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# Applications of differential subordination for certain subclasses of meromorphically univalent functions defined by rapid operator

Bolineni Venkateswarlu, P. Thirupathi Reddy, Settipalli Sridevi and Galla Swapna

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**Abstract.** In this work, we investigate some applications of differential subordination for the class of meromorphic univalent functions defined by rapid operator and obtained coefficient bounds, integral representations, weighted and arithmetic mean for the class $\Sigma(A, B, \mu, \theta)$.

**Keywords.** Meromorphic, coefficient bound, arithmetic mean, subordination.

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## 1 Introduction

Let $\Sigma$ denote the class of functions $f$ of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \quad a_n \geq 0 \tag{1}$$

which are analytic in the punctured open disk

$$U^* := \{z : z \in \mathbb{C},\ 0 < |z| < 1\} = U \setminus \{0\}. \tag{2}$$

Let $\Sigma_s, \Sigma^*(\gamma)$ and $\Sigma_k(\gamma)$ $(0 \leq \gamma < 1)$ denote the subclasses of $\Sigma$ that are meromorphic univalent, meromorphically starlike functions of order $\gamma$ and meromorphically convex functions of order $\gamma$ respectively. Analytically, $f \in \Sigma^*(\gamma)$ if and only if $f$ is of the form (1) and satisfies

$$-\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \gamma,\ z \in U. \tag{3}$$

Similarly, $f \in \Sigma_k(\gamma)$ if and only if $f$ is of the form (1) and satisfies

$$-\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \gamma,\ z \in U \tag{4}$$

and similar other classes of meromorphically univalent functions have been extensively studied by Altintaş et al. [1], Aouf [2], Mogra et al. [5], Urlegaddi and Ganigi [9] and others (see [4, 6, 10]).

Given two functions $f$ and $g$, which are analytic in $U$, the function $f$ is said to be subordinate to $g$, written as

$$f \prec g \text{ and } f(z) \prec g(z), \quad z \in U,$$

if there exists a Schwarz function $w$ analytic in $U$, with

$$w(0) = 0 \text{ and } |w(z)| < 1, \quad z \in U$$

and such that

$$f(z) = g(w(z)), \quad z \in U.$$

If $g$ is univalent in $U$, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$ ([7, p. 36]).

In [3], Athsan and Kulkarni introduced rapid operator for analytic functions and Rosy and Sunil Varma [8] modified their operator to meromorphic functions as follows.

**Lemma 1.1.** *For $f \in \Sigma$ given by* (1), $0 \le \mu \le 1$ *and* $0 \le \theta \le 1$, *if the operator $\mathscr{S}_\mu^\theta : \Sigma \to \Sigma$ is defined by*

$$\mathscr{S}_\mu^\theta f(z) = \frac{1}{[(1-\mu)^\theta \Gamma(\theta+1)]} \int_0^\infty t^{\theta+1} e^{\frac{-t}{1-\mu}} f(tz)dt \tag{5}$$

*then*

$$\mathscr{S}_\mu^\theta f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} L(n,\mu,\theta) a_n z^n, \tag{6}$$

*where*

$$L(n,\mu,\theta) = (1-\mu)^{n+1} \frac{\Gamma(n+\theta+2)}{\Gamma(\theta+1)} \tag{7}$$

*and $\Gamma$ is the familiar Gamma function.*

**Definition 1.2.** Let $A$ and $B$, $(-1 \le B < A \le 1)$ be defined parameters. We say that a function $f \in \Sigma$ is in the class $\Sigma(A, B, \mu, \theta)$ if it satisfies the following subordination condition by (6)

$$-z^2(\mathscr{S}_\mu^\theta f(z))' \prec \frac{1+Az}{1+Bz}, \quad z \in U. \tag{8}$$

By the definition of the differential subordination (8) is equivalent to the following condition

$$\left| \frac{1 + z^2(\mathscr{S}_\mu^\theta f(z))'}{A + Bz^2(\mathscr{S}_\mu^\theta f(z))'} \right| < 1, \ z \in U. \tag{9}$$

In particular, we can write $\Sigma(1-2\beta, -1, ) = \Sigma(\beta)$, where $\Sigma(\beta)$ denotes class of the functions in $\Sigma$ satisfying the following condition:

$$\Re \left( -z^2(\mathscr{S}_\mu^\theta f(z))' \right) > \beta, \ \ 0 \le \beta < 1, \ z \in U. \tag{10}$$

The aim of this study is to determine some usual properties of the geometric function theory such as coefficient bounds, integral representation, weighted mean and arithmetic mean for the class $\Sigma(A, B, \mu, \theta)$.

