz}{y+z} = \frac{\frac{3xyz}{xy+yz+xz} \cdot z}{t\frac{3xyz}{xy+yz+xz} + (1-t)\,z}.$$

For simplicity set, $u = \frac{3xyz}{xy+yz+xz}$, summing the reciprocal of the previous two equations

$$\frac{x+z}{2xz} + \frac{y+z}{2yz} = \frac{(s+t)\,\frac{3xyz}{xy+yz+xz} + (2-s-t)\,z}{\frac{3xyz}{xy+yz+xz} \cdot z} = \frac{3\,(s+t)\,u + (2-s-t)\,z}{3u \cdot z}.$$

Simplifying the above equation and reverse it back to the original form (taking the reciprocal again), we get

$$\frac{u}{u+z} = \frac{u}{2\,(s+t)\,u + \frac{2}{3}\,(2-s-t)\,z},$$

since $y, x, z > 0$, this yields that $x = y = z$ and thus Popoviciu's inequality holds, or $s + t = \frac{1}{2}$ and in this case since f is $\mathrm{H_t A_h}$-convex, we have

$$\begin{aligned}
f\left(\frac{2xz}{x+z}\right) &= f\left(\frac{\frac{3xyz}{xy+yz+xz} \cdot z}{s\frac{3xyz}{xy+yz+xz} + (1-s)\,z}\right) \\
&\le h\,(s)\,f\,(z) + h\,(1-s)\,f\left(\frac{3xyz}{xy+yz+xz}\right), \\
f\left(\frac{2yz}{y+z}\right) &= f\left(\frac{\frac{3xyz}{xy+yz+xz} \cdot z}{t\frac{3xyz}{xy+yz+xz} + (1-t)\,z}\right) \\
&\le h\,(t)\,f\,(z) + h\,(1-t)\,f\left(\frac{3xyz}{xy+yz+xz}\right), \\
f\left(\frac{2xy}{x+y}\right) &\le h\,(1/2)\,[f\,(x) + f\,(y)]\,.
\end{aligned}$$

Summing up these inequalities we get

$$
\begin{aligned}
& f\left(\frac{2xz}{x+z}\right)+f\left(\frac{2yz}{y+z}\right)+f\left(\frac{2xy}{x+y}\right) \\
& \leq [h(s)+h(t)]\, f(z)+[h(1-s)+h(1-t)]\, f\left(\frac{3xyz}{xy+yz+xz}\right) \\
& \qquad + h(1/2)\,[f(x)+f(y)] \\
& \leq h(s+t)\, f(z)+h(2-s-t)\, f\left(\frac{3xyz}{xy+yz+xz}\right)+h(1/2)\,[f(x)+f(y)] \\
& = h(3/2)\, f\left(\frac{3xyz}{xy+yz+xz}\right)+h(1/2)\,[f(x)+f(y)+f(z)],
\end{aligned}
$$

which proves the inequality in (30). □

Remark 4.2. In (30), setting $z=y$ then we get

$$
2f\left(\frac{2xy}{x+y}\right)+f(y)\leq\ (\geq)\, h(3/2)\, f\left(\frac{3xy}{2x+y}\right)+h(1/2)\,[f(x)+2f(y)],
$$

for all $x,y\in I$.

Corollary 4.3. *If $f: I\to(0,\infty)$ is* $\mathrm{H_tA_t}$*-convex (concave) function, then*

$$
\begin{aligned}
& \frac{2}{3}\left[f\left(\frac{2xz}{x+z}\right)+f\left(\frac{2yz}{y+z}\right)+f\left(\frac{2xy}{x+y}\right)\right] \\
& \qquad\qquad \leq\ (\geq)\, f\left(\frac{3xyz}{xy+yz+xz}\right)+\frac{f(x)+f(y)+f(z)}{3},
\end{aligned}
$$

for all $x,y,z\in I$. The equality holds with $f(x)=\frac{1}{x}$, $x>0$.

