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About one problem of optimal control synthesis

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Abstract. This paper tackles the problem of characterizing the natural class, or Riccati rule space, of solutions to a specific equation. Despite the significant theoretical and practical implications, there is limited research exploring the application of spectral decomposition of non-self-adjoint differential operators to solve explicitly this nonlinear Riccati equation. Therefore, investigating operator Riccati equations holds potential to validate the dynamic programming method and address the synthesis problem.

Keywords. Bellman equation, operator Riccati equation, energy class.

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1 Introduction. Formulation of the problem and Bellman equation

Numerous works have been published on optimal control of systems with distributed parameters and several monographs have been published (see the bibliography in [1]). Nevertheless, today one of the pressing problems is the justified application of known optimal control methods to problems of optimal control of systems with distributed parameters. It should be noted that in practice, to solve the problem of synthesizing optimal control in systems with distributed parameters, the dynamic programming method has found wide application. It is known that the problem of synthesizing optimal control with a minimum of a strictly convex quadratic functional for a linear equation of a controlled object leads to the solution of the nonlinear operator Riccati equation. The importance and necessity of a complete study of this equation is dictated by the practical applicability of this equation [1, 2]. One of the research problems is to determine the natural class - the natural class, that is, the Riccati rule space of solutions to this equation. We know little of the work devoted to the application of the spectral decomposition of non-self-adjoint differential operators to the explicit determination of the solution of the nonlinear Riccati equation, which has great theoretical and practical significance. This implies the relevance of the problem of studying operator Riccati equations to substantiate the dynamic programming method and solve the synthesis problem.

Let H be a real Hilbert space. Let us consider a controlled object, the state of which is described by the following dynamic equation with Cauchy data [1]:

$$\ddot{x}(t) - a\dot{x}(t) - A_1 A x(t) = u_1(t) + q u_2(t) + f(t),\ 0 < t < T \leq \infty, \tag{1}$$

$$x(0) = x_0 \in H(\lambda) \bigcap D(A), \quad \dot{x}(0) = x_1 \in H, \tag{2}$$

where $a \leq 0$ is a number, the operator $A_1 A$ satisfies the following conditions, the totality of which we denote by (A): A and A_1 are linear unbounded operators, with dense domains of definition $D(A) \subset H$ and $D(A_1) \subset H$, respectively; the conjugate operator $A^* A_1^*$ also has a dense domain of definition $D(A^* A_1^*) \subset H$. It is assumed that q and $f(t)$ are given elements from H and $L_2((0,T);H)$, respectively, and $u_1(t) \in L_2((0,T);H)$, $u_2(t) \in L_2(0,T)$ are control functions. The quadratic functional is minimized $(t_0 = 0)$

$$\mathcal{I}[t_0, u_1(\cdot), u_2(\cdot)] = a_0 \|x(T) - \xi_0\|^2 + a_1 \|\dot{x}(T) - \xi_1\|^2$$
$$+ \int_{t_0}^{T} [a_2(t) \|x(t) - \psi_0(t)\|^2 + a_3(t) \|\dot{x}(t) - \psi_1(t)\|^2$$
$$+ a_4 \|u_1(t)\|^2 + a_5 u_2^2(t)] dt, \tag{3}$$

where a_0, a_1 are given numbers, $a_2(t), a_3(t) \in L_2(0,T)$ are given non-negative functions, such that $a_0 + a_1 + a_2(t) + a_3(t) \neq 0$, a_4, a_5 are non-negative numbers satisfying $a_4 + a_5 \neq 0$, the elements $\psi_0, \psi_1 \in L_2((0,T);H)$ and $\xi_0, \xi_1 \in H$ are also specified. Moreover, if $T = \infty$, then $a_0 = a_1 = 0$.

