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On ideal convergence of rough triple sequence

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Abstract. In this paper, we present the ideal convergence of triple sequences for rough variables. Furthermore, sequence convergence plays an extremely important role in the fundamental theory of mathematics. This paper presents two types of ideal convergence of rough triple sequence: Convergence in trust and convergence in mean. Some mathematical properties of those new convergence concepts are also given. In addition, we introduce ideal Cauchy triple sequence in rough spaces.

Keywords. Rough variable, rough space, ideal convergence, ideal Cauchy sequence.

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1 Introduction and background

Rough set theory, initialized by Pawlak [17], has been proved to be an excellent mathematical tool dealing with vague description of objects. A fundamental assumption in rough set theory is that any object from a universe is perceived through available information, and such information may not be sufficient to characterize the object exactly. One way is the approximation of a set by other sets. Thus a rough set may be defined by a pair of crisp sets, called the lower and the upper approximations, that are originally produced by an equivalence relation (reflexive, symmetric, and transitive). Slowinski and Vanderpooten [23] extended the equivalence relation to more general case and proposed a binary similarity relation that has not symmetry and transitivity but reflexivity. Liu [12] characterized a rough variable from rough space to the set of real numbers and he presented the description of the lower and upper approximation of the rough variable. Considering sequence convergence plays a key role in rough theory, Liu [13] presented four kinds of convergence concept for rough variables: convergence in trust, convergence almost surely, convergence in mean, convergence in distribution.

In this study we aim to present ideal convergence of rough variables in rough spaces. Ideal convergence is a general type of the usual convergence with respect to a hereditary family of subsets of the natural number set stable under finite unions. This family is called an ideal of the natural number set and it is indicated by $\mathcal{I}$. The notion of $\mathcal{I}$-convergence as a generalization of the usual convergence was established in [11]. Pointwise $\mathcal{I}$-convergence and $\mathcal{I}$-convergence in measure for the function sequences have been worked in [10]. $\mathcal{I}$-Cauchy sequence and some other features of the $\mathcal{I}$-convergence were examined in [16].

Influenced by this, in this study, a further research into the mathematical features of ideal convergence for rough variables will be presented. In addition, we plan to work the notion $\mathcal{I}$-convergence of a triple sequence of rough variables and to construct fundamental properties of the $\mathcal{I}$-convergence of triple sequences in trust.

In order to provide an axiomatic theory to describe rough variable, Liu [12] gave a definition of rough space.

Definition 1.1. Let Λ be a nonempty set, $\mathcal{A}$ be a σ-algebra of subsets of Λ, Δ be an element in $\mathcal{A}$, and π be nonnegative, real-valued, additive set function. Then $(\Lambda, \Delta, \mathcal{A}, \pi)$ is called a rough space.

Alternately, if the real-valued set function satisfies (i) $\pi\{\emptyset\} = 0$; (ii) $\pi\{A\} \leq \pi\{B\}$ whenever $A, B \in \mathcal{A}$ and $A \subset B$, then $(\Lambda, \Delta, \mathcal{A}, \pi)$ is called a generalized rough space.

Definition 1.2. A rough variable ξ on the rough space $(\Lambda, \Delta, \mathcal{A}, \pi)$ is a function from Λ to the real line $\mathbb{R}$ such that for every Borel set B of $\mathbb{R}$ we have

$$\{\lambda \in \Lambda : \xi(\lambda) \in B\} \in \mathcal{A}.$$

The lower and the upper approximations of the rough variable ξ are then defined as $\underline{\xi} = \{\xi(\lambda) \mid \lambda \in \Delta\}$ and $\overline{\xi} = \{\xi(\lambda) \mid \lambda \in \Lambda\}$, respectively.

