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A new finite difference algorithm for boundary value problems involving transmission conditions

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Abstract. The finite difference method (FDM) is used to find an approximate solution to ordinary and partial differential equations of various type using finite difference equations to approximate derivatives. The idea is to replace ordinary or partial derivatives appearing in the boundary-value problem by finite differences that approximate them. There is an extensive literature on this topic. But, as a rule, ordinary differential equations or partial differential equations were studied without an internal singular point and without corresponding transmission conditions .It is our main goal here to develop finite difference method to deal with an boundary value problem involving additional transmission conditions at the interior singular point.
In this study, we have proposed a new modification of classical FDM for the solution of boundary value problems which are defined on two disjoint intervals and involved additional transmission conditions at an common end of these intervals. The proposed modification of FDM differs from the classical FDM in calculating the iterative terms of numerical solutions. To demonstrate the efficiency and reliability of the proposed modification of FDM an illustrative example is solved b y this method. The obtained results are compared with those obtained by the standard FDM and by the analytical method. Corresponding graphical illustrations are also presented.

Keywords. Finite difference method, transmission conditions, boundary value problem.

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1 Introduction

A lot of mechanical and physical processes are modeled by linear or nonlinear differential equations, whose exact solutions are impossible to find by using analytical methods. Many researchers have tried to do this in various semi-analytical, numerical and approximate methods, such as the finite element method, the Adomian decomposition method, the differential transform method, the explicit Euler method, the Taylor's expansion method, etc. One of them is the finite difference method (FDM), which can be applied to wide class of problems appearing in mathematical physics and engineering. Many important theoretical and numerical

results have been obtained during the last seven decades regarding the stability, accuracy and convergence of the FDM for different type initial and/or boundary value problems (see, [1, 2, 5, 6, 14] and references cited therein).
The standard FDM is intended for solving one-interval initial and/or boundary value problems without transmission conditions (see, [7, 8, 15, 16]).
Based on FDM, we have developed a new technique for solving two-interval boundary value problems (BVP), that included additional transmission conditions across the common endpoint of these intervals. We note that some important theoretical aspects of BVP with transmission conditions were studied in [3, 4, 9–13] and corresponding references cited therein.

2 Analysis of the method

Let us consider a linear boundary-value problem for second order ordinary differential equation defined on two disjoint intervals, given by

$$u^{''} + p(x)u^{'} + q(x)u = f(x), \quad x \in [a, c) \cup (c, b] \tag{1}$$

together with the boundary conditions (BC's), given by

$$u(a) = \alpha, \quad u(b) = \beta \tag{2}$$

where $p(x)$, $q(x)$ and $f(x)$ are continuous functions on $[a, c) \cup (c, b]$ having finite limit values $p(c\pm0)$, $q(c\pm0)$ and $f(c\pm0)$, respectively, and α, β are real numbers. To discretize the problem (1)-(2), the definition range $[a, b]$ is divided into N equal ranges $[x_0, x_1]$, $[x_1, x_2]$, ..., $[x_{N-1}, x_N]$, that is

$$x_k = a + kh, \quad h = \frac{b-a}{N} \quad k = 0, 1, 2, ..., N.$$

By using the Taylor expansion

$$u(x_{ik} + h) \approx u(x_k) + hu^{'}(x_k) + \frac{h^2}{2!}u^{''}(x_k) + \cdots,$$

we can express the first derivative in the ordinary differential equation using one of the following approximate expressions, so-called finite differences

$$D_+u(x) \approx \frac{u(x+h) - u(x)}{h},$$

$$D_-u(x) \approx \frac{u(x) - u(x-h)}{h},$$

$$D_0 u(x) \approx \frac{1}{2}\left(D_- u(x) + D_+ u(x)\right),$$

where $D_+u(x)$, $D_-u(x)$ and $D_0u(x)$ denotes the forward finite difference, backward finite difference and centered finite difference of the unknown solution $u(x)$, respectively.