## 2 Coefficient bounds

**Theorem 2.1.** *Let the function $f$ of the form* (1) *be in* $\Sigma$. *Then the function $f$ belongs to the class $\Sigma(A, B, \mu, \theta)$ if and only if*

$$(1-B)\sum_{n=1}^{\infty} nL(n,\mu,\theta)a_n \le (A-B), \tag{11}$$

$$\textit{where} \quad -1 \le B < A \le 1, \ \ 0 \le \mu \le 1 \ \textit{and} \ 0 \le \theta \le 1.$$

*The result is sharp for the function $f$ is given by*

$$f(z) = \frac{1}{z} + \frac{A-B}{(1-B)nL(n,\mu,\theta)} z^n. \tag{12}$$

*Proof.* Assume that the condition (11) is true.
We must show that $f \in \Sigma(A, B, \mu, \theta)$ or equivalently prove that

$$\left| \frac{1 + z^2(\mathscr{S}_\mu^\theta f(z))'}{A + Bz^2(\mathscr{S}_\mu^\theta f(z))'} \right| < 1.$$

Since

$$\begin{aligned}
\left|\frac{1+z^2(\mathscr{S}_\mu^\theta f(z))'}{A+Bz^2(\mathscr{S}_\mu^\theta f(z))'}\right| &= \left|\frac{1+(-1+\sum\limits_{n=1}^{\infty} nL(n,\mu,\theta)a_n z^{n+1})}{A+B(-1+\sum\limits_{n=1}^{\infty} nL(n,\mu,\theta)a_n z^{n+1})}\right| \\
&= \left|\frac{\sum\limits_{n=1}^{\infty} nL(n,\mu,\theta)a_n z^{n+1}}{A-B+B\sum\limits_{n=1}^{\infty} nL(n,\mu,\theta)a_n z^{n+1}}\right| \\
&\leq \left|\frac{\sum\limits_{n=1}^{\infty} nL(n,\mu,\theta)}{A-B+B\sum\limits_{n=1}^{\infty} nL(n,\mu,\theta)a_n}\right| \\
&< 1,
\end{aligned}$$

last inequality is true by (11).

Conversely, suppose that $f \in \Sigma(A, B, \mu, \theta)$.
We must to show that the condition (11) holds. We have

$$\begin{aligned}
&\left|\frac{1+z^2(\mathscr{S}_\mu^\theta f(z))'}{A+Bz^2(\mathscr{S}_\mu^\theta f(z))'}\right| < 1 \\
\Rightarrow &\left|\frac{\sum\limits_{n=1}^{\infty} nL(n,\mu,\theta)a_n z^{n+1}}{A-B+B\sum\limits_{n=1}^{\infty} nL(n,\mu,\theta)a_n z^{n+1}}\right| < 1
\end{aligned}$$

and since $\Re(z) < |z|$, we have

$$\Re\left\{\frac{\sum\limits_{n=1}^{\infty} nL(n,\mu,\theta)a_n z^{n+1}}{A-B+B\sum\limits_{n=1}^{\infty} nL(n,\mu,\theta)a_n z^{n+1}}\right\} < 1. \tag{13}$$

We choose the values of $z$ on the real axis and letting $z \to 1^-$, then we obtain

$$\left\{ \frac{\sum\limits_{n=1}^{\infty} nL(n,\mu,\theta)a_n}{A - B + B\sum\limits_{n=1}^{\infty} nL(n,\mu,\theta)a_n} \right\} < 1$$

$$\Rightarrow (1-B)\sum_{n=1}^{\infty} nL(n,\mu,\theta)a_n < (A-B).$$

The result is sharp for the function $f$ is given by (12). □

**Corollary 2.2.** *Let* $f \in \Sigma(A,B,\mu,\theta)$. *Then*

$$a_n \le \frac{(A-B)}{n(1-B)L(n,\mu,\theta)},\ n \ge 1.$$

## 3 Integral representation

In the next theorem, we obtain an integral representation for $\mathscr{S}_\mu^\theta f(z)$.