Definition 1.3. Let $(\Lambda, \Delta, \mathcal{A}, \pi)$ be a rough space. Then, the lower and upper trust $\underline{Tr}\{A\}$ and $\overline{Tr}\{A\}$ of an event A is respectively defined by

$$\underline{Tr}\{A\} = \frac{\pi\{A \cap \Delta\}}{\pi\{\Delta\}} \text{ and } \overline{Tr}\{A\} = \frac{\pi\{A\}}{\pi\{\Lambda\}}. \tag{1}$$

The trust $Tr\{A\}$ of the event A is defined by

$$Tr\{A\} = \frac{1}{2}\left(\underline{Tr}\{A\} + \overline{Tr}\{A\}\right). \tag{2}$$

Definition 1.4. Let ξ be a rough variable on the rough space $(\Lambda, \Delta, \mathcal{A}, \pi)$. The expected value $E|\xi|$ is defined by

$$E|\xi| = \int_{0}^{+\infty} Tr\{\xi \geq r\}\,dr - \int_{-\infty}^{0} Tr\{\xi \leq r\}\,dr \tag{3}$$

provided that at least one of the two integrals is finite.

Let $\mathbb{N} = \{1, 2, ...\}$ and let $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be an ideal, i.e. a hereditary family of subsets of $\mathbb{N}$ stable under finite unions. We say that a sequence $(x_n)_{n\in\mathbb{N}}$ of points in a metric space (X, d) is $\mathcal{I}$-convergent to $\gamma \in X$ if for each $\varepsilon > 0$ the set $\{n \in \mathbb{N} : d(x_n, \gamma) \geq \varepsilon\}$ belongs to $\mathcal{I}$. We show it by $x_n \to_{\mathcal{I}} \gamma$. If $\mathcal{I}$ is a proper ideal, then the limit γ is uniquely determined. If $\mathcal{I} = \mathcal{I}_{fin}$ is the ideal of all finite subsets of $\mathbb{N}$, then we obtain the usual convergence $x_n \to \gamma$. If $\mathcal{I}_{fin} \subset \mathcal{I}$, then the usual convergence gives the $\mathcal{I}$-convergence. When one considers $\mathcal{I}$-convergence, it is reasonable to assume that $\mathcal{I}_{fin} \subset \mathcal{I} \neq \mathcal{P}(\mathbb{N})$. Several generalizations and applications of statistical convergence have been presented, see [1,3,4,8,9,19,21, 22].

Another way to speak about $\mathcal{I}$-convergence is utilizing the notion of filter convergence. Let $\mathcal{F} \subset \mathbb{N}$ be a filter. We say that a sequence $(x_n)_{n\in\mathbb{N}}$ of points in a metric space (X, d) is $\mathcal{F}$-convergent to $\gamma \in X$ if $\{n \in \mathbb{N} : d(x_n, \gamma) < \varepsilon\} \in \mathcal{F}$ for every $\varepsilon > 0$. Both points of view are clearly equivalent: If $\mathcal{I}$ is an ideal and $\mathcal{F} = \{A : \mathbb{N} \backslash A \in \mathcal{I}\}$ is its dual filter, then the notions of $\mathcal{I}$-convergence and $\mathcal{F}$-convergence coincide.

Recently, Mursaleen and Edely [15] presented the idea of statistical convergence for multiple sequences, and there are several papers dealing with the statistical and ideal convergence of double and triple sequences (see literature [5–7]). Also, the readers should refer to the monographs [2] and [14] for the background on the sequence spaces and related topics.

By the convergence of a triple sequence we mean the convergence in the Pringsheim sense, i.e. a triple sequence $x = (x_{ijk})$ has Pringsheim limit L (denoted by $P - \lim x = L$) provided that given $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $|x_{ijk} - L| < \varepsilon$ whenever $i, j, k > n$, (see [19]). We shall write more briefly as $P-$convergent. The triple sequence $x = (x_{ijk})$ is bounded if there exists a positive number K such that $|x_{ijk}| < K$ for all i, j and k.

Definition 1.5. Let $\mathcal{I}$ be an admissible ideal on $\mathbb{N}$ and $x = (x_k)$ be a real sequence. We say that the sequence x is $\mathcal{I}$-convergent to $L \in \mathbb{R}$ if for each $\varepsilon > 0$, the set $A(\varepsilon) = \{n \in \mathbb{N} : |x_k - L| \geq \varepsilon\} \in \mathcal{I}$.