Example 4.4. Let $f(x)=\arctan(x)$, then f is $\mathrm{H_tA_t}$-convex on $(0,\infty)$. Applying Corollary 4.3, then we get

$$
\begin{aligned}
& \frac{2}{3}\left[\arctan\left(\frac{2xz}{x+z}\right)+\arctan\left(\frac{2yz}{y+z}\right)+\arctan\left(\frac{2xy}{x+y}\right)\right] \\
& \qquad \leq \arctan\left(\frac{3xyz}{xy+yz+xz}\right)+\frac{\arctan(x)+\arctan(y)+\arctan(z)}{3},
\end{aligned}
$$

Corollary 4.5. *If* $f : I \to (0,\infty)$ *is* $\mathrm{H_tA_{1/t}}$*-concave function, then*

$$\frac{3}{2}\left[f\left(\frac{2xz}{x+z}\right)+f\left(\frac{2yz}{y+z}\right)+f\left(\frac{2xy}{x+y}\right)\right] \geq f\left(\frac{3xyz}{xy+yz+xz}\right)+3\left[f\left(x\right)+f\left(y\right)+f\left(z\right)\right],$$

for all $x, y, z \in I$.

Example 4.6. Let $f\left(x\right)=x^2$, therefore f is $\mathrm{H_tA_{1/t}}$-concave on $x<0$. Applying Corollary 4.5, we get

$$\left(\frac{xz}{x+z}\right)^2+\left(\frac{yz}{y+z}\right)^2+\left(\frac{xy}{x+y}\right)^2 \geq \frac{3}{2}\left(\frac{xyz}{xy+yz+xz}\right)^2+\frac{1}{18}\left(x^2+y^2+z^2\right),$$

for all $x, y, z < 0$.

Corollary 4.7. *If* $f : I \to (0,\infty)$ *is* $\mathrm{H_tA_1}$*-concave function, then*

$$f\left(\frac{2xz}{x+z}\right)+f\left(\frac{2yz}{y+z}\right)+f\left(\frac{2xy}{x+y}\right) \geq f\left(\frac{3xyz}{xy+yz+xz}\right)+\left[f\left(x\right)+f\left(y\right)+f\left(z\right)\right],$$

for all $x, y, z \in I$.

Example 4.8. Let $f\left(x\right)=x^2$, therefore f is $\mathrm{H_tA_1}$-concave on $(-\infty,0)$. Applying Corollary 4.7, then we get

$$\left(\frac{xz}{x+z}\right)^2+\left(\frac{yz}{y+z}\right)^2+\left(\frac{xy}{x+y}\right)^2 \geq \frac{9}{4}\left[\frac{x^2+y^2+z^2}{9}+\left(\frac{xyz}{xy+yz+xz}\right)^2\right],$$

for all $x, y, z < 0$.

Corollary 4.9. *In Theorem* 4.1*,*

(i) *if* $f : I \to (0,\infty)$ *is an* $\mathrm{H_tA_h}$*-convex and superadditive, then*

$$\begin{aligned}
&2\left[f\left(\frac{xz}{x+z}\right)+f\left(\frac{yz}{y+z}\right)+f\left(\frac{xy}{x+y}\right)\right]\\
&\le f\left(\frac{2xz}{x+z}\right)+f\left(\frac{2yz}{y+z}\right)+f\left(\frac{2xy}{x+y}\right)\\
&\le h\left(3/2\right)f\left(\frac{3xyz}{xy+yz+xz}\right)+h\left(1/2\right)\left[f\left(x\right)+f\left(y\right)+f\left(z\right)\right]\\
&\le h\left(3/2\right)f\left(\frac{3xyz}{xy+yz+xz}\right)+h\left(1/2\right)f\left(x+y+z\right),
\end{aligned}$$

for all $x,y,z\in I$*. If* f *is an* $\mathrm{H_tA_h}$*-concave and subadditive, then the inequality is reversed;*