**Theorem 3.1.** *Let* $f \in \Sigma(A,B,\mu,\theta)$. *Then*

$$\mathscr{S}_\mu^\theta f(z) = \int\limits_0^z \frac{(A\phi(t)-1)}{t^2(1-B\phi(t))}dt,\ where\ |\phi(z)| < 1,\ z \in U. \tag{14}$$

*Proof.* Let $f \in \Sigma(A,B,\mu,\theta)$. Letting $-z^2(\mathscr{S}_\mu^\theta f(z))' = q(z)$.
We have

$$q(z) \prec \frac{1+Az}{1+Bz}$$

or we can write

$$\left|\frac{q(z)-1}{Bq(z)-A}\right| < 1,$$

so that consequently, we have

$$\frac{q(z)-1}{Bq(z)-A} = \phi(z),\ \ |\phi(z)| < 1,\ z \in U.$$

We can write

$$-z^2(\mathscr{S}_\mu^\theta f(z))' = \frac{1-A\phi(z)}{1-B\phi(z)},$$

$$-(\mathscr{S}_{\mu}^{\theta} f(z))' = \frac{1}{z^2}\frac{1-A\phi(z)}{1-B\phi(z)}$$

$$\mathscr{S}_{\mu}^{\theta} f(z) = \int_0^z \frac{(A\phi(t)-1)}{t^2(1-B\phi(t))}dt.$$

Hence, the proof of theorem is completed. □

## 4 Linear combination

In this section, we prove a linear combination for the class $\Sigma(A, B, \mu, \theta)$.

**Theorem 4.1.** *Let*

$$f_i(z) = \sum_{i=1}^{k} c_i f_i(z) \in \Sigma(A, B, \mu, \theta) \tag{15}$$

$$\textit{where } \sum_{i=1}^{k} c_i = 1.$$

*Proof.* By Theorem 2.1, we can write for every $i \in \{1, 2, \cdots, k\}$,

$$\sum_{n=1}^{\infty}\frac{n(1-B)L(n,\mu,\theta)}{A-B}a_{n,i} < 1.$$

Therefore,

$$\begin{aligned} F(z) &= \sum_{i=1}^{k} c_i \left(z^{-1} + \sum_{n=1}^{\infty} a_{n,i}z^n\right) \\ &= z^{-1} + \sum_{i=1}^{k}\sum_{n=1}^{\infty} c_i a_{n,i} z^n \\ &= z^{-1} + \sum_{n=1}^{\infty}\left(\sum_{i=1}^{k} c_i a_{n,i}\right) z^n. \end{aligned}$$

However,

$$\sum_{n=1}^{\infty}\frac{n(1-B)L(n,\mu,\theta)}{A-B}\left(\sum_{i=1}^{k} c_i a_{n,i}\right)$$

$$= \sum_{i=1}^{k}\left[\sum_{n=1}^{\infty}\frac{n(1-B)L(n,\mu,\theta)}{A-B}a_{n,i}\right] c_i \leq 1.$$

Then $F(z) \in \Sigma(A, B, \mu, \theta)$. The proof of theorem completed. $\square$

## 5 Weighted mean

**Definition 5.1.** Let $f$ and $g$ belongs to $\Sigma$. Then the weighted mean $h_j(z)$ of $f$ and $g$ is given as

$$h_j(z) = \frac{1}{2}\left[(1-j)f(z) + (1+j)g(z)\right]. \tag{16}$$

In the following theorem we will show the weighted mean for the class $\Sigma(A, B, \mu, \theta)$.