We define the following two classes of arbitrary functions $x(t)$ $(T \leq \infty)$:

$$B((0,T);H) = \Big\{ x(t) \ : \ x \in W_2^1((0,T);H) \bigcap C^1((0,T);H);\ x(t) \in D(A), \forall t \in [0,T] \Big\},$$

$$B^*((0,T);H) = \Big\{ x(t) \ : \ x \in W_2^1((0,T);H) \bigcap C^1((0,T);H);\ x(t) \in D(A_1^*), \forall t \in [0,T] \Big\}.$$

We say that $x(t)$ is a solution to problem (1)-(2) from the energy class (e.c.), if $x \in B((0,T);H)$,

$$\lim_{t \to 0} \|x(t) - x_0\|_{H(\lambda)} = 0, \quad \lim_{t \to 0} \|\dot{x}(t) - x_1\|_H = 0$$

and the function $x(t)$ satisfies equation (1) in the sense of the integral identity

$$\Big(\dot{x}(t),\omega(t)\Big)\Big|_{t_1}^{t_2} = \int_{t_1}^{t_2}\Big[\Big(\dot{x}(t),\dot{\omega}(t)\Big)+\Big(\dot{x}(t),\omega(t)\Big)+\Big(A\dot{x}(t),A_1^*\omega(t)\Big)$$
$$+\Big(u_1(t)+qu_2(t)+f(t),\omega(t)\Big)\Big]dt,\ t_1<t_2,\ \forall\omega\in B^*\left((0,T);H\right) \qquad (4)$$

considered $\forall t_1,t_2\in[0,T]$ (if $T=\infty$, then $t_2<T$).

Let us assume that problem (1)-(2) for any $u_1\in L_2\left((0,T);H\right)$ and $u_2\in L_2(0,T)$ has a unique solution from the e.c. If the operator A_1A is such that the solution to problem (1)-(2) under any controls belongs to the e.c. only on a finite segment $[0,T]$, then problem (1)-(3) should be interpreted as the problem of finding a control such that the solution to problem (1)-(2) belonged to the e.c. and at the same time, functional (3) took a minimum value. Then problem (1)-(3), due to the strong convexity of functional (3), has a unique solution, i.e. the only pair (u_1,u_2), $u_1\in L_2\left((0,T);H\right)$, $u_2\in L_2(0,T)$ of controls that implements the minimum of functionality (3).

The problem of optimal control synthesis is to find controls $u_n^0,\ n=1,2$ that satisfy the following conditions:

(i) $\forall t\in[0,T]$ controls u_n^0 are functions of $w(t)=\{x(t),\dot{x}(t)\}$, i.e. $u_n^0=u_n^0[t]\equiv u_n^0\left(t,w(t)\right)$;

(ii) controls $u_1^0[t]\in L_2\left((0,T);H\right)$, $u_2^0[t]\in L_2(0,T)$, $\forall x\in B\left((0,T);H\right)$;

(iii) when $u_n(t)=u_n^0[t]$ problem (1)-(2) has a unique solution from $B\left((0,T);H\right)$;

(iv) on this pair (u_1,u_2) of controls $u_n(t)=u_n^0[t],\ n=1,2$, functional (3) reaches its minimum value.

Let us start solving the formulated problem using the dynamic programming method. Let us denote

$$V\equiv V\left[t,x(t),\dot{x}(t)\right]\equiv V\left[t,w(t)\right]=\min_{u_n}\mathcal{I}\left[t,u_1(\cdot),u_2(\cdot)\right],\ t\geq 0.$$

Then, by virtue of the optimality principle, we obtain $\forall t\in[0,T]$

$$V\left[t,w(t)\right] = \min_{u_n(\tau)}\Big\{\int_t^{t+\Delta t}\Big[a_2(t)\left\|x(\tau)-\Psi_0(\tau)\right\|^2+a_3(t)\left\|\dot{x}(\tau)-\Psi_1(\tau)\right\|^2$$
$$+a_4\left\|u_1(\tau)\right\|^2+a_5u_2^2(\tau)\Big]d\tau+V\left[t+\Delta t,w(t+\Delta t)\right]\Big\}. \qquad (5)$$

Note that by definition the functional V is defined $\forall t \in [0,T]$ on $B\left((0,T);H\right)$ and is a continuous function of $t \in [0,T]$. Let us assume that $\forall t \in [0,T]$ the functional V is almost strongly differentiable with respect to $w(t)$ in the norm of $H \oplus H$ and have the usual summable derivative with respect to t. Then, we have

$$V[t+\Delta t, w+\Delta w] - V[t,w] = \frac{\partial V}{\partial t}\Delta t + \Phi(t,w,\Delta w) + o_1, \tag{6}$$

where o_1 is an infinitesimal quantity depending on t and $w(t)$ in the norm of space $H \oplus H$, $\frac{o_1}{\Delta t} \to 0$ as $\Delta t \to 0$, Φ is the Frechet derivative, calculated at the point (t,w) and being a continuous functional in $H \oplus H$, i.e. $\forall t \in [0,T]$, Φ is representable in the form

