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Analysis of stochastic pantograph differential equations with generalized derivative of arbitrary order

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Abstract. In this paper, we mainly study the existence of analytical solutions of stochastic pantograph differential equations. The standard Picard's iteration method is used to obtain the theory.

Keywords. Pantograph differential equation, ψ-type fractional derivative, existence, Picard's iteration method.

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1 Introduction

In recent years, many researchers have been interested in fractional differential equations (FDEs). This is due, first, to their widespread applications in diverse fields of engineering and natural sciences, and secondly to the thorough improvement of the theory of fractional calculus (see [1, 17, 18]).In much of the literature we can see various complicated fractional operators in which of them the well-known Caputo, the Riemann-Liouville, Hadamard, Caputo-Hadamard, Hilfer and Fabrizio-Caputo fractional operators have been utilized by many researchers (see for example, [11–14]).

The pantograph equation is one of the most famous classes of differential equations and this type of equation is taken for as proportional delay functional differential equations and have many applications in pure and applied mathematics as it appears in various contexts such as control systems, probability, electrodynamics, quantum mechanics, etc. Furthermore, FDEs with delays have been proven to be more realistic in the description of natural phenomena than those without delays. Therefore, the study of these equations has drawn much attention (see [3, 11–14]). Stochastic delay differential equations played an important role in application areas, such as physics, biology, economics, and finance [2, 4]. Stochastic pantograph differential equations are particular cases of stochastic unbounded delay differential equations, Ockendon and Tayler [16] found how the electric current is collected by the pantograph of an electric locomotive, therefore one speaks of stochastic pantograph differential equations (see [10, 20]). In recent years, as one of the most important characteristics of stochastic systems, the existence and stability analysis has caused much more attention [4, 5, 7, 8, 19].

Very recently, Almeida [6] introduced a new fractional derivative named by ψ-fractional derivative with respect to another function, which extended the classical fractional derivative and also studied some properties like semigroup law, Taylor's Theorem and so on. Thereafter, Vivek et al. [20] initially studied a Cauchy problem for pantograph equations including Hilfer fractional derivative.

Inspired by the papers [6, 15, 20], we consider the stochastic pantograph differential equations (SPDEs) involving ψ-Caputo fractional derivative of the form

$$[l]^{c}D^{\alpha;\psi}x(t) = Ax(t) + f(t, x(t), x(\lambda t)) + \sigma(t, x(t), x(\lambda t))\dot{W}(t),$$
$$t \in J := [0, T], \tag{1}$$
$$x^{(k)}(0) = x_0^{(k)}, \quad k = 0, 1, 2, ..., m-1, \tag{2}$$

where $0 < \lambda < 1$, $n - 1 < \alpha \leq n$ and f, σ are given functions and A is the generator of strongly continuous semigroup $\{\tau(t) : t \geq 0\}$ on a Hilbert space $\mathscr{H}$.

Observing that (1)-(2) is equivalent to the Volterra integral equation as follows:

$$x(t) = \begin{cases} \sum_{k=0}^{\lceil\alpha\rceil-1} \frac{x^{(k)}(0)}{k!}(\psi(t))^k \\ +\frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}Ax(s)ds \\ +\frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}f(s, x(s), x(\lambda s))ds \\ +\frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\sigma(s, x(s), x(\lambda s))dW(s), \end{cases} \tag{3}$$

where $n - 1 < \alpha \leq n$ and $t \geq 0$.

2 Prerequisite

Throughout this paper, we define $(\Omega, \Im, \mathbb{P})$ be a completely probability space, for a separable Hilbert space $\mathscr{H}$ with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$. Then, $\mathscr{L}_2(\Omega, \mathscr{H})$ is Hilbert space of $\mathscr{H}$-valued random variables with the inner product $\mathbb{E}(\cdot, \cdot)$ and the norm $\left(\mathbb{E}\|\cdot\|^2\right)^{\frac{1}{2}}$ in which $\mathbb{E}$ denotes the expectation.