The first and second derivative expressions in the boundary value problem can be expressed in the same way, as

$$u'(x) \approx \frac{D_+u(x) + D_-u(x)}{2} = \frac{u(x+h) - u(x-h)}{2h} \tag{3}$$

and

$$u''(x) \approx \frac{D_+u(x) - D_-u(x)}{h} = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}. \tag{4}$$

Let us define the finite difference solution for $u(x)$ at all grid points $x_0, x_1, \cdots, x_N$ by $u_k = u(x_k)$. Substituting (3) and (4) in the boundary value problem (1)-(2), we have the following linear system of algebraic equations

$$\left(1 - \tfrac{1}{2}hp_k\right)u_{k-1} + \left(-2 + h^2 q_k\right)u_k + \left(1 + \tfrac{1}{2}hp_k\right)u_{k+1} = h^2 f\left(x_k\right)$$

$$1 \leq k \leq N-1, \qquad k = 1, 2, 3, ..., N-1,$$

where

$$u_0 = \alpha, \quad u_N = \beta.$$

Note that, each equation of this system involves solution values at three nodal points x_{k-1}, x_k and x_{k+1}. The linear system of algebraic equations can be written in the matrix and vector form

$$Mu = B \tag{5}$$

where M is a tridiagonal matrix of size $(N-1) \times (N-1)$, given by

$$M = \begin{pmatrix} -2 + h^2 q_1 & 1 + \frac{1}{2}hp_1 & 0 & \cdots & 0 & 0 \\ 1 - \frac{1}{2}hp_2 & -2 + h^2 q_2 & 1 + \frac{1}{2}hp_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \cdots & \cdots \\ 0 & 0 & 1 & \cdots & 1 + \frac{1}{2}hp_{N-2} & 0 \\ 0 & 0 & 0 & \cdot & 1 - \frac{1}{2}hp_{N-1} & -2 + h^2 q_{N-1} \end{pmatrix},$$

$$y=\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-2} \\ u_{N-1} \end{pmatrix} \quad and \quad B=\begin{pmatrix} h^2 f(x_1)-\left(1-\frac{1}{2}hp_1\right)\alpha \\ h^2 f(x_2) \\ \vdots \\ h^2 f(x_{N-2}) \\ h^2 f(x_{N-1})-\left(1-\frac{1}{2}hp_1\right)\beta \end{pmatrix}.$$

This is the tridiagonal linear system of algebraic equations (5) and therefore can be solved by using MATLAB/Octave.

2.1 Modification of the FDM for solving transmission problems

Now, consider the BVP (1)-(2) together with additional transmission conditions at the interior point of singularity $x=c$, given by

$$u(c^-)=mu(c^+), u'(c^-)=nu'(c^+) \tag{6}$$

where m,n are real constants. Let the definition range $[a,b]$ is divided into N equal ranges by the grid points $x_k=a+kh,\ k=0,1,...,N$ and let the singular point $x=c$ lies between x_t and x_{t+1} that is $x\in[x_t,x_{t+1}]$.
If we apply the transmission conditions (6), then we have two additional algebraic equations.
Since u_t is closest to $x=c$ and lies to the left of $x=c$, we have identified u_t with $u(c^-)$ and similarly y_{t+1} is closest to $x=c$ and lies to the right of $x=c$, we have identified u_{t+1} with $u(c^+)$.
Therefore the transmission condition

$$u(c^-)=mu(c^+)$$

is transformed to the finite difference equation

$$u_t-mu_{t+1}=0 \tag{7}$$

and the transmission condition

$$u'(c^-)=nu'(c^+).$$

is transformed to the finite difference equation

$$u_{t-2} - u_t - nu_{t+1} + nu_{t+3} = 0. \tag{8}$$

Note that the equation of this system involves solution values at four nodal points x_{t-2} , x_t x_{t+1} and x_{t+3}.
By adding equations (7) and (8) to the system of equations (2), a linear equation system is obtained, in the matrix and vector form

$$\tilde{M}u = \tilde{B}.$$

This system of equations is not tridiagonal.The solution of this linear system of algebraic equations can be found by using MATLAB-Octave or Mathematica.

3 Convergence and error estimates of FDM

When the FDM is used to find a numerical solution to differential equations, it is important to know how accurately the numerical solution approximates the exact solution.

Definition 3.1 (Global Error). Let $\tilde{U} = (u_1, u_2, \ldots, u_n)$ denote the finite difference solution and $\tilde{u} = (u(x_1), u(x_2), \ldots, u(x_n))$ is the exact solution at the grid points $x_1, x_2, \ldots, x_n$. Then the vector

$$\tilde{E} = (u_1 - u(x_1), u_2 - u(x_2), \ldots, u_n - u(x_n)) = \tilde{U} - \tilde{u}$$

is said to be the global error vector.