$$\Phi(t,w,\Delta w) = \left(\vartheta_1(t), \Delta x(t)\right) + \left(\vartheta_2(t), \Delta \dot{x}(t)\right), \tag{7}$$

where $\vartheta_n(t) \in H$, $n = 1,2$, $\forall t \in [0,T]$. Let us additionally assume that $\vartheta_2 \in B^*\left((0,T);H\right)$. For $\left(\vartheta_2(t), \Delta\dot{x}(t)\right)$ we obtain an expression that takes into account identity (4):

$$\begin{aligned}\Big(\Delta\dot{x}(t), \vartheta_2(t)\Big) = \int_t^{t+\Delta t} \Big[\Big(\dot{x}(\tau), \dot{\vartheta}_2(\tau)\Big) + \alpha\Big(\dot{x}(t), \vartheta_2(t)\Big) + \Big(Ax(\tau), A_1^*\vartheta_2(\tau)\Big) \\ +\Big(u_1(\tau) + qu_2(\tau) + f(\tau), u_2(\tau)\Big)\Big]d\tau - \Big(\dot{x}(t+\Delta t), \Delta\vartheta_2(t)\Big). \end{aligned} \tag{8}$$

Now, passing to the limit at $\Delta t \to 0$, from (5)-(8), we obtain the following problem for determining the functional V (almost $\forall t \in [0,T]$):

$$\begin{aligned}\frac{\partial V}{\partial t} = \min_{u_n(t)} \Big[F\left(t, w(t), \vartheta(t)\right) + a_4 \left\|u_1(t)\right\|^2 + a_5 u_2^2(t) \\ +\Big(u_1(t) + qu_2(t) + f(t), \vartheta_2(t)\Big)\Big],\end{aligned}$$

$$\begin{aligned}F\left(t, w(t), \vartheta(t)\right) = \quad & \Big(\dot{x}(t), \vartheta_1(t)\Big) + \Big(Ax(t), A_1^*\vartheta_2(t)\Big) + a_2\left\|x(t) - \psi_0(t)\right\|^2 \\ & +a_3\left\|\dot{x}(t) - \psi_1(t)\right\|^2, \ \vartheta(t) = \left\{\vartheta_1(t), \vartheta_2(t)\right\},\end{aligned} \tag{9}$$

$$V\left[T, w(T)\right] = a_0\left\|x(T) - \xi_0\right\|^2 + a_1\left\|\dot{x}(T) - \xi_1\right\|^2. \tag{10}$$

Under the above assumptions, equation (9) should satisfy almost $\forall t \in [0,T]$. Equation (9) is a nonlinear equation in private functional industries. If there is an optimal triple (V^0, u_1^0, u_2^0), then in order to justify the above diagram of the dynamic programming method, it is necessary to establish execution for V^0 the following conditions:

1) $V^0[t, w(t)] > 0,\ \forall t \in [0, T],\ \forall x \in B\left((0, T); H\right)$;

2) $V^0[t, w(t)],\ \forall t \in [0, T]$ is a continuous functional on $B\left((0, T); H\right)$ and continuously depends on t;

3) $V^0[t, w(t)]$ is Frechet differentiable; $\frac{\partial V}{\partial t}$ is summable by $[0, T]$; $\vartheta_1(t) \in H,\ \forall t \in [0, T],\ \vartheta_2 \in B^*\left((0, T); H\right)$.

From equation (9), it is easy to determine the law of "optimal" control:

$$u_1(t) = -\frac{1}{2a_4}\vartheta_2(t), \quad u_2(t) = -\frac{1}{2a_5}(q, \vartheta_2(t)). \tag{11}$$

2 Systems of operator equations and methods for their solutions

We search the solution of problem (9)-(10) in the form

$$V\left[t, w(t)\right] = \left(K(t)w(t), w(t)\right)_{H\oplus H} + \left(\varphi(t), w(t)\right)_{H\oplus H} + \eta(t), \tag{12}$$

where $\eta(t)$ is a scalar function, the operator matrix $K(t)$ and vector $\varphi(t)$ have the form:

$$K(t) = \left(\begin{pmatrix} K_{11}(t) \\ K_{12}(t) \end{pmatrix} \quad \begin{pmatrix} K_{12}(t) \\ K_{22}(t) \end{pmatrix}\right), \quad \varphi(t) = \left\{\varphi_1(t), \varphi_2(t)\right\}.$$