Further, we consider the ψ-type Caputo fractional derivative of order α for a vector-valued function $x(t)$, and the initial value problem (IVP) of an abstract SPDEs (1)-(2), where $f(t, x(t), x(\lambda t))$, $\sigma(t, x(t), x(\lambda t)) : J \times \mathscr{R}^d \times \mathscr{R}^d \to \mathscr{R}^d$ and the dimension $d \geq 1$. The term $\dot{W}(t) = \frac{dW}{dt}$ describes a state dependent random noise, $\{W(t)\}_{t\geq 0}$ is a standard scalar Brownian motion or Wiener process defined on a given filtered probability space$(\Omega, \Im, \Im_t, \mathbb{P})$ with a normal filteration $\{\Im_t\}_{t\geq 0}$, which is an increasing and continuous family of σ-algebra of $\Im$, contains the $\mathbb{P}$-null sets, and $W(t)$ is $\Im_t$-measurable for all $t \geq 0$.

Let us start by giving the definition of ψ-type fractional derivatives. Further details of related basic properties used in the text can be found in [6].

Definition 2.1. [9] For $u \in \mathscr{L}_2(\Omega, \mathscr{H})$, there holds the following $It\widehat{o}$ isometry property:

$$\mathbb{E}\left\|\int_0^t u(s)dW(s)\right\|^2 = \int_0^t E\left\|u(s)\right\|^2 ds, \tag{4}$$

where $\{W(t)\}_{t\geq 0}$ is the Wiener (Brownian motion) process.

Definition 2.2. The ψ-type fractional integral of Riemann-Liouville of order $\alpha > 0$ of a function f is defined as follows:

$$I^{\alpha,\psi} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \psi^{'}(s) \left(\psi(t) - \psi(s)\right)^{\alpha-1} f(s)ds, \quad \text{a.e} \quad t \in J,$$

where the symbol $\Gamma(\cdot)$ stands for the Euler's gamma function.

Definition 2.3. The ψ-Caputo-type derivative of order α for a function f can be defined as

$${}^{c}D^{\alpha;\psi} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \psi^{'}(s) \left(\psi(t) - \psi(s)\right)^{n-\alpha-1} f^{(n)}(s)ds,$$

where $t > 0$, $n-1 < \alpha \leq n$.

Remark 2.4. The relationship between the ψ-type Riemann-Liouville derivative and the ψ-type Caputo derivative can be defined as

$${}^{c}D^{\alpha;\psi} f(t) = {}^{R}D^{\alpha;\psi} f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} (\psi(t))^k.$$

Lemma 2.5. *Every solution of the equation* (3) *is also a solution of the IVP* (1)-(2) *for* $\alpha \in (0, 1]$, *and vice versa.*

In particular, when $0 < \alpha \leq 1$, the Volterra integral equation (3) can be written as

$$x(t) = \begin{cases} x(0) + \frac{1}{\Gamma(\alpha)} \int_0^t \psi^{'}(s)(\psi(t) - \psi(s))^{\alpha-1} Ax(s)ds \\ +\frac{1}{\Gamma(\alpha)} \int_0^t \psi^{'}(s)(\psi(t) - \psi(s))^{\alpha-1} f(s, x(s), x(\lambda s))ds \\ +\frac{1}{\Gamma(\alpha)} \int_0^t \psi^{'}(s)(\psi(t) - \psi(s))^{\alpha-1} \sigma(s, x(s), x(\lambda s))dW(s). \end{cases} \tag{5}$$

Lemma 2.6. *The IVP* (1)-(2) *is equivalent to the integral equation* (5)*, for* $\alpha \in (0, 1]$. *In other words, every solution of the integral equation* (5) *is also a solution of the IVP* (1)-(2) *and vice versa.*

Proof. For proof, see e.g. [1]. □

First of all, in order to consider the existence and uniqueness of the solution for the IVP (1)-(2) for $\alpha \in (0, 1]$, we impose the following hypotheses.

(H1) Let $\tau(\cdot)$ be a C_0-semigroup generated by the unbounded operator A, let $M = \max_{t \in J} \|\tau(t)\|_{\mathscr{H}}$.

(H2) The function f and σ are measurable and continuous in $\mathscr{H}$ for each fixed $t \in J$ and there exists a bounded function $\mathbb{L} : J \times [0, \infty) \times [0, \infty) \to [0, \infty)$, $(t, x, y) \to \mathbb{L}(t, x, y)$ such that

$$\mathbb{E}\left(\|f(t, u, v)\|^2\right) + \mathbb{E}\left(\|\sigma(t, u, v)\|^2\right) \leq \mathbb{L}\left(t, \mathbb{E}\left(\|u\|^2\right), \mathbb{E}\left(\|v\|^2\right)\right), \tag{6}$$

for all $t \in \mathscr{R}$ and $u, v \in \mathscr{L}_2(\Omega, \mathscr{H})$.