You usually want to find an admissible upper bound for this error with respect to the maximum norm , defined by

$$\| \tilde{E} \| = \max_{1 \leqslant k \leqslant n} | u_k - u(x_k) |$$

or p-norm ($p \geq 1$), defined by

$$\| \tilde{E} \|_p = \left(\sum_{k=1}^{n} | u_{x_k} - u(k) |^p (x_{k+1} - x_k) \right)^{1/2}.$$

Definition 3.2. Denote

$$h_i := \max_{1 \leqslant k \leqslant n} (x_{k+1} - x_k).$$

If $\| \tilde{E} \|_p$ converges to zero as h approaches 0 $(h \to 0)$, then a finite difference method is called convergent with respect to the p-norm $(p \geq)$. Moreover, if there is $c \geq 0$ such that

$$\| \tilde{E} \|_p \leq Ch^q, \; q > 0,$$

then FDM is called q-th order accurate.

Definition 3.3. A FDM is called convergent with respect to the maximum norm if

$$\lim_{h \to 0} \| \tilde{E} \|_\infty = 0.$$

4 Local truncation errors

We shall show that the FDM solution converges to the exact solution of the BVP (1)-(2) when h converges to zero. Using formulas (3) and (4), one can show that the exact solution $\tilde{u} = (u(x_1), u(x_2), \ldots, u(x_n))$ satisfies the following linear system of equation

$$\frac{u(x_{k+1}) - 2u(x_k) + u(x_{k-1})}{h^2} - \frac{h^2}{12} u^{(4)}(\xi_k) + p_k \frac{u(x_{k+1}) - u(x_{k-1})}{2h}$$
$$- \frac{h^2}{6} u^{(3)}(\eta_k) + q_k u(x_k) = f(x_k), \quad 1 \leqslant k \leqslant n$$

for same $\xi_k \in [a, b]$.
On the other hand, the FDM solution $\tilde{U} = (u_1, u_2, \ldots, u_n)$ satisfies the linear system of equation

$$\frac{u_{k+1} - 2u_k + u_{k-1}}{h^2} + p_k \frac{u_{k+1} - u_{k-1}}{2h} + q_k u_k = f_k, \quad 1 \leqslant k \leqslant n.$$

Subtracting these equation one from the other, we get

$$\frac{E_{k+1} - 2E_k + E_{k-1}}{h^2} + p_k \frac{E_{k+1} - E_{k-1}}{2h} + q_k E_k = h^2 f_k, \quad 1 \leqslant k \leqslant n, \quad (9)$$

where E_k is the global error $E_k := u(x_k) - u_k$ and $h^2 f_k$ is the local truncation error at the grid point $x = x_k$ and

$$f_k = \frac{1}{12} u^{(4)}(\xi_k) - \frac{1}{6} u^{(3)}(\eta_k).$$

After multiplying both sides of (9) by h^2 and then collecting the corresponding terms, we have

$$\left(1-\frac{h}{2}p_k\right)E_{k-1}+\left(-2+h^2q_k\right)E_k+\left(1+\frac{h}{2}p_k\right)E_{k+1}=h^4f_k. \qquad (10)$$

We will apply the infinite norm (i.e, maximum norm) $\| \tilde{E} \|_\infty$, because it is used to measure grid functions and is easily estimated.
The equation (10) can be written as

$$(2+h^2q_k)E_k=(1-\frac{h}{2}p_k)E_{k+1}-(1+\frac{h}{2}p_k)E_k+h^4f_k.$$

Consequently

$$\begin{aligned} \mid 2+h^2q_k \mid\mid E_k \mid &\le\mid 1-\frac{h}{2}p_k \mid\mid E_{k+1} \mid + \mid 1+\frac{h}{2}p_k \mid\mid E_k \mid +h^4 \mid f_k \mid \\ &\le\mid 1-\frac{h}{2}p_k \mid\| \tilde{E} \|_\infty + \mid 1+\frac{h}{2}p_k \mid\| \tilde{E} \|_\infty +h^4 \| \tilde{f} \|_\infty, \end{aligned}$$

where $\| \tilde{f} \|_\infty = \max\limits_{1\leqslant k\leqslant n} \mid f_k \mid$.

