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Numerical algorithm for solving parabolic identification problem with mixed boundary condition

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Abstract. A source identification problem for a parabolic equation with mixed boundary condition is studied. Stability estimates for the solution of identification problem with mixed boundary conditions are obtained. Numerical algorithms for solving this inverse problem are proposed. Stability estimates for difference schemes are established. The numerical result in test example is presented.

Keywords. Mixed boundary condition, source identification problem, difference schemes, stability, well-posedness, stability estimates.

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1 Introduction

Source identification problems (SIPs) for parabolic equations (PEs) are fundamental to the success and accuracy of modeling efforts for many real processes. They underpin the ability to understand, predict, and manage complex systems and phenomena effectively.

Methods of solving SIPs and approximations of nonlocal problems for PEs were investigated intensively by several authors (see [1–21, 23] and references therein).

In this paper, we study source identification problem for multi-dimensional PE. Let $\Omega = (0,1) \times (0,1) \times \cdots \times (0,1)$ with boundary $S = S_1 \cup S_2$, $\overline{\Omega} = \Omega \cup S$, where

$$\begin{aligned}
&S = \{y = (y_1, \cdots, y_n) \mid y_i = 0 \text{ or } y_i = 1,\ 0 \le y_s \le 1,\ s \ne i,\ 1 \le i \le n\}, \\
&S_1 = \{y = (y_1, \cdots, y_n) \mid y_i = 0,\ 0 \le y_s \le 1,\ s \ne i,\ 1 \le i \le n\}, \\
&S_2 = \{y = (y_1, \cdots, y_n) \mid y_i = 1,\ 0 < y_s \le 1,\ s \ne i,\ 1 \le i \le n\}.
\end{aligned}$$

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Let $L_2(\Omega)$ and $W_2^2(\Omega)$ be the Hilbert spaces of integrable functions $u(y)$, defined on Ω, equipped with the suitable norms

$$\|u\|_{L_2(\Omega)} = \left\{ \int\limits_{y\in\Omega} |u(y)|^2 \, dy_1 \dots dy_n \right\}^{\frac{1}{2}},$$
$$\|u\|_{W_2^2(\Omega)} = \left\{ \int\limits_{y\in\Omega} \left(|u(y)|^2 + \sum_{i=1}^{n} \sum_{j=1}^{n} \left| u_{y_i y_j}(y) \right|^2 \right) dy_1 \dots dy_n \right\}^{\frac{1}{2}}.$$

Assume that $\varphi \in L_2(\Omega)$, $\psi \in W_2^2(\Omega)$, $f \in C^{\alpha}(L_2(\Omega))$ are given functions, and $a_i : \Omega \to R^+,\ i = 1, \dots, n$ are known smooth functions.

In the region $[0, 1] \times \Omega$, we study the following SIP for multi-dimensional PE with mixed boundary conditions

$$\begin{cases} u_t(t,x) - \sum\limits_{i=1}^{n} (a_i(x) u_{x_i}(t,x))_{x_i} + \sigma u(t,x) = f(t,x) + p(x), \\ x = (x_1, \cdots, x_n) \in \Omega,\ 0 < t < 1, \\ \frac{\partial}{\partial \overrightarrow{n}} u(t,x) = 0,\ x \in S_1,\ u(t,x) = 0,\ x \in S_2,\ 0 < t < 1, \\ u(0,x) = \sum\limits_{k=1}^{r} \mu_k u(s_k, x) + \psi(x),\ u(1,x) = \varphi(x),\ x \in \overline{\Omega}, \end{cases} \tag{1}$$

where $\overrightarrow{n}$ is the normal vector to Ω at the corresponding boundary point.

The differential expression

$$A^x v(x) = -\sum_{i=1}^{n} (a_i(x) v_{x_i}(x))_{x_i} + \sigma v(x)$$

defines the self-adjoint positive definite (SAPD) operator A^x, acting on the Hilbert space $L_2(\Omega)$, with the domain

$$D(A^x) = \left\{ v \mid v \in W_2^2(\Omega),\ \frac{\partial v}{\partial \overrightarrow{n}}(x) = 0 \text{ on } S_1,\ v(x) = 0 \text{ on } S_2 \right\}.$$

Therefore, the SIP (1) for the multi-dimensional PE can be reduced to the abstract problem (5), (7), (8) in paper [12] for $H = L_2(\Omega)$. By using stability estimates of Theorem 1.1, we can formulate the following theorem on stability of SIP (1).

Theorem 1.1. *Let $s_1, \mu_1, s_2, \mu_2, ..., s_r, \mu_r$ be given numbers so that*

$$\sum_{k=1}^{r} |\mu_k| < 1,\ 0 \le s_1 < s_2 < \dots < s_r < 1.$$

Assume that $\varphi,\psi\in W_2^2(\Omega)$ and $f\in C^{\alpha}(L_2(\Omega))$ are given. Then, for the solution of SIP (1) for multi-dimensional PE, the following stability estimates hold

$$\|p\|_{L_2(\Omega)}\le M\left[\|\varphi\|_{W_2^2(\Omega)}+\|\psi\|_{W_2^2(\Omega)}+\frac{1}{\alpha}\|f\|_{C^{\alpha}(L_2(\Omega))}\right],$$

$$\|v\|_{C(L_2(\Omega))}\le M\left[\|\varphi\|_{L_2(\Omega)}+\|\psi\|_{L_2(\Omega)}+\|f\|_{C(L_2(\Omega))}\right],$$

where positive number M does not depend on f, ψ, φ and α.