Take for $\mathcal{I}$ the class $\mathcal{I}_f$ of all finite subsets of $\mathbb{N}$. Then, $\mathcal{I}_f$ is a non-trivial admissible ideal and $\mathcal{I}_f$-convergence coincides with the usual convergence. For more information about $\mathcal{I}$-convergence, see the references in [16].

Definition 1.6. ([20]) Let $\mathcal{I}_3$ be an admissible ideal on $\mathbb{N}^3$, then a triple sequence

(x_{jkl}) is said to be $\mathcal{I}_3$-convergent to L in Pringsheim's sense if for every $\varepsilon > 0$,

$$\left\{(j,k,l) \in \mathbb{N}^3 : |x_{jkl} - L| \geq \varepsilon\right\} \in \mathcal{I}_3,$$

and is written as $\mathcal{I}_3$-$\lim x_{jkl} = L$.

Remark 1.7. The following statements hold:
(i) Let $\mathcal{I}_3\left(f\right)$ be the family of all finite subsets of $\mathbb{N}^3$. Then, $\mathcal{I}_3\left(f\right)$ convergence coincides with the convergence of triple sequences in [18].
(ii) Let $\mathcal{I}_3\left(\delta\right) = \left\{A \subset \mathbb{N}^3 : \delta_3\left(A\right) = 0\right\}$. Then, $\mathcal{I}_3\left(\delta\right)$ convergence coincides with the statistical convergence in [18].

2 Main Results

In this section, based on existing ideal convergence, we study the ideal convergence of a triple sequence in a rough space and the ideal Cauchy sequence of a triple sequence in a rough space. In order to better explain our results, we quote some required definitions.

Definition 2.1. Assume that $\{\mu_{jkl}\}$ be a triple sequence of rough variables. The triple sequence $\{\mu_{jkl}\}$ converges in trust to the rough variable μ if

$$\lim_{j,k,l\to\infty} \text{Tr}\left\{|\mu_{jkl} - \mu| \geq \varepsilon\right\} = 0$$

for each $\varepsilon > 0$, and is written as (Tr)-$\lim \mu_n = \mu$.

Definition 2.2. Let $\mu, \mu_{jkl}, j, k, l \in \mathbb{N}$, be a rough variables. The sequence $\{\mu_{jkl}\}$ is said to be ideal convergent in trust to the rough variable μ if

$$\left\{(j,k,l) \in \mathbb{N}^3 : \text{Tr}\left\{|\mu_{jkl} - \mu| \geq \varepsilon\right\} \geq \delta\right\} \in \mathcal{I}_3$$

for each $\varepsilon > 0$ and $\delta > 0$, and is written as $\mathcal{I}_3$ (Tr)-$\lim \mu_n = \mu$.

Theorem 2.3. *Let* $\mu, \mu_{jkl}, j, k, l \in \mathbb{N}$, *be a rough variables. If* (Tr)-$\lim \mu_n = \mu$, *then* $\mathcal{I}_3$ (Tr)-$\lim \mu_n = \mu$.

Proof. Since $\{\mu_{jkl}\}$ converges in trust to the rough variable μ, we have for each $\varepsilon > 0$,

$$\lim_{j,k,l\to\infty} \text{Tr}\left\{|\mu_{jkl} - \mu| \geq \varepsilon\right\},$$

i.e., for any $\delta > 0$ there exists $N \in \mathbb{N}$ such that

$$\text{Tr}\left\{|\mu_{jkl} - \mu| \geq \varepsilon\right\} \leq \delta$$

for all $j > N, k > N, l > N$. So,

$$\left\{(j,k,l) \in \mathbb{N}^3 : \text{Tr}\left\{|\mu_{jkl} - \mu| \geq \varepsilon\right\} \geq \delta\right\} \subseteq N$$

for all $j, k, l \in \mathbb{N}$. Hence we deduce

$$\left\{(j,k,l) \in \mathbb{N}^3 : \text{Tr}\left\{|\mu_{jkl} - \mu| \geq \varepsilon\right\} \geq \delta\right\} \in \mathcal{I}_3,$$