From this inequality it follows immediately that

$$\mid 2+h^2q_k \mid\| \tilde{E} \|_\infty \le \left(\mid 1-\frac{h}{2}p_k \mid + \mid 1+\frac{h}{2}p_k \mid\right)\| \tilde{E} \|_\infty +h^4 \| \tilde{f} \|_\infty . \quad (11)$$

Obviously, one can choose $h>0$ small enough to satisfy

$$\mid 1-\frac{h}{2}p_k \mid + \mid 1+\frac{h}{2}p_k \mid = 2$$

and

$$\mid 2+h^2q_k \mid = 2+h^2 \mid q_k \mid$$

for all $k=1,2,\ldots,n$.
Consequently, for sufficiently small $h>0$ we have from (11) that

$$\mid q_k \mid\| \tilde{E} \|_\infty \le h^2 \| \tilde{f} \|_\infty .$$

Denoting

$$C=\frac{\| \tilde{f} \|_\infty}{\min\limits_{1\leqslant k\leqslant n} \mid q_k \mid},$$

we obtain

$$\| \tilde{E} \|_\infty \le Ch^2.$$

Hence, the FDM is convergent and 2-order accurate.

5 Numerical examples

Example 5.1. Consider the following two-interval BVP consisting of the differential equation

$$u^{''}-2\left(2x+1\right)u^{'}+\left((2x+1)^2-2\right)u=0, \quad x\in[-1,0)\cup(0,1]\,, \tag{12}$$

subject to the boundary conditions at the endpoints $x=-1$ and $x=1$, given by

$$u\left(-1\right)=0, \quad u\left(1\right)=3, \tag{13}$$

together with transmission conditions across the common endpoint $x=0$, given by

$$u(0^-)=2u(0^+), \quad u'(0^-)=3u'(0^+). \tag{14}$$

At first we consider the problem (12)-(14) without transmission conditions (14). It is easy to verify that the function

$$u=\frac{3}{2}e^{x^2+x-2}(x+1) \tag{15}$$

satisfies the equation (12) on whole $[-1,0)\cup(0,1]$ and both boundary conditions (13). For simplicity we will use the uniform cartesian grid

$$x_k=1+kh, \quad k=0,1,\ldots,32$$

for $h=\frac{2}{32}=\frac{1}{16}$. In particular we have $x_0=0, x_{32}=3$.
The central finite difference (CFD) approximation of the derivatives $u^{'}$ and $u^{''}$ are defined by

$$u'(x)\approx\frac{1}{2}\left(D_+u(x)+D_-u(x)\right)$$

and

$$u''(x)\approx\frac{1}{h}\left(D_+u(x)-D_-u(x)\right),$$

where $D_+u(x)$ and $D_-u(x)$ denotes the forward finite difference and backward finite difference of $u(x)$. By applying the CFD to the differential equation (12) at a typical grid point $x=x_k$ and denoting $u_k=u(x_k)$, we have the following finite difference equations

$$\begin{gathered}(2+(4x_k+2)h)u_{k-1}+(-4+2((2x_k+1)^2-2)h^2)u_k\\+(2-(4x_k+2)h)u_{k+1}=0, k=1,2,\ldots,31.\end{gathered} \tag{16}$$

That is, we have the linear algebraic system of equations with respect to the variables $u_1, u_2, \ldots, u_{31}$. The system of linear algebraic equations (16) can be written in a tridiagonal matrix-vector form

$$Mu = b.$$

The solution of this system can be found by using MATLAB-Octave. The obtained numerical FDM solutions are graphically compared with the exact solution (15) (see, Figures 1,2,3 and 4).

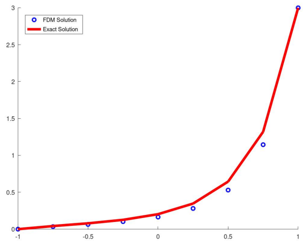

Figure 1. The FDM-solution and exact solution for the problem (12)-(13) where N=8

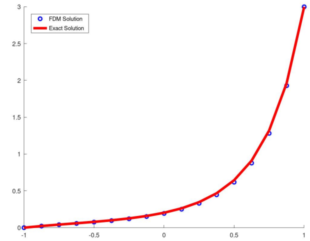

Figure 2. The FDM-solution and exact solution for the problem (12)-(13) where N=16

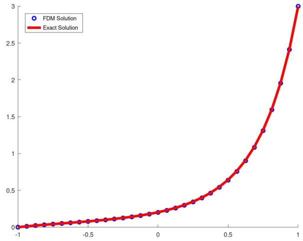

Figure 3. The FDM-solution and exact solution for the problem (12)-(13) where N=32