(H3) There exists a bounded function $K : J \times [0, \infty) \times [0, \infty) \to [0, \infty)$ such that

$$\begin{aligned} &\mathbb{E}\left(\|f(t, u, v) - f(t, \overline{u}, \overline{v})\|^2\right) + \mathbb{E}\left(\|\sigma(t, u, v) - \sigma(t, \overline{u}, \overline{v})\|^2\right) \\ &\leq K\left(t, \mathbb{E}\left(\|u - \overline{u}\|^2\right), \mathbb{E}\left(\|v - \overline{v}\|^2\right)\right), \end{aligned} \tag{7}$$

for all $t \in \mathscr{R}$ and $u, \overline{u}, v, \overline{v} \in \mathscr{L}_2(\Omega, \mathscr{H})$.

Lemma 2.7. *(*[9]*) If the function* $\mathbb{L}(t, x(\cdot), x(\cdot))$ *is locally integrable in t for each fixed* $x \in [0, \infty)$ *and is continuous non-decreasing in x for each fixed* $t \in J$*, for all* $\delta >$*,* $x_0 \geq 0$*, then the integral equation*

$$x(t) = x_0 + \delta \int_0^t \mathbb{L}(s, x(s), x(\lambda s))ds,$$

has a global solution on J.

Lemma 2.8. *(*[9]*) The function $K(t, x(\cdot), x(\cdot))$ is locally integrable in t for each fixed $x \in [0, \infty)$ and is continuous non-decreasing in x for each fixed $t \in J$, for $K(t, 0, 0) = 0$ and $\gamma > 0$, if a non-negative continuous function $\phi(t)$ satisfies*

$$\phi(t) \leq \gamma \int_0^t K(s, x(s), x(\lambda s))ds, \quad t \in \mathscr{R},$$
$$\phi(0) = 0,$$

then $\phi(0) = 0$ for all $t \in J$.

In order to consider the existence and uniqueness of the solution of equation (5), we use the Picard's iteration method. The sequence of stochastic process $\{x_n\}_{n \geq 0}$ is constructed as follows:

$$x_0(t) = x_0$$
$$x_{n+1}(t) = x_0 + G_1(x_n)(t) + G_2(x_n)(t), \quad n \geq 1,$$

in which

$$G_1(x_n)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s, x_n(s), x_n(\lambda s))ds,$$
$$G_2(x_n)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \sigma(s, x_n(s), x_n(\lambda s))dW(s).$$

Lemma 2.9. *(*[9]*) The sequence of stochastic processes $\{x_n\}_{n \geq 0}$ is bounded in $\mathscr{L}_2(\Omega, \mathscr{H})$.*

Proof. From the inequality

$$(a + b + c)^n \leq 3^{n-1}(a^n + b^n + c^n), \quad n \geq 1.$$

We have

$$\mathbb{E} \|x_{n+1}(t)\|^2 \leq 3\mathbb{E} \|x_0\|^2 + 3\mathbb{E} \|G_1(x_n)(t)\|^2 + 3\mathbb{E} \|G_2(x_n)(t)\|^2 . \tag{8}$$

Using the $H\ddot{o}lder's$ inequality, the hypothesis (H2) and $\alpha > \frac{1}{2}$, we can obtain

$$\begin{aligned}
\mathbb{E} \|G_1(x_n)(t)\|^2 &\leq \frac{1}{\Gamma^2(\alpha)} \mathbb{E} \left\| \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s, x_n(s), x_n(\lambda s))ds \right\|^2 \\
&\leq \frac{1}{\Gamma^2(\alpha)} \frac{(\psi(t))^{2\alpha-1}}{2\alpha - 1} \int_0^t \mathbb{E} \|f(s, x_n(s), x_n(\lambda s))\|^2 ds \\
&\leq k_1 \int_0^t \mathbb{L} \left(s, \|x_n(s)\|^2_{\mathscr{L}_2(\Omega,\mathscr{H})}, \|x_n(s)\|^2_{\mathscr{L}_2(\Omega,\mathscr{H})} \right) ds,
\end{aligned}$$

where $k_1 = \frac{1}{\Gamma^2(\alpha)} \frac{(\psi(T))^{2\alpha-1}}{2\alpha-1}$.