as desired. □

Example 2.4. Define $\Lambda = \{B_1, B_2\}$, $\mathcal{A} = \mathcal{P}(\Lambda)$, where $\mathcal{P}$ is the power set, $\Delta = \Lambda$ and $\pi\{B_1\} = 8 = \pi\{B_2\}$. Then it can be easily seen $(\Lambda, \Delta, \mathcal{A}, \pi)$ is a rough space. Define $\text{Tr}\{B_i\} = 1/2$ for $i = 1, 2$ and the rough variables are identified by

$$\mu_{jkl}(B) = \begin{cases} -1, & j,k,l = m^2 \wedge B = B_1 \\ 1, & j,k,l = m^2 \wedge B = B_2 \\ 0, & j,k,l \neq m^2 \wedge B = B_1 \\ 1, & j,k,l \neq m^2 \wedge B = B_2 \end{cases}$$

for $m = 1, 2, ...$ and $\mu = 0$. Then, for $0 < \varepsilon < 1$ and $\delta \in \left(\frac{1}{2}, 1\right)$, we have

$$\text{Tr}\left\{|\mu_{jkl} - \mu| \geq \varepsilon\right\} = \begin{cases} 1, & j,k,l = m^2 \\ \frac{1}{2}, & \text{otherwise} \end{cases}.$$

Therefore we get

$$\lim_{j,k,l \to \infty} \text{Tr}\left\{|\mu_{jkl} - \mu| \geq \varepsilon\right\} \neq 0$$

and

$$\left\{(j,k,l) \in \mathbb{N}^3 : \text{Tr}\left\{|\mu_{jkl} - \mu| \geq \varepsilon\right\} \geq \delta\right\} \in \mathcal{I}_3.$$

Therefore, we show that a triple sequence which is ideal convergent doesn't need to be convergent in trust.

Take μ_1, μ_2, μ_{jkl} as rough variables defined on rough space $(\Lambda, \Delta, \mathcal{A}, \pi)$.

(U) The uniqueness of limit: If $\mathcal{I}_3$ (Tr)-lim $\mu_{jkl} = \mu_1$ and $\mathcal{I}_3$ (Tr)-lim $\mu_{jkl} = \mu_2$, at that case $\mu_1 = \mu_2$ in trust.

Theorem 2.5. *Ideal convergence in trust satisfies the axiom **(U)**.*

Proof. Now, we examine that ideal convergence in trust supplies the axiom (**U**). Presume that $\mathcal{I}_3$ (Tr)-lim $\mu_{jkl} = \mu_1$ and $\mathcal{I}_3$ (Tr)-lim $\mu_{jkl} = \mu_2$, then for any $\gamma, \varepsilon > 0$ there exists $\delta > 0$ such that $\delta + \delta < \gamma$. We make the subsequent marks:

$$B_1 = \left\{(j,k,l) \in \mathbb{N}^3 : \text{Tr}\left\{|\mu_{jkl} - \mu_1| \geq \frac{\varepsilon}{2}\right\} \geq \frac{\delta}{2}\right\} \in \mathcal{I}_3,$$

and

$$B_2 = \left\{(j,k,l) \in \mathbb{N}^3 : \text{Tr}\left\{|\mu_{jkl} - \mu_2| \geq \frac{\varepsilon}{2}\right\} \geq \frac{\delta}{2}\right\} \in \mathcal{I}_3.$$

Now let $(p,q,r) \in B_1^c \cap B_2^c$. Then, we get

$$\text{Tr}\left\{|\mu_{pqr} - \mu_1| \geq \frac{\varepsilon}{2}\right\} < \frac{\delta}{2} \wedge \text{Tr}\left\{|\mu_{pqr} - \mu_2| \geq \frac{\varepsilon}{2}\right\} < \frac{\delta}{2}.$$