2 Difference schemes

We will use the set of uniform grid points

$$[0,1]_\tau=\{t_k=k\tau,\ k=0,1,\cdots,N,\ N\tau=1\}.$$

To discretize the problem (1) we use an algorithm with two steps. Firstly, we define grid spaces

$$\begin{aligned}&\widetilde{\Omega}_h=\{x=x_m=(h_1m_1,\cdots,h_nm_n);\ m=(m_1,\cdots,m_n),\\&\qquad m_j=0,\cdots,N_j,\ h_jN_j=1,\ j=1,\cdots,n\},\\&\Omega_h=\widetilde{\Omega}_h\cap\Omega,\ S_1^h=\widetilde{\Omega}_h\cap S_1,\ S_2^h=\widetilde{\Omega}_h\cap S_2.\end{aligned}$$

Let us introduce the difference operator A_h^x by formula

$$A_h^xv^h(x)=-\sum_{i=1}^{n}\left(a_i(x)v_{\overline{x}_i}^h(x)\right)_{x_i,j_i}+\sigma v^h(x), \tag{2}$$

which acts in space of grid functions $v^h(x)$ and satisfies the condition $v^h(x)=0$ for all $x\in S_2^h$ and $Dv^h(x)=0$ for all $x\in S_1^h$.

Applying A_h^x, we arrive at the multi-point nonlocal boundary value problem (BVP) for some infinite system of ordinary differential equations. Secondly, by using [12, Eqn. (26), p.1922], we get the first order of accuracy difference scheme (ADS)

$$\begin{cases}\tau^{-1}\left(v_k^h(x)-v_{k-1}^h(x)\right)+A_h^xv_k^h(x)=f^h(t_k,x)+p^h(x),\\ 1\le k\le N,\ x\in\widetilde{\Omega}_h,\\ v_N^h(x)=\varphi^h(x),\ v_0^h(x)=\sum\limits_{i=1}^{r}\mu_iv_{l_i}^h(x)+\psi^h(x),\ x\in\widetilde{\Omega}_h.\end{cases} \tag{3}$$

Applying [12, Eqns. (37),(38),(39), p.1925], we get the second order of ADS

$$\begin{cases} \tau^{-1}\left(v_k^h(x) - v_{k-1}^h(x)\right) + A_h^x\left(I + \frac{\tau A_h^x}{2}\right)v_k^h(x) \\ \quad = \left(I + \frac{\tau A_h^x}{2}\right)\left(f^h(t_{k-\frac{\tau}{2}}, x) + p^h(x)\right),\ 1 \le k \le N,\ x \in \widetilde{\Omega}_h, \\ v_N^h(x) = \varphi^h(x),\ x \in \widetilde{\Omega}_h, \\ v_0^h(x) = \sum_{i=1}^{r}\left\{\mu_i(1-\rho_i)v_{l_i}^h(x) + \mu_i\rho_i v_{l_i+1}^h(x)\right\} + \psi^h(x),\ x \in \widetilde{\Omega}_h. \end{cases} \tag{4}$$

Denote by $L_{2h} = L_2(\widetilde{\Omega}_h)$ and $W_{2h}^2 = W_2^2(\widetilde{\Omega}_h)$ the spaces of the grid functions $u^h(x) = \{u(h_1m_1, \cdots, h_nm_n)\}$ defined on $\widetilde{\Omega}_h$, equipped with the corresponding norms

$$\left\|u^h\right\|_{L_{2h}} = \left(\sum_{x\in\widetilde{\Omega}_h} |u^h(x)|^2 h_1\cdots h_n\right)^{1/2},$$

$$\left\|u^h\right\|_{W_{2h}^2} = \left\|u^h\right\|_{L_{2h}} + \left(\sum_{x\in\widetilde{\Omega}_h}\sum_{r=1}^{n}\left|(u^h(x))_{x_r\overline{x_r},m_r}\right|^2 h_1\cdots h_n\right)^{1/2},$$

and by $\mathcal{C}_\tau(L_{2h}) = \mathcal{C}([0,1]_\tau, L_{2h})$ the Banach space of L_{2h}-valued grid functions $u^\tau = \{u_k\}_1^N$ with the suitable norm $\|u^\tau\|_{\mathcal{C}_\tau(L_{2h})} = \max_{1\le k\le N}\|u_k\|_{L_{2h}}$.