Applying the $It\hat{o}$ isometry property (4), the $H\ddot{o}lder's$ inequality and the hypothesis (H2) and $\alpha > \frac{1}{2}$, we have

$$\begin{aligned}\mathbb{E}\left\|G_2(x_n)(t)\right\|^2 &\le \frac{1}{\Gamma^2(\alpha)}\mathbb{E}\left\|\int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\sigma(s,x_n(s),x_n(\lambda s))ds\right\|^2 \\ &\le \frac{1}{\Gamma^2(\alpha)}\frac{(\psi(t))^{2\alpha-1}}{2\alpha-1}\int_0^t \mathbb{E}\left\|\sigma(s,x_n(s),x_n(\lambda s))\right\|^2 ds \\ &\le k_1\int_0^t \mathbb{L}\left(s, \|x_n(s)\|^2_{\mathscr{L}_2(\Omega,\mathscr{H})}, \|x_n(s)\|^2_{\mathscr{L}_2(\Omega,\mathscr{H})}\right)ds.\end{aligned}$$

Hence, using the above relation into the inequality (8), we have

$$\|x_{n+1}(t)\|^2_{\mathscr{L}_2(\Omega,\mathscr{H})} \le c_1 + c_2\int_0^t \mathbb{L}\left(s, \|x_n(s)\|^2_{\mathscr{L}_2(\Omega,\mathscr{H})}, \|x_n(s)\|^2_{\mathscr{L}_2(\Omega,\mathscr{H})}\right)ds, \quad (9)$$

in which $c_1 = 3\mathbb{E}\|x_0\|^2$ and $c_2 = 6k_1$.
Then, we consider the following integral equation:

$$u(t) = c_1 + c_2\int_0^t \mathbb{L}\left(s,u(s),u(\lambda s)\right)ds. \quad (10)$$

This equation has a globe solution via the Lemma 2.7 and we can use the mathematical induction to prove $\|x_n(t)\|^2_{\mathscr{L}_2(\Omega,\mathscr{H})} \le x(t)$ for all $t \in J$. Particularly, we have

$$\sup_{n\ge 0}\|x_n(t)\|_{\mathscr{L}_2(\Omega,\mathscr{H})} \le [x(T)]^{\frac{1}{2}}.$$

□

Lemma 2.10. *The sequence of stochastic processes* $\{x_n\}_{n\ge 0}$ *is a Cauchy sequence.*

3 Main results

In this part, we prove the existence and uniqueness of the solution of the problem (1)-(2).

Theorem 3.1. *Under the condition* (6) *and* (7)*, by using Lemma* 2.7 *and Lemma* 2.8*, there exists a unique solution of equation* (5)*.*

Proof. Existence: If we denote $x(t)$ by the limit of the sequence $\{x_n(t)\}_{n\geq 0}$ and by using Lemma 2.10 then we can see that the right hand side in the second Picard's iteration tend to

$$\begin{cases} x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} f(s,x(s),x(\lambda s))ds \\ +\frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} \sigma(s,x(s),x(\lambda s))dW(s), \end{cases}$$

which is just a solution of equation (5).
Uniqueness: Let $x(t)$ and $y(t)$ are two solution's of equation (5), using Lemma 2.9, we have

$$\|x(t)-y(t)\|^2_{\mathscr{L}_2(\Omega,\mathscr{H})} \leq c_3 \int_0^t K\left(s, 2\|x(s)-y(s)\|^2_{\mathscr{L}_2(\Omega,\mathscr{H})}\right) ds.$$

Using Lemma 2.8, we can obtain $\|x(t)-y(t)\|^2_{\mathscr{L}_2(\Omega,\mathscr{H})} = 0$ for all $t \in J$, which implies that $x(t) = y(t)$. □

4 Conclusion

In the last decades, stochastic pantograph differential equations have played an important role in application areas, such as physics, biology, economics, and finance. In this paper, we employed the standard Picard's iteration method to study the existence and uniqueness of analytical solutions of stochastic pantograph differential equations involving ψ-Caputo fractional derivatives in Hilbert space.

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