Therefore

$$\begin{aligned}\text{Tr}\{|\mu_1 - \mu_2| \geq \varepsilon\} &= \text{Tr}\{|\mu_1 - \mu_{pqr} + \mu_{pqr} - \mu_2| \geq \varepsilon\} \\ &\leq \text{Tr}\left\{|\mu_{pqr} - \mu_1| \geq \frac{\varepsilon}{2}\right\} + \text{Tr}\left\{|\mu_{pqr} - \mu_2| \geq \frac{\varepsilon}{2}\right\} < 2\delta < \gamma.\end{aligned}$$

Since $\gamma > 0$ is arbitrary, we acquire

$$\left\{(j,k,l) \in \mathbb{N}^3 : \text{Tr}\{|\mu_1 - \mu_2| \geq \varepsilon\} \geq \gamma\right\} \in \mathcal{I}_3,$$

which gives $\mu_1 = \mu_2$ in trust. □

Definition 2.6. Suppose that $\{\mu_{jkl}\}$ is a triple sequence of rough variables with finite expected values. We say that the triple sequence $\{\mu_{jkl}\}$ ideal converges in mean to the rough variable μ if

$$\left\{(j,k,l) \in \mathbb{N}^3 : E\left[|\mu_n - \mu|\right] \geq \varepsilon\right\} \in \mathcal{I}_3$$

for each $\varepsilon > 0$.

Theorem 2.7. *Let $\{\mu_{jkl}\}$ be a triple sequence of rough variables. If the triple sequence $\{\mu_{jkl}\}$ converges in mean to a rough variable μ, then $\{\mu_{jkl}\}$ converges in trust to μ.*

Proof. Let the rough variable triple sequence $\{\mu_{jkl}\}$ be ideal convergent in mean to a rough variable μ. For any taken $\varepsilon, \delta > 0$ with the aid of Markov inequality, we obtain

$$\begin{aligned}&\left\{(j,k,l) \in \mathbb{N}^3 : \text{Tr}\{|\mu_{jkl} - \mu| \geq \varepsilon\} \geq \delta\right\} \\ &\subseteq \left\{(j,k,l) \in \mathbb{N}^3 : \tfrac{E\left[|\mu_{jkl} - \mu|\right]}{\varepsilon} \geq \delta\right\} \in \mathcal{I}_3.\end{aligned}$$

Thus, $\{\mu_{jkl}\}$ ideal converges in trust to μ. □

We define

$$T_n(\mathrm{Tr}) = \frac{1}{nmo} \sum_{j,k,l=1,1,1}^{n,m,o} \mathrm{Tr}\left\{ |\mu_{jkl} - \mu| \geq \varepsilon \right\}.$$

Next, we give the following definitions.

Definition 2.8. Take μ, μ_{jkl} as rough variables. Then, $\{\mu_{jkl}\}$ is called to be ideal $T_n(\mathrm{Tr})$-summable to μ if

$$\left\{ (j,k,l) \in \mathbb{N}^3 : \sum_{j,k,l=1,1,1}^{n,m,o} \mathrm{Tr}\left\{ |\mu_{jkl} - \mu| \geq \varepsilon \right\} \geq \delta \right\} \in \mathcal{I}_3,$$

for any $\varepsilon, \delta > 0$, and is written as $\mathcal{I}_3(T_n(\mathrm{Tr}))$-lim $\mu_n = \mu$.

Theorem 2.9. *Take μ, μ_{jkl} as rough variables. If $\{\mu_{jkl}\}$ is an ideal $T_n(\mathrm{Tr})$-summable to μ, then it is an ideal convergent to the rough variable μ in trust.*

Proof. For all $\varepsilon > 0$ and $\delta > 0$, we get

$$\mathrm{Tr}\left\{ |\mu_{jkl} - \mu| \geq \varepsilon \right\} \leq \sum_{j,k,l=1,1,1}^{n,m,o} \mathrm{Tr}\left\{ |\mu_{jkl} - \mu| \geq \varepsilon \right\},$$

and

$$\begin{aligned} &\left\{ (j,k,l) \in \mathbb{N}^3 : \mathrm{Tr}\left\{ |\mu_{jkl} - \mu| \geq \varepsilon \right\} \geq \delta \right\} \\ &\subseteq \left\{ (j,k,l) \in \mathbb{N}^3 : \sum_{j,k,l=1,1,1}^{n,m,o} \mathrm{Tr}\left\{ |\mu_{jkl} - \mu| \geq \varepsilon \right\} \geq \delta \right\}. \end{aligned}$$