**Theorem 5.2.** *If $f$ and $g$ are in the class $\Sigma(A, B, \mu, \theta)$ then the weighted mean of $f$ and $g$ are also in $\Sigma(A, B, \mu, \theta)$.*

*Proof.* By definition of $h_j(z)$, we get

$$\begin{aligned} h_j(z) &= \frac{1}{2}\left[(1-j)\left(z^{-1} + \sum_{n=1}^{\infty} a_n z^n\right) + (1+j)\left(z^{-1} + \sum_{n=1}^{\infty} b_n z^n\right)\right] \\ &= z^{-1} + \frac{1}{2}\sum_{n=1}^{\infty}\left[(1-j)a_n + (1+j)b_n\right] z^n. \end{aligned}$$

Since $f, g \in \Sigma(A, B, \mu, \theta)$, by Theorem 2.1, we must prove that

$$\begin{aligned} &\sum_{n=1}^{\infty} n(1-B)L(n,\mu,\theta)\left[\frac{1}{2}(1-j)a_n + \frac{1}{2}(1+j)b_n\right] \\ &= \frac{1}{2}(1-j)(1-B)\sum_{n=1}^{\infty} nL(n,\mu,\theta)a_n + \frac{1}{2}(1+j)(1-B)\sum_{n=1}^{\infty} nL(n,\mu,\theta)b_n \\ &\le \frac{1}{2}(1-j)(A-B) + \frac{1}{2}(1+j)(A-B) \\ &\le (A-B). \end{aligned}$$

Hence the proof of theorem is completed. $\square$

## 6 Arithmetic mean

**Definition 6.1.** Let $f_1(z), f_2(z), \cdots, f_k(z) \in \Sigma(A, B, \mu, \theta)$. Then the arithmetic mean $h(z)$ of $f_i(z)$ is given by $h(z) = \frac{1}{k}\sum\limits_{i=1}^{k} f_i(z)$.

Next, we will prove the arithmetic mean for the class $\Sigma(A, B, \mu, \theta)$.

**Theorem 6.2.** *If $f_1(z), f_2(z), \cdots, f_k(z)$ are in the class $\Sigma(A, B, \mu, \theta)$ then the arithmetic mean $h(z)$ of $f_i(z)$ is given by*

$$h(z) = \frac{1}{k} \sum_{i=1}^{k} f_i(z) \tag{17}$$

*is also in the class $\Sigma(A, B, \mu, \theta)$.*

*Proof.* We have for $h(z)$ by definition

$$\begin{aligned} h(z) &= \frac{1}{k} \sum_{i=1}^{k} \left[ z^{-1} + \sum_{n=1}^{\infty} a_{n,i} z^n \right] \\ &= z^{-1} + \sum_{n=1}^{\infty} \left[ \frac{1}{k} \sum_{i=1}^{k} a_{n,i} \right] z^n. \end{aligned}$$

Since $f_i(z) \in \Sigma(A, B, \mu, \theta)$, for every $i \in \{1, 2, \cdots, k\}$, by using Theorem 2.1, we prove that

$$\begin{aligned} (1-B) \sum_{n=1}^{\infty} nL(n, \mu, \theta) \left( \frac{1}{k} \sum_{i=1}^{k} a_{n,i} \right) &= \frac{1}{k} \sum_{i=1}^{k} \left( \sum_{n=1}^{\infty} n(1-B)L(n, \mu, \theta) a_{n,i} \right) \\ &\leq \frac{1}{k} \sum_{i=1}^{k} (A - B). \end{aligned}$$

The proof of theorem completed. $\square$

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**Author information**

Bolineni Venkateswarlu, Department of Mathematics, GSS, GITAM University, Bengaluru Rural, India.
E-mail: bvlmaths@gmail.com

P. Thirupathi Reddy, Department of Mathematics, Kakatiya University, Warangal-506 009, Telangana, India.
E-mail: reddypt2@gmail.com

Settipalli Sridevi, Department of Mathematics, GSS, GITAM University, Bengaluru Rural, India.
E-mail: siri_settipalli@yahoo.co.in

Galla Swapna, Department of Mathematics, GSS, GITAM University, Bengaluru Rural, India.
E-mail: swapna.priya38@gmail.com