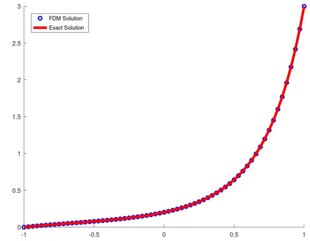

Figure 4. The FDM-solution and exact solution for the problem (12)-(13) where N=64

Remark. In Figures 1,2,3 and 4, the exact solution (15) is compared with the numerical FDM solutions for $N = 8, 16, 32, 64$ respectively. It can be seen from these graphical illustrations that, the error between the FDM solutions and the exact solution decreases as the number of grid points N increases.

Table 1. Maximum absolute error (MAE) for the problem (12)-(13)

N	h	$\parallel E \parallel_\infty$	N	h	$\parallel E \parallel_\infty$
4	$\frac{2}{4}$	0.64464	128	$\frac{2}{128}$	0.00057676
8	$\frac{2}{8}$	0.17501	256	$\frac{2}{256}$	0.00014415
16	$\frac{2}{16}$	0.038225	512	$\frac{2}{512}$	0.000036035
32	$\frac{2}{32}$	0.0092668	1024	$\frac{2}{1024}$	0.0000090086
64	$\frac{2}{64}$	0.0023117	2048	$\frac{2}{2048}$	0.0000022521

Example 5.2. Now, we will investigate the boundary value problem (12)-(13) under additional transmission conditions (14). If we select $N = 64$ and apply the transmission conditions (14), then we have two additional algebraic equations.

Since u_{32} is closest to $x = 0$ and lies to the left of $x = 0$, we have identified u_{32} with $u(0^-)$ and similarly u_{33} is closest to $x = 0$ and lies to the right of $x = 0$, we have identified y_{33} with $y(0^+)$. Therefore, the transmission condition

$$u(0^-) = 2u(0^+), \quad u'(0^-) = 3u'(0^+).$$

is transformed to the finite difference equation

$$u_{32} - 2u_{33} = 0 \tag{17}$$

and the transmission condition

$$u'(0^-) = 3u'(0^+)$$

is transformed to the equality

$$u_{30} - u_{32} - 3u_{33} + 3u_{35} = 0. \tag{18}$$

Table 2. Maximum absolute error (MAE) for transmission problem

N	h	$\parallel E \parallel_\infty$
8	$\frac{2}{8}$	1.7634
16	$\frac{2}{16}$	0.57588
32	$\frac{2}{32}$	0.23828
64	$\frac{2}{64}$	0.10978

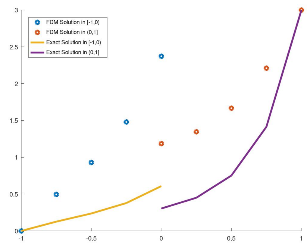

Figure 5. The FDM-solution and exact solution for the problem (12)-(14) where N=8

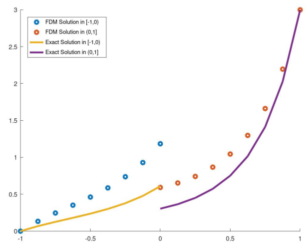

Figure 6. The FDM-solution and exact solution for the problem (12)-(14) where N=16

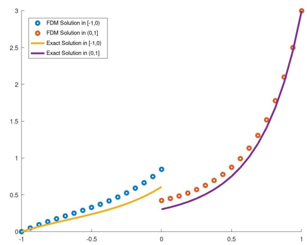

Figure 7. The FDM-solution and exact solution for the problem (12)-(14) where N=32

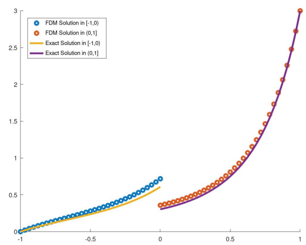

Figure 8. The FDM-solution and exact solution for the problem (12)-(14) where N=64

Note that the equation of this system involves solution values at four nodal points x_{30}, x_{32}, x_{33} and x_{35}.
By adding equations (17), (18) to the system of equations (16), a linear equation system is obtained in the form

$$\tilde{M}y = \tilde{B}.$$

The solution of this linear system of algebraic equations can be found by using MATLAB/Octave.
In Figures 5,6,7 and 8, the finite difference solution of the problem (12)-(14) is graphically compared with the exact solution.

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