(ii) *if* $f : I \to (0,\infty)$ *is an* $\mathrm{H_tA_h}$*-convex and subadditive, then*

$$\begin{aligned}
&f\left(\frac{2xz}{x+z}+\frac{2yz}{y+z}+\frac{2xy}{x+y}\right)\\
&\le f\left(\frac{2xz}{x+z}\right)+f\left(\frac{2yz}{y+z}\right)+f\left(\frac{2xy}{x+y}\right)\\
&\le h\left(3/2\right)f\left(\frac{3xyz}{xy+yz+xz}\right)+h\left(1/2\right)\left[f\left(x\right)+f\left(y\right)+f\left(z\right)\right]\\
&\le 3h\left(3/2\right)f\left(\frac{xyz}{xy+yz+xz}\right)+h\left(1/2\right)\left[f\left(x\right)+f\left(y\right)+f\left(z\right)\right],
\end{aligned}$$

for all $x,y,z\in I$*. If* f *is an* $\mathrm{H_tA_h}$*-concave and superadditive, then the inequality is reversed.*

4.2 The case when *f* is $\mathrm{H_tG_h}$-convex

Theorem 4.10. *Let* $h : I \to (0,\infty)$ *be a nonnegative super(sub)additive. If* $f : I \to (0,\infty)$ *is* $\mathrm{H_tG_h}$*-convex (concave) function, then*

$$\begin{aligned}
&f\left(\frac{2xz}{x+z}\right)f\left(\frac{2yz}{y+z}\right)f\left(\frac{2xy}{x+y}\right)\\
&\le\ (\ge)\ \left[f\left(\frac{3xyz}{xy+yz+xz}\right)\right]^{h(3/2)}\left[f\left(x\right)f\left(y\right)f\left(z\right)\right]^{h(1/2)}, \qquad (31)
\end{aligned}$$

for all $x,y,z\in I$*.*

Proof. f is $\mathrm{H_tG_h}$-convex iff the inequality

$$f\left(\frac{\alpha\beta}{t\alpha+(1-t)\beta}\right)\le [f(\alpha)]^{h(1-t)}[f(\beta)]^{h(t)},\qquad 0\le t\le 1.$$

holds for all $\alpha,\beta\in I$. As in the proof of Theorem 4.1, if $x=y=z$, then the inequality holds. If $s+t=\frac{1}{2}$ since f is $\mathrm{H_tG_h}$-convex, we have

$$\begin{aligned}
f\left(\frac{2xz}{x+z}\right) &= f\left(\frac{\frac{3xyz}{xy+yz+xz}\cdot z}{s\frac{3xyz}{xy+yz+xz}+(1-s)z}\right)\\
&\le [f(z)]^{h(s)}\left[f\left(\frac{3xyz}{xy+yz+xz}\right)\right]^{h(1-s)},\\
f\left(\frac{2yz}{y+z}\right) &= f\left(\frac{\frac{3xyz}{xy+yz+xz}\cdot z}{t\frac{3xyz}{xy+yz+xz}+(1-t)z}\right)\\
&\le [f(z)]^{h(t)}\left[f\left(\frac{3xyz}{xy+yz+xz}\right)\right]^{h(1-t)},\\
f\left(\frac{2xy}{x+y}\right) &\le [f(x)f(y)]^{h(1/2)}.
\end{aligned}$$