Since $\mathcal{I}_3(T_n(\mathrm{Tr}))$-lim $\mu_n = \mu$, we have

$$\left\{ (j,k,l) \in \mathbb{N}^3 : \mathrm{Tr}\left\{ |\mu_{jkl} - \mu| \geq \varepsilon \right\} \geq \delta \right\} \in \mathcal{I}_3$$

as desired. □

Now, we present the notion of ideal Cauchy sequence in trust.

Definition 2.10. Take $\mu_{jkl}, j,k,l \in \mathbb{N}$, as rough variables. We say that the triple sequence $\{\mu_{jkl}\}$ is an ideal Cauchy sequence in trust, if for any $\varepsilon > 0$, $\delta > 0$, there exists N_1, N_2 and N_3 such that for $j,p \geq N_1$, $k,q \geq N_2$, $l,r \geq N_3$,

$$\left\{ (j,k,l) \in \mathbb{N}^3 : \mathrm{Tr}\left\{ |\mu_{jkl} - \mu_{pqr}| \geq \varepsilon \right\} \geq \delta \right\} \in \mathcal{I}_3.$$

Theorem 2.11. *Let μ, μ_{jkl} be rough variables. If $\{\mu_{jkl}\}$ is ideal convergent to the rough variable μ in trust, then $\{\mu_{jkl}\}$ is an ideal Cauchy sequence in trust.*

Proof. If $\{\mu_{jkl}\}$ is ideal convergent to the rough variable μ in trust, then, we have

$$\left\{(j,k,l) \in \mathbb{N}^3 : \text{Tr}\left\{|\mu_{jkl} - \mu| \geq \frac{\varepsilon}{2}\right\} \geq \frac{\delta}{2}\right\} \in \mathcal{I}_3$$

for each $\varepsilon > 0$ and $\delta > 0$. Let

$$A = \left\{(j,k,l) \in \mathbb{N}^3 : \text{Tr}\left\{|\mu_{jkl} - \mu| \geq \frac{\varepsilon}{2}\right\} \geq \frac{\delta}{2}\right\},$$

and

$$B = \left\{(j,k,l) \in \mathbb{N}^3 : \text{Tr}\left\{|\mu_{jkl} - \mu_{pqr}| \geq \varepsilon\right\} \geq \delta\right\}.$$

Thus

$$A^c = \left\{(j,k,l) \in \mathbb{N}^3 : \text{Tr}\left\{|\mu_n - \mu| \geq \frac{\varepsilon}{2}\right\} < \frac{\delta}{2}\right\}.$$

Next we prove $B \subset A$. Presume in contrast that $A \subseteq B$ and $(j,k,l) \in B \setminus A$. Then

$$\text{Tr}\left\{|\mu_{jkl} - \mu| \geq \frac{\varepsilon}{2}\right\} < \frac{\delta}{2}, \; \text{Tr}\left\{|\mu_{jkl} - \mu_{pqr}| \geq \varepsilon\right\} \geq \delta.$$

Let $(p,q,r) \in A^c$, we get $\text{Tr}\left\{|\mu_{pqr} - \mu| \geq \frac{\varepsilon}{2}\right\} < \frac{\delta}{2}$. Hence

$$\begin{aligned}\delta &\leq \text{Tr}\left\{|\mu_{jkl} - \mu_{pqr}| \geq \varepsilon\right\} \\ &\leq \text{Tr}\left\{|\mu_{pqr} - \mu| \geq \tfrac{\varepsilon}{2}\right\} + \text{Tr}\left\{|\mu_{jkl} - \mu| \geq \tfrac{\varepsilon}{2}\right\} \\ &< \tfrac{\delta}{2} + \tfrac{\delta}{2} = \delta,\end{aligned}$$

which is a contradiction. Therefore, $B \subset A$. So, we get

$$\left\{(j,k,l) \in \mathbb{N}^3 : \text{Tr}\left\{|\mu_{jkl} - \mu_{pqr}| \geq \varepsilon\right\} \geq \delta\right\} \in \mathcal{I}_3.$$

Hence, $\{\mu_{jkl}\}$ is ideal Cauchy sequence in trust. □

Definition 2.12. A rough space is named as ideal complete in trust if every ideal Cauchy triple sequence in trust ideal converges in trust.