It is assumed that $\forall t \in [0, T]$ the operators $K_{ij},\ i, j = 1, 2$ are self-adjoint in H and $K_{11}(t) > 0,\ K_{22}(t) > 0.$ Since the calculations given below are formal, we do not clarify the smoothness of operators $K_{ij}(t)$ and functions for now $\varphi(t),\ \eta(t).$ According to formulas (6) and (7), we easily find

$$\vartheta(t) = 2K(t)w(t) + \varphi(t). \tag{13}$$

Substituting the values for $\frac{\partial V}{\partial t}$ and $\vartheta(t)$ from (12) and (13) into (9) and (10), we obtain the following systems of differential operator Riccati equations and linear equations ($\forall x \in D(A),\ b_i = a_5^{-1} q\left(q, K_{i2}x\right),\ i = 1, 2;\ K_{12}y \in D\left(A_1^*\right),\ K_{22}y \in D\left(A_1^*\right),\ \forall y \in H$)

$$\begin{cases} (K_{11}'x, y) + 2\left(Ax, A_1^* K_{12}y\right) \\ -\left(a_4^{-1}K_{12}x + b_1, K_{12}y\right) + a_2x, y) = 0, \\ (K_{12}'x, y) + \left(Ax, A_1^* K_{22}y\right) + a\left(x, K_{12}y\right) + \left(K_{12}x, y\right) \\ -\left(a_4^{-1}K_{12} + b_1, K_{22}y\right) = 0, \\ (K_{22}'x, y) + 2a\left(x, K_{22}y\right) + 2\left(K_{12}x, y\right) \\ -\left(a_4^{-1}K_{22}x + b_2, K_{22}y\right) + a_3(x, y) = 0, \end{cases} \tag{14}$$

$$(K_{11}(T)x,y)=a_0(x,y),\ \ (K_{12}(T)x,y)=0,\ \ (K_{22}(T)x,y)=a_1(x,y), \quad (15)$$

$$\begin{cases} (\varphi_1',x)+(A_1^*\varphi_2,Ax)-\left(a_4^{-1}\varphi_2,K_{12}x\right) \\ -(\varphi_2,b_1)-2a_2(\psi_0,x)+2(f,K_{12}x)=0, \\ (\varphi_2'+\varphi_1-2a_3\psi_1,y)+a(\varphi_2,y)-\left(a_4^{-1}\varphi_2,K_{22}y\right) \\ -(\varphi_2,b_2)+2(f,K_{22}y)=0, \end{cases} \quad (16)$$

$$(\varphi_1(T),x)=-2a_0(\xi_0,x)\,;\ \ (\varphi_2(T),y)=2a_1(\xi_1,y)\,,\ \forall y\in H, \quad (17)$$

$$\begin{aligned} \eta(t) &= a_0\|\xi_0\|^2+a_1\|\xi_1\|^2 \\ &\quad -\int_T^t\Big[a_2(\tau)\|\psi_0,(\tau)\|^2+a_3(\tau)\|\psi_1,(\tau)\|^2e+(f(\tau),\varphi_2(\tau)) \\ &\quad -\frac{1}{4a_4}\|\varphi_2(\tau)\|^2-\frac{1}{4a_5}(q,\varphi_2(\tau))^2\Big]d\tau. \end{aligned} \quad (18)$$

Thus, to determine the optimal pair (u_1,u_2), one first needs to solve the nonlinear problem (14)-(15), and then with obtained K_{12} and K_{22} solve the linear problem (16)-(17). With known operators $K_{ij}(t)$ and functions $\varphi_i(t)$ we find the required vector $\vartheta(t)$. Finally, substituting the already found value $\vartheta_2(t)$ from (13) into (11), we obtain the law of the synthesizing optimal pair of (u_1,u_2) controls.

In system (14)-(18), the main difficulty is the choice of a function space and the study of the regularity of the operator $K(t)$ in it. In this paper, using the method of spectral decomposition of non-self-adjoint operators, explicit representations of the operator $K(t)$ are constructed, its properties and related issues of optimal control synthesis are studied. If $f=\psi_0=\psi_1=\xi_0=\xi_1=0$, then formally we assume that $\eta(t)=\varphi_i(t)=0$, $i=1,2$. Consequently, system (14)-(15) must be solved independently of the others.