Multiplying these inequalities we get

$$\begin{aligned}
&f\left(\frac{2xz}{x+z}\right)f\left(\frac{2yz}{y+z}\right)f\left(\frac{2xy}{x+y}\right)\\
&\le [f(z)]^{h(s)}\left[f\left(\frac{3xyz}{xy+yz+xz}\right)\right]^{h(1-s)}[f(z)]^{h(t)}\\
&\qquad\times\left[f\left(\frac{3xyz}{xy+yz+xz}\right)\right]^{h(1-t)}[f(x)f(y)]^{h(1/2)}\\
&\le [f(z)]^{h(s)+h(t)}\left[f\left(\frac{3xyz}{xy+yz+xz}\right)\right]^{h(1-s)+h(1-t)}[f(x)f(y)]^{h(1/2)}\\
&\le [f(z)]^{h(s+t)}\left[f\left(\frac{3xyz}{xy+yz+xz}\right)\right]^{h(2-s-t)}[f(x)f(y)]^{h(1/2)}\\
&= [f(z)]^{h(1/2)}\left[f\left(\frac{3xyz}{xy+yz+xz}\right)\right]^{h(3/2)}[f(x)f(y)]^{h(1/2)}\\
&= \left[f\left(\frac{3xyz}{xy+yz+xz}\right)\right]^{h(3/2)}[f(x)f(y)f(z)]^{h(1/2)},
\end{aligned}$$

which proves the inequality in (31). □

Remark 4.11. In (31), setting $z = y$ we get that

$$2f\left(\frac{2xy}{x+y}\right)f(y) \leq (\geq) \left[f\left(\frac{3xy}{2x+y}\right)\right]^{h(3/2)}\left[f(x)f^2(y)\right]^{h(1/2)},$$

for all $x, y \in I$.

Corollary 4.12. *If $f : I \to (0,\infty)$ is* $\mathrm{H_tG_t}$*-convex (concave) function, then*

$$\begin{aligned}&f\left(\frac{2xz}{x+z}\right)f\left(\frac{2yz}{y+z}\right)f\left(\frac{2xy}{x+y}\right)\\&\leq (\geq) \left[f\left(\frac{3xyz}{xy+yz+xz}\right)\right]^{3/2}\left[f(x)f(y)f(z)\right]^{1/2},\end{aligned}$$

for all $x, y, z \in I$. The equality holds with $f(x) = \mathrm{e}^{\frac{1}{x}}$, $x > 0$.

Example 4.13. Let $f(x) = \exp(x)$, $x > 0$. Then, f is $\mathrm{H_tG_t}$-convex on $(0,\infty)$. Applying Corollary 4.12 we get

$$\frac{4xz}{x+z} + \frac{4yz}{y+z} + \frac{4xy}{x+y} \leq \frac{9xyz}{xy+yz+xz} + xyz,$$

for all $x, y, z > 0$.

Corollary 4.14. *If $f : I \to (0,\infty)$ is* $\mathrm{H_tG_{1/t}}$*-concave, then*

$$\begin{aligned}&f\left(\frac{2xz}{x+z}\right)f\left(\frac{2yz}{y+z}\right)f\left(\frac{2xy}{x+y}\right)\\&\geq \left[f\left(\frac{3xyz}{xy+yz+xz}\right)\right]^{2/3}\left[f(x)f(y)f(z)\right]^{2},\end{aligned}$$

for all $x, y, z \in I$.

Example 4.15. Let $f(x) = \exp(-x)$, $x > 0$. Then, f is $\mathrm{H_tG_{1/t}}$-concave on $(0,\infty)$. Applying Corollary 4.14, we get

$$\frac{xz}{x+z} + \frac{yz}{y+z} + \frac{xy}{x+y} \leq \frac{xyz}{xy+yz+xz} + xyz,$$

for all $x, y, z > 0$.

Corollary 4.16. *If* $f : I \to (0,\infty)$ *is* $\mathrm{H_tG_1}$*-concave function, then*

$$f\left(\frac{2xz}{x+z}\right) f\left(\frac{2yz}{y+z}\right) f\left(\frac{2xy}{x+y}\right) \geq f\left(\frac{3xyz}{xy+yz+xz}\right) f(x) f(y) f(z),$$

for all $x,y,z \in I$.

Example 4.17. Let $f(x) = \exp(-x)$, $x > 0$. Then, f is $\mathrm{H_tG_1}$-concave on $(0,\infty)$. Applying Corollary 4.16 we get

$$\frac{2xz}{x+z} + \frac{2yz}{y+z} + \frac{2xy}{x+y} \leq \frac{3xyz}{xy+yz+xz} + x + y + z$$

for all $x,y,z > 0$.