Theorem 2.13. *Rough space $(\Lambda, \Delta, \mathcal{A}, \pi)$ is ideal complete in trust.*

Proof. Take $\{\mu_{jkl}\}$ as an ideal Cauchy sequence in trust. Then, for any $\varepsilon > 0$, $\delta > 0$, there exists N_1, N_2 and N_3 such that for $j, p \geq N_1$, $k, q \geq N_2$, $l, r \geq N_3$,

$$\left\{(j,k,l) \in \mathbb{N}^3 : \text{Tr}\left\{|\mu_{jkl} - \mu_{pqr}| \geq \varepsilon\right\} \geq \delta\right\} \in \mathcal{I}_3.$$

Assume in contrast that it is not ideal convergent in trust. Then, we have

$$\left\{(j,k,l) \in \mathbb{N}^3 : \text{Tr}\left\{|\mu_{jkl} - \mu| \geq \frac{\varepsilon}{2}\right\} \geq \frac{\delta}{2}\right\} \notin \mathcal{I}_3$$

for each $\varepsilon > 0$ and $\delta > 0$. Let

$$B = \left\{(j,k,l) \in \mathbb{N}^3 : \text{Tr}\left\{|\mu_{jkl} - \mu| \geq \frac{\varepsilon}{2}\right\} \geq \frac{\delta}{2}\right\}$$

and

$$C = \left\{(j,k,l) \in \mathbb{N}^3 : \text{Tr}\left\{|\mu_{jkl} - \mu_{pqr}| \geq \varepsilon\right\} \geq \delta\right\}.$$

Thus

$$B^c = \left\{(j,k,l) \in \mathbb{N}^3 : \text{Tr}\left\{|\mu_{jkl} - \mu| \geq \frac{\varepsilon}{2}\right\} < \frac{\delta}{2}\right\}.$$

Next we prove $B \subseteq C$. Assume $C \subseteq B$ and $(j,k,l) \in B^c \cap C$. Then

$$\text{Tr}\left\{|\mu_{jkl} - \mu| \geq \frac{\varepsilon}{2}\right\} < \frac{\delta}{2}, \text{Tr}\left\{|\mu_{jkl} - \mu_{pqr}| \geq \varepsilon\right\} \geq \delta.$$

Let $(p,q,r) \in B^c$, we obtain

$$\text{Tr}\left\{|\mu_{pqr} - \mu| \geq \frac{\varepsilon}{2}\right\} < \frac{\delta}{2}.$$

Hence, there exists N_1, N_2 and N_3 such that for $j, p \geq N_1$, $k, q \geq N_2$, $l, r \geq N_3$,

$$\begin{aligned}\delta &\leq \text{Tr}\left\{|\mu_{jkl} - \mu_{pqr}| \geq \varepsilon\right\} \\ &\leq \text{Tr}\left\{|\mu_{pqr} - \mu| \geq \tfrac{\varepsilon}{2}\right\} + \text{Tr}\left\{|\mu_{jkl} - \mu| \geq \tfrac{\varepsilon}{2}\right\} \\ &\leq \tfrac{\delta}{2} + \tfrac{\delta}{2} = \delta,\end{aligned}$$

which is impossible. Observe that $B \subseteq C$. This gives that

$$\left\{(j,k,l) \in \mathbb{N}^3 : \text{Tr}\left\{|\mu_{jkl} - \mu| \geq \frac{\varepsilon}{2}\right\} \geq \frac{\delta}{2}\right\} \in \mathcal{I}_3,$$

i.e.,

$$\left\{(j,k,l) \in \mathbb{N}^3 : \text{Tr}\left\{|\mu_{jkl} - \mu| \geq \varepsilon\right\} \geq \delta\right\} \in \mathcal{I}_3.$$