3 Application of the spectral decomposition method

Additionally, we propose that the eigen and associated elements (e.a.e.) x_h^k of the formal operator A_1A satisfy the equations

$$\left(Ax_h^k,A_1^*y\right)+\left(\lambda_k^2x_h^k+x_{h-1}^k,y\right)=0,\quad \forall x_h^k\in D(A),\quad \forall y\in D\left(A_1^*\right),$$

where is $k=1,2,\ldots$ and $h=0,\ldots,m_k-1$, $\sup_k m_k<\infty$, with m_k being the multiplicity of the proper element x_0^k and it is assumed that $x_i^k=0$ for $i<0$, $k=1,2,\ldots$; eigenvalues λ_k^2 are real and $\lambda_k\geq 0$, $k=1,2,\ldots$;

$\lambda_k \to \infty$ as $k \to \infty$; e.a.e. ϑ_ν^l conjugate operator $A^* A_1^*$ satisfy the following relations: $\vartheta_\nu^e \in D\left(A_1^*\right)$ $\left(\vartheta_i^e = 0, i < 0, e = 1, 2, \ldots\right)$,

$$(Ax, A_1^*\vartheta_\nu^e) + \left(x, \lambda_k^2\vartheta_\nu^e + \vartheta_{\nu-1}^e\right) = 0, \quad \forall x \in D(A);$$

and the system $\left(x_h^k, \vartheta_\nu^e\right)$ is biorthogonal and forms a Riesz basis in H. If $A_1^* = A^* = A$, then A^2 the operator is considered to be itself conjugate, $k = 1, 2, \ldots$ its eigenvectors φ^k form a complete orthonormal system.

Let us assume that $A_1^* = A^* = A$, $q = a_5 = a = 0$, then we search the solution to system (14)-(15) in the form

$$K(t) = \sum_{k=1}^{\infty} S_k(t)\varphi^k \otimes \varphi^k, \quad K(t) = \|K_{ij}(t)\|, \quad K_{21}(t) = K_{12}(t),$$

$$S_k(t) = \begin{pmatrix} \alpha_k(t) & \beta_k(t) \\ \beta_k(t) & \gamma_k(t) \end{pmatrix}, \quad i, j = 1, 2. \tag{19}$$

Then, assuming $x = \varphi^k$, $y = \varphi^l$, from (14) and (15) we obtain the following Cauchy problem for a countable system of ordinary differential equations of Riccati type, after replacing the variable $t \to T - t$ which (we keep the previous notations for the unknowns α_k, β_k, γ_k) has the form:

$$\begin{cases} \alpha_k'(t) + 2\lambda_k^2\beta_k(t) + a_4^{-1}\beta_k^2(t) - a_2(t) = 0 \\ \beta_k'(t) + \lambda_k^2\gamma_k(t) - \alpha_k(t) + a_4^{-1}\gamma_k(t)\beta_k(t) = 0 \\ \gamma_k'(t) - 2\beta_k(t) + a_4^{-1}\gamma_k^2(t) - a_3(t) = 0, \end{cases} \tag{20}$$

$$\alpha_k(0) = a_0, \quad \beta_k(0) = 0, \quad \gamma_k(0) = a_1. \tag{21}$$

The local solvability of the system of nonlinear equations (20) is proven. Let us introduce a Banach space $mC[0, T]$ over a set of arbitrary ones with continuous $[0, T]k$ vector functions $(k = 1, 2, \ldots,\ h = 0, \ldots, m_k - 1)$
$z(t) = \left\{z_h^k(t)\right\} = \left\{\alpha_h^k(t), \beta_h^k(t), \gamma_h^k(t)\right\}$ such that

$$\|z\|_{mc} \equiv \sup_{k,h} \max_t \left|\lambda_k^{-2}\alpha_h^k(t)\right| + \sup_{k,h} \max_t \left|\lambda_k^{-1}\beta_h^k(t)\right| + \sup_{k,h} \max_t \left|\gamma_h^k(t)\right| < \infty.$$