Corollary 4.18. *In Theorem* 4.10*.*

(i) *If* $f : I \to (0,\infty)$ *is an* $\mathrm{H_tG_h}$*-convex and superadditive, then*

$$\begin{aligned} &2\left[f\left(\frac{xz}{x+z}\right) + f\left(\frac{yz}{y+z}\right) + f\left(\frac{xy}{x+y}\right)\right] \\ &\leq f\left(\frac{2xz}{x+z}\right) + f\left(\frac{2yz}{y+z}\right) + f\left(\frac{2xy}{x+y}\right) \\ &\leq \left[f\left(\frac{3xyz}{xy+yz+xz}\right)\right]^{h(3/2)} [f(x) f(y) f(z)]^{h(1/2)}, \end{aligned}$$

for all $x,y,z \in I$*. If* f *is an* h*-*$\mathrm{H_tG_t}$*-concave and subadditive, then the inequality is reversed.*

(ii) *If* $f : I \to (0,\infty)$ *is an* $\mathrm{H_tG_h}$*-convex and subadditive, then*

$$\begin{aligned} &f\left(\frac{2xz}{x+z} + \frac{2yz}{y+z} + \frac{2xy}{x+y}\right) \\ &\leq f\left(\frac{2xz}{x+z}\right) + f\left(\frac{2yz}{y+z}\right) + f\left(\frac{2xy}{x+y}\right) \\ &\leq \left[f\left(\frac{3xyz}{xy+yz+xz}\right)\right]^{h(3/2)} [f(x) f(y) f(z)]^{h(1/2)} \\ &\leq \left[3f\left(\frac{xyz}{xy+yz+xz}\right)\right]^{h(3/2)} [f(x) f(y) f(z)]^{h(1/2)}, \end{aligned}$$

for all $x,y,z \in I$*. If* f *is an* $\mathrm{H_tG_h}$*-concave and superadditive, then the inequality is reversed.*

4.3 The case when f is $\mathrm{H_tH_h}$-convex

Theorem 4.19. *Let $h : I \to (0,\infty)$ be a nonnegative super(sub)additive. If $f : I \to (0,\infty)$ is* $\mathrm{H_tH_h}$*-concave (convex) function, then*

$$f\left(\frac{2xz}{x+z}\right)+f\left(\frac{2yz}{y+z}\right)+f\left(\frac{2xy}{x+y}\right) \leq (\geq)\, h\left(\frac{1}{2}\right)\left[\frac{1}{f(x)}+\frac{1}{f(y)}+\frac{1}{f(z)}\right]+\frac{h(3/2)}{f\left(\frac{3xyz}{xy+yz+xz}\right)}, \tag{32}$$

for all $x,y,z \in I$.

Proof. f is $\mathrm{H_tH_h}$-convex iff the inequality

$$f\left(\frac{\alpha\beta}{t\alpha+(1-t)\beta}\right) \leq \frac{f(\alpha)f(\beta)}{h(t)f(\alpha)+h(1-t)f(\beta)}, \qquad 0 \leq t \leq 1$$

holds for all $\alpha,\beta \in I$. As in the proof of Theorem 4.1, if $x=y=z$, then the inequality holds. If $s+t=\frac{1}{2}$ since f is $\mathrm{H_tH_h}$-convex, we have

$$\begin{aligned}
f\left(\frac{2xz}{x+z}\right) &= f\left(\frac{\frac{3xyz}{xy+yz+xz}\cdot z}{s\frac{3xyz}{xy+yz+xz}+(1-s)z}\right) \\
&\geq \frac{f\left(\frac{3xyz}{xy+yz+xz}\right)\cdot f(z)}{h(s)f\left(\frac{3xyz}{xy+yz+xz}\right)+h(1-s)f(z)}, \\
f\left(\frac{2yz}{y+z}\right) &= f\left(\frac{\frac{3xyz}{xy+yz+xz}\cdot z}{t\frac{3xyz}{xy+yz+xz}+(1-t)z}\right) \\
&\geq \frac{f\left(\frac{3xyz}{xy+yz+xz}\right)\cdot f(z)}{h(t)f\left(\frac{3xyz}{xy+yz+xz}\right)+h(1-t)f(z)}, \\
f\left(\frac{2xy}{x+y}\right) &\geq \frac{f(x)f(y)}{h(1/2)\left[f(x)+f(y)\right]},
\end{aligned}$$