Thus, the triple sequence $\{\mu_{jkl}\}$ have to be ideal convergent in trust. This means that rough space is ideal complete in trust. □

Theorem 2.14. *If $\{\mu_{jkl}\}$ is ideal convergent to μ in trust and $f : \mathbb{R} \to \mathbb{R}$ is a convex function, then $\{f(\mu_{jkl})\}$ is ideal convergent in trust to $f(\mu)$.*

Proof. For that reason f is a convex function, there is a constant k such that

$$|f(x) - f(y)| \leq k|x - y|,$$

for any $x, y \in \mathbb{R}$. Since $\{\mu_{jkl}\}$ ideal converges to μ in trust, we acquire

$$\left\{(j,k,l) \in \mathbb{N}^3 : \text{Tr}\left\{|\mu_{jkl} - \mu| \geq \frac{\varepsilon}{k}\right\} \geq \delta\right\} \in \mathcal{I}_3,$$

for every $\varepsilon > 0$. Thus

$$\left\{(j,k,l) \in \mathbb{N}^3 : \text{Tr}\left\{|\mu_{jkl} - \mu| < \frac{\varepsilon}{k}\right\} < \delta\right\} \in \mathcal{F}(\mathcal{I}_3).$$

Then

$$|f(\mu_{jkl}) - f(\mu)| \leq k|\mu_{jkl} - \mu| < k.\frac{\varepsilon}{k} = \varepsilon.$$

Therefore

$$\left\{(j,k,l) \in \mathbb{N}^3 : \text{Tr}\left\{|f(\mu_{jkl}) - f(\mu)| < \varepsilon\right\} < \delta\right\} \in \mathcal{F}(\mathcal{I}_3).$$

Thus

$$\left\{(j,k,l) \in \mathbb{N}^3 : \text{Tr}\left\{|f(\mu_{jkl}) - f(\mu)| \geq \varepsilon\right\} \geq \delta\right\} \in \mathcal{I}_3.$$

Hence, $\{f(\mu_{jkl})\}$ ideal converges to $f(\mu)$ in trust. □

Theorem 2.15. *If $\{\mu_{jkl}\}$ is ideal convergent to μ in trust and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, then $\{f(\mu_{jkl})\}$ is ideal convergent to $f(\mu)$ in trust.*

Proof. If $\{\mu_{jkl}\}$ ideal converges to μ in trust, then for every $\varepsilon > 0$ and $\delta > 0$,

$$\left\{(j,k,l) \in \mathbb{N}^3 : \text{Tr}\left\{|\mu_{jkl} - \mu| \geq \varepsilon\right\} \geq \delta\right\} \in \mathcal{I}_3. \tag{4}$$

For the reason that f is a continuous function, for every $\varepsilon > 0$, there exists $\delta_1 > 0$ such that $|\mu_{jkl} - \mu| < \delta_1$ implies $|f(\mu_{jkl}) - f(\mu)| < \varepsilon$. Therefore,

$$\begin{aligned}&\left\{(j,k,l) \in \mathbb{N}^3 : \text{Tr}\left\{|f(\mu_{jkl}) - f(\mu)| \geq \varepsilon\right\} \geq \delta\right\}\\ &\subseteq \left\{(j,k,l) \in \mathbb{N}^3 : \text{Tr}\left\{|\mu_{jkl} - \mu| \geq \delta_1\right\} \geq \delta\right\},\end{aligned}$$

From (4), we have

$$\left\{(j,k,l) \in \mathbb{N}^3 : \text{Tr}\left\{|f(\mu_{jkl}) - f(\mu)| \geq \varepsilon\right\} \geq \delta\right\} \in \mathcal{I}_3.$$

That means, $\{f(\mu_{jkl})\}$ ideal converges to $f(\mu)$ in trust. □

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