Moreover, if $m_k = 1$, then $h = 0$. Let us put $z(t) = \left\{z_h^k(t)\right\}$,

$$z(t) = \left\{\alpha_h^k(t), \beta_h^k(t), \gamma_h^k(t)\right\}, \ \alpha_k(t) = \alpha_0^k(t),$$

$$\beta_k(t) = \beta_0^k(t), \ \gamma_k(t) = \gamma_0^k(t).$$

Let $\left\{\varphi_h^k\right\}$ be a complete orthonormal system in H. By $H\left(\lambda^\alpha\right)$ we denote the Hilbert space of all elements x defined by series of the form $x = \sum_{k=1}^{\infty}\sum_h \lambda_k^\alpha a_h^k \varphi_h^k$, $\forall a = \left\{a_h^k\right\} \in l_2^\alpha$. For any two elements x and y from $H(\lambda)$ we set

$$(x,y)_{H(\lambda^\alpha)} = \sum_{k=1}^{\infty}\sum_h \lambda_k^{2\alpha} a_h^k b_h^k = (a,b)_{l_2^\alpha}; \quad \|x\|^2_{H(\lambda^\alpha)} = (x,x)_{H(\lambda^\alpha)}.$$

The following has been proven.

Theorem 3.1. *Let the conditions (A) hold and $A_1^* = A^* = A$, $q = a = a_5 = 0$, $T < \infty$. Then there is an interval $[T_0, T] \subseteq [0,T]$ in which the problem (14)-(15) has a unique positive-definite solution $K(t) = \|K_{ij}(t)\|$ in $H \oplus H$ represented by the series from (19). The indicated series for $K_{11}(t)$, $K_{12}(t)$ and $K_{22}(t)$ converge uniformly in $t \in [T_0,T]$, respectively, in the norms $\mathcal{L}\left(H\left(\lambda^2\right), H\right)$, $\mathcal{L}\left(H(\lambda), H\right)$ and $\mathcal{L}(H,H)$, their sums are self-adjoint operators in H; $K_{11}(t)$, $K_{22}(t)$, $\forall t \in [T_0,T]$ are positive definite in H. The relations are valid $\Delta = [T_0,T]$,*

$$K_{11}(t),\ K_{12}(t) \notin \mathcal{L}(H,H),\ K_{11}(t) \in C\left(\Delta; \mathcal{L}\left(H\left(\lambda^2\right), H\right)\right),$$

$$K_{12}(t) \in C\left(\Delta; \mathcal{L}\left(H(\lambda), H\right)\right),$$

$$K_{22}(t) \in C\left(\Delta; \mathcal{L}(H,H)\right),\ K(t) \in C\left(\Delta; \mathcal{L}\left(H\left(\lambda^2\right) \oplus H(\lambda), H \oplus H\right)\right),$$

$$K_{11}'(t) \in L_2\left(\Delta; \mathcal{L}\left(H\left(\lambda^3\right), H\right)\right),\ K_{12}'(t) \in L_2\left(\Delta; \mathcal{L}\left(H\left(\lambda^2\right), H\right)\right),$$

$$K_{22}'(t) \in L_2\left(\Delta; \mathcal{L}\left(H(\lambda), H\right)\right),$$

$$K'(t) \in L_2\left(\Delta; \mathcal{L}\left(H\left(\lambda^3\right) \oplus H\left(\lambda^2\right), H \oplus H\right)\right).$$

In the particular case, important in practice, when in the problem $a_2(t) = a_3(t) = 0$, the solution to problem (2.18)-(2.19) is obtained in the explicit form:

$$\alpha_k(t) = \frac{z_k(t)}{\Delta_k(t)}, \quad \beta_k(t) = -\frac{y_k(t)}{\Delta_k(t)}, \quad \gamma_k(t) = \frac{x_k(t)}{\Delta_k(t)}, \tag{22}$$

$$z_k(t) = \left(\frac{\lambda_k^2}{a_0} - \frac{1}{a_1}\right)\sin^2\lambda_k(T-t) + \frac{1}{4a_4\lambda_k}\sin 2\lambda_k(T-t) + \frac{T-t}{2a_4} + a_1^{-1} > 0,$$

$$y_k(t) = \frac{1}{2\lambda_k}\left(\frac{\lambda_k^2}{a_0} - \frac{1}{a_1}\right)\sin 2\lambda_k(T-t) - \frac{1}{2a_4\lambda_k^2}\sin^2\lambda_k(T-t),$$

$$x_k(t) = \frac{T-t}{2a_4\lambda_k^2} - \frac{1}{\lambda_k^2}\left(\frac{\lambda_k^2}{a_0} - \frac{1}{a_1}\right)\sin^2\lambda_k(T-t) - \frac{1}{4a_4\lambda_k^3}\sin 2\lambda_k(T-t) + \frac{1}{a_0} > 0,$$