Therefore, by summing the reciprocal of the above inequalities we get

$$\begin{aligned}
&\frac{1}{f\left(\frac{2xz}{x+z}\right)}+\frac{1}{f\left(\frac{2yz}{y+z}\right)}+\frac{1}{f\left(\frac{2xy}{x+y}\right)}\\
&\le \frac{h(s)\,f\left(\frac{3xyz}{xy+yz+xz}\right)+h(1-s)\,f(z)+h(t)\,f\left(\frac{3xyz}{xy+yz+xz}\right)+h(1-t)\,f(z)}{f\left(\frac{3xyz}{xy+yz+xz}\right)\cdot f(z)}\\
&\quad+\frac{h(1/2)\left[f(x)+f(y)\right]}{f(x)\,f(y)}\\
&\le \frac{\left[h(s)+h(s)\right]f\left(\frac{3xyz}{xy+yz+xz}\right)+\left[h(1-s)+h(1-t)\right]f(z)}{f\left(\frac{3xyz}{xy+yz+xz}\right)\cdot f(z)}\\
&\quad+\frac{h(1/2)\left[f(x)+f(y)\right]}{f(x)\,f(y)}\\
&\le \frac{h(s+t)\,f\left(\frac{3xyz}{xy+yz+xz}\right)+h(2-s-t)\,f(z)}{f\left(\frac{3xyz}{xy+yz+xz}\right)\cdot f(z)}+\frac{h(1/2)\left[f(x)+f(y)\right]}{f(x)\,f(y)}\\
&=\frac{h(1/2)\,f\left(\frac{3xyz}{xy+yz+xz}\right)+h(3/2)\,f(z)}{f\left(\frac{3xyz}{xy+yz+xz}\right)\cdot f(z)}+\frac{h(1/2)\left[f(x)+f(y)\right]}{f(x)\,f(y)}\\
&=\frac{h(1/2)\,f\left(\frac{3xyz}{xy+yz+xz}\right)+h(3/2)\,f(z)}{f\left(\frac{3xyz}{xy+yz+xz}\right)\cdot f(z)}+\frac{h(1/2)\left[f(x)+f(y)\right]}{f(x)\,f(y)}\\
&=h\left(\frac{1}{2}\right)\left[\frac{1}{f(x)}+\frac{1}{f(y)}+\frac{1}{f(z)}\right]+\frac{h(3/2)}{f\left(\frac{3xyz}{xy+yz+xz}\right)},
\end{aligned}$$

which proves the inequality in (32). □

Remark 4.20. In (32), setting $z=y$ then we get

$$2f\left(\frac{2xy}{x+y}\right)+f(y)\le\ (\ge)\,h\left(\frac{1}{2}\right)\left[\frac{1}{f(x)}+\frac{2}{f(y)}\right]+\frac{h(3/2)}{f\left(\frac{3xy}{2x+y}\right)},$$

for all $x,y,z\in I$.

Corollary 4.21. *If $f : I \to (0,\infty)$ is* $\mathrm{H_tH_t}$*-concave (convex) function, then*

$$\frac{2}{3}\left[\frac{1}{f\left(\frac{2xz}{x+z}\right)}+\frac{1}{f\left(\frac{2yz}{y+z}\right)}+\frac{1}{f\left(\frac{2xy}{x+y}\right)}\right]$$
$$\leq\ (\geq)\ \frac{1}{3}\left[\frac{1}{f\left(x\right)}+\frac{1}{f\left(y\right)}+\frac{1}{f\left(z\right)}\right]+\frac{1}{f\left(\frac{3xyz}{xy+yz+xz}\right)},$$

for all $x,y,z \in I$. The equality holds with $f\left(x\right)=x$, $x>1$.

Example 4.22. Let $f\left(x\right)=\arctan\left(x\right)$, $x>0$. Then f is $\mathrm{H_tH_t}$-concave on $(0,\infty)$. Applying Corollary 4.21, then we get

$$\frac{2}{3}\left[\frac{1}{\arctan\left(\frac{2xz}{x+z}\right)}+\frac{1}{\arctan\left(\frac{2yz}{y+z}\right)}+\frac{1}{\arctan\left(\frac{2xy}{x+y}\right)}\right]$$
$$\leq\frac{1}{3}\left[\frac{1}{\arctan\left(x\right)}+\frac{1}{\arctan\left(y\right)}+\frac{1}{\arctan\left(z\right)}\right]+\frac{1}{\arctan\left(\frac{3xyz}{xy+yz+xz}\right)},$$

for all $x,y,z>0$.

Corollary 4.23. *If $f : I \to (0,\infty)$ is* $\mathrm{H_tH_{1/t}}$*-convex function, then*

$$\frac{3}{2}\left[\frac{1}{f\left(\frac{2xz}{x+z}\right)}+\frac{1}{f\left(\frac{2yz}{y+z}\right)}+\frac{1}{f\left(\frac{2xy}{x+y}\right)}\right]$$
$$\geq 3\left[\frac{1}{f\left(x\right)}+\frac{1}{f\left(y\right)}+\frac{1}{f\left(z\right)}\right]+\frac{1}{f\left(\frac{3xyz}{xy+yz+xz}\right)},$$

for all $x,y,z \in I$.

Example 4.24. Let $f\left(x\right)=-\log\left(x\right)$, $x>1$. Then f is $\mathrm{H_tH_{1/t}}$-convex on $(0,\infty)$. Applying Corollary 4.23, we get

$$\frac{3}{2}\left[\frac{1}{\log\left(\frac{2xz}{x+z}\right)}+\frac{1}{\log\left(\frac{2yz}{y+z}\right)}+\frac{1}{\log\left(\frac{2xy}{x+y}\right)}\right]$$
$$\leq 3\left[\frac{1}{\log\left(x\right)}+\frac{1}{\log\left(y\right)}+\frac{1}{\log\left(z\right)}\right]+\frac{1}{\log\left(\frac{3xyz}{xy+yz+xz}\right)},$$

for all $x,y,z>0$.

Corollary 4.25. *If* $f : I \to (0, \infty)$ *is* $\mathrm{H_tH_1}$*-convex function, then*

$$\frac{1}{f\left(\frac{2xz}{x+z}\right)} + \frac{1}{f\left(\frac{2yz}{y+z}\right)} + \frac{1}{f\left(\frac{2xy}{x+y}\right)}$$
$$\geq \left[\frac{1}{f(x)} + \frac{1}{f(y)} + \frac{1}{f(z)}\right] + \frac{1}{f\left(\frac{3xyz}{xy+yz+xz}\right)},$$

for all $x, y, z \in I$.

Example 4.26. Let $f(x) = -\log(x)$, $x > 0$. Then f is $\mathrm{H_tH_1}$-convex on $(0, \infty)$. Applying Corollary 4.25, then we get

$$\frac{1}{\log\left(\frac{2xz}{x+z}\right)} + \frac{1}{\log\left(\frac{2yz}{y+z}\right)} + \frac{1}{\log\left(\frac{2xy}{x+y}\right)}$$
$$\leq \left[\frac{1}{\log(x)} + \frac{1}{\log(y)} + \frac{1}{\log(z)}\right] + \frac{1}{\log\left(\frac{3xyz}{xy+yz+xz}\right)},$$

for all $x, y, z > 0$.

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