$$\Delta_k(t) \equiv x_k(t)z_k(t) - y_k^2(t) = \frac{1}{4a_4\lambda_k^3}\left(\frac{\lambda_k^2}{a_0} - \frac{1}{a_1}\right)\sin 2\lambda_k(T-t)$$

$$+\lambda_k^{-2}\left(\frac{\lambda_k^2}{a_0} + \frac{T-t}{2a_4}\right)\left(\frac{T-t}{2a_4} + \frac{1}{a_0}\right) - \frac{1}{4a_4\lambda_k^4}\sin^2\lambda_k(T-t) > 0,$$

$$a_k(t)\gamma_k(t) - \beta_k^2(t) \equiv 1 > 0, \quad k = 1, 2, \ldots$$

Using the above mentioned method for solving the problem (20)-(21), an explicit solution to the system of Riccati equations can also be constructed in the case when $q \neq 0$, $a_2(t) = a_3(t) = 0$. We will not dwell on this here.

Note that for the operators $S(t)$and $K(t)$, formed using the solution (22) of problem (14)-(15), when $a_2(t) = a_3(t) = 0$ asserting the theorem, are valid throughout the entire space $mC[0,T]$, which shows the naturalness of the introduced space $mC[0,T]$ for the solvability of problem (14)-(15).

In the case when the operator A_1A is not self-adjoint and its s.e. forms the Riesz basis, the application of the spectral decomposition method to the solution of problems (14)-(18) requires a special construction, but in this case it is possible to prove similar theorems on the solvability of the Riccati operator equations, and finally, it is possible to substantiate the dynamic programming method.

4 Application to the problem of optimal design of a circular arch

Let us consider a curved thin rod of constant cross-section, the axis of which is an arc with radius a. The rod is subject to uniform unilateral external pressure p; α is bending rigidity of the rod [3]. Then, the dynamics of displacement $u = u(t,x)$ of a particle of a curved thin rod (convex circular arch) in the presence of an additional external force $F(t,x)$ can be descibed as:

$$u_{tt} - Au = q_1(x)p_1(t) + q_2(x)p_2(t) + f(t,x) \equiv F(t,x), \tag{23}$$

where A is a sixth-order differential operator defined as $Au \equiv u^{(6)} + \alpha_1 u^{(4)} + \alpha_2 u''$, $\alpha_1 = \frac{2}{a^2} + \frac{pa}{\alpha}$, $\alpha_2 = \frac{1}{a^4} + \frac{p}{\alpha a}$. The initial and boundary conditions are

$$\begin{cases} u(0,x) = \varphi_1(x), \quad u_t(0,x) = \varphi_2(x), \\ u(t,0) = u_x(t,0) = u_{xxx}(t,0) = 0, \quad x = 0, \\ u(t,l) = u_x(t,l) = u_{xx}(t,l) = 0, \quad x = l. \end{cases} \tag{24}$$

An investigation shows that the operator A associated with the boundary value problem (23)-(24) is a self-adjoint operator in $L_2(0,l)$ and has in it a linearly independent orthonormal system of basis functions corresponding to the eigenvalues. Assuming $v = e^{\lambda x}$, $v^{(k)} = \lambda^k e^{\lambda x}$, $k = 0, 1, \ldots, 6$, from boundary value problem $Av + \mu^6 v = 0$, $\alpha_1 > 0$, $\alpha_2 > 0$, $v(0) = v'(0) = v'''(0) = 0$; $v(l) = v'(l) = v''(l) = 0$ we obtain the following characteristic equation for λ:

$$\lambda^6 + \alpha_1 \lambda^4 + \alpha_2 \lambda^2 + \mu^6 = 0. \tag{25}$$

Let us put $\lambda^2 = \gamma$, $\gamma = \nu - \frac{\alpha_1}{3}$. Then, equation (25) is reduced to the following cubic equation:

$$\nu^3 + p\nu + q = 0, \ p = \alpha_2 - \frac{\alpha_1^2}{3}, \ q = \frac{2\alpha_1^3}{27} - \frac{\alpha_1 \alpha_2}{3} + \mu^6, \tag{26}$$

where positive numbers α_1 and α_2 are determined from equation (23). Now, using the Cardan formula, we can write out the solution to equation (26):

$$\nu = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}.$$

If: 1) $D = -108\left(\frac{q^2}{4} + \frac{p^3}{27}\right) < 0$, then equation (26) has one real and two conjugate complex roots; 2) $D = 0$, then all roots are real, and two of them are equal to each other; 3) $D > 0$, then equation (26) has three distinct real roots.

Since μ is an eigenvalue of the operator A and $\mu_k \to \infty$ as $k \to \infty$, we can assume that after a certain index k the discriminant $D < 0$. Then, equation (26) has one real root and two mutually conjugate complex roots. However, note that the expression $\frac{q^2}{4} + \frac{p^3}{27}$ and number q after some index k are always positive, then the corresponding root of equation (26) will be a negative real number. The other two roots are mutually conjugate complex. On the other hand, $\gamma = \nu - \frac{\alpha_1}{3}$ then γ, corresponding to negative ν will also be negative, and complex conjugate roots will correspond to complex conjugate roots. So we conclude that one of the roots γ is negative, the other two are complex conjugate.

Now from the substitution $\lambda^2 = \gamma$ it follows that for negative γ we have: $\lambda = \pm i\sqrt{-\gamma}$. Consequently, all roots λ are complexly conjugate. Thus, we obtain three series of complex conjugate roots of the characteristic equation (25). Let us denote them by $\lambda_{nk} = \xi_{nk} + i\theta_{nk},\ n = 1,2,3;\ k = 1,2,3,\ldots$. The solution of the differential equation $Av + \mu^6 v = 0$ corresponding to λ_{nk} has the form:

$$v_k(x) = e^{\xi_{1k}}\left(C_{1k}\cos\theta_{1k}x + C_{2k}\sin\theta_{1k}x\right)$$

$$+e^{\xi_{2k}}\left(C_{3k}\cos\theta_{2k}x + C_{4k}\sin\theta_{2k}x\right) + e^{\xi_{3k}}\left(C_{5k}\cos\theta_{3k}x + C_{5k}\sin\theta_{3k}x\right).$$

Constant parameters C_{mk}, $m = 1,2,3,4,5,6$ are determined from the boundary conditions.

In the problem (23)-(24), we take as control functions $p_1(t)$, $p_2(t)$, $f(t,x)$. The functions $q_1(x)$ and $q_2(x)$ on the right side of (23) are considered given and characterize the shape (geometric) of external forces acting on the arch along the x-axis. The function $f(t,x)$ expresses an arbitrary external force. Note that in many control problems with a boundary (inhomogeneous boundary conditions), using a special substitution the problem is reduced to a homogeneous one, but with the right-hand side of the type of $F(t,x)$, such images we can assume that the case when control is carried out from the boundary is also considered. The integral is taken as an optimality criterion:

$$I\left[t_0, p_1(t), p_2(t), f(t,\cdot)\right] = \int_{t_0}^{T}\int_0^l \left[\alpha_1 u^2 + \alpha_2 u_t^2 + \beta_0 f^2(t,x)\right] dxdt$$

$$+\int_{t_0}^{T}\left[\beta_1 p_1^2(t) + \beta_2 p_2^2(t)\right] dt, \quad \alpha_1^2 + \alpha_2^2 \neq 0,\ \beta_0^2 + \beta_1^2 + \beta_2^2 \neq 0. \qquad (27)$$

Required to find control functions $f(t,x) = f(t,w)$, $p_1(t) = p_1(t,w)$, $p_2(t) = p_2(t,w)$ as a vector function of the state $w = w(t,x) = \{u(t,x), u_t(t,x)\}$ of the solution to problem (23)-(24) and such that the functional (27) takes the minimum possible value ($T - fixed$).

Problem (23), (24), (27) is a special case of problem (1)-(4), therefore its solution is obtained from the above diagram.

In structures of sufficiently large height or length, determining the parameters of stable modes and studying a model for the optimal design of a circular arch are an important task of modern applied science.

Bibliography

[1] M. Rahimov, Optimal Modeling of Heat Transfer and Vibration Processes. Methods of Dynamic Programming and Spectral Decomposition, Scientific Monograph, LAP, LAMBERT Academic Publishing, ISBN: 978-620-3-30910-2.

[2] A.I. Egorov, Riccati Equations, M. Fizmatlit, 2001. −319 p.

[3] L. Collatz, Eigenvalue problems, “Science”, main editorial office of physical and mathematical literature, Moscow, 1968.

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