Denote by $C([0,1]_\tau, L_{2h})$ the linear space of grid functions $w^\tau = \{w_k\}_1^N$ with values in the Hilbert space L_{2h}, and by $C_\tau(H) = C([0,1]_\tau, H)$, $C_\tau^\alpha(H) = C^\alpha([0,1]_\tau, H)$ the Banach spaces of bounded grid functions with the norms

$$\|w^\tau\|_{C_\tau(L_{2h})} = \max_{1\le k\le N}\|w_k\|_{L_{2h}},$$

$$\|w\|_{C_\tau^\alpha(L_{2h})} = \|w^\tau\|_{C_\tau(L_{2h})} + \max_{1\le k<k+r\le N}\frac{\|w_{k+r} - w_k\|_{L_{2h}}}{(r\tau)^\alpha}.$$

Theorem 2.1. *Suppose that τ and $|h| = \sqrt{h_1^2 + \cdots + h_n^2}$ are sufficiently small positive numbers, $\varphi^h \in L_{2h}$, $\psi^h \in W_{2h}^2$ and $\left\{f_k^h\right\}_1^N \in \mathcal{C}_\tau^\alpha(L_{2h})$. Then, for the solution of DSs (3) and (4), the following stability estimates hold*

$$\left\|p^h\right\|_{\mathcal{C}_\tau(L_{2h})} \le M\left[\left\|\varphi^h\right\|_{L_{2h}} + \left\|\psi^h\right\|_{W_{2h}} + \frac{1}{\alpha}\left\|\left\{f_k^h\right\}_1^N\right\|_{\mathcal{C}_\tau^\alpha(L_{2h})}\right],$$

$$\left\| \left\{ v_k^h \right\}_1^N \right\|_{C_\tau(L_{2h})} \le M \left[\left\| \varphi^h \right\|_{L_{2h}} + \left\| \psi^h \right\|_{L_{2h}} + \left\| \left\{ f_k^h \right\}_1^N \right\|_{C_\tau(L_{2h})} \right],$$

where M is independent of $\left\{ f_k^h \right\}_1^N$, $\psi^h(x)$, $\varphi^h(x)$ and τ.

The proof of Theorem 2.1 is based on Theorems 3.1 and 3.2 of paper [12] on stability estimate for solutions of corresponding DSs for approximate solution of abstract SIP (5), (7), (8) and the theorem on the coercivity inequality for the solution of the elliptic difference problem in L_{2h} (see [22]).

3 Numerical Algorithm

For test example, we consider the SIP

$$\begin{cases} v_t(t,x) - (1+x^2)v_{xx}(t,x) - 2x \cdot v_x(t,x) + v(t,x) \\ \quad = f(t,x) + p(x),\ 0 < x < \pi,\ 0 < t < 1, \\ v(1,x) = \varphi(x),\ v(0,x) = v(0.3,x) + \psi(x),\ 0 \le x \le \pi, \\ v(t,0) = 0,\ v_x(t,\pi) = 0,\ 0 \le t \le 1 \end{cases} \tag{5}$$

for one-dimensional parabolic PDE. Here,

$$\begin{aligned} &f(t,x) = \left(e^{-t} - e^{-1}\right)\left((1+x^2)\cos x + 2x \sin x\right) + e^{-1}(\cos x + 1), \\ &\quad 0 < x < \pi,\ 0 < t < 1, \\ &\varphi(x) = e^{-1}\left(\cos x + 1\right),\ \psi(x) = \left(1 - e^{-0.3}\right)\left(\cos x + 1\right),\ 0 \le x \le \pi. \end{aligned}$$

It is easy to check that the pair of functions

$$\left\{ e^{-1}\left((1+x^2)\cos x + 2x\sin x + \cos x + 1\right),\ e^{-t}(\cos x + 1) \right\}$$

satisfies SIP (5).

An algorithm of finding the solution of problem (5) consists of three stages. In the first stage, we search the solution of SIP in the form

$$v(t,x) = u(t,x) + (A^x)^{-1}\left(p(x)\right) + v(1,x),$$

where $u(t,x)$ is solution of the following auxiliary nonlocal BVP

$$\begin{cases} u_t(t,x) - (1+x^2)u_{xx}(t,x) + 2x \cdot u_x(t,x) + u(t,x) \\ = e^{-1}\left((1+x^2)\cos x + 2x\sin x + \cos x + 1\right) + f(t,x), \\ 0 < x < \pi,\ 0 < t < 1, \\ u(1,x) - u(0.3,x) = \psi(x),\ 0 \le x \le \pi, \\ u_x(t,0) = 0,\ u_x(t,\pi) = 0,\ 0 \le t \le 1. \end{cases} \tag{6}$$

Then, in the second stage, we find $p(x)$ by

$$p(x) = e^{-1}\left((1+x^2)\cos x + 2x\sin x + \cos x + 1\right).$$

In the third stage, we put $p(x)$ in the right side of equation (5) and solve that problem for $v(t,x)$.

We introduce the set of grid points

$$[0,1]_\tau \times [0,\pi]_h = \{(t_k, x_n) \mid t_k = k\tau,\ k = 1,\cdots,N-1,\ N\tau = 1,\\ x_n = nh,\ n = 1,\cdots,M-1,\ Mh = \pi\}.$$

We use notation $l = \left[\frac{\gamma}{\tau}\right]$ for greatest integer value of $\frac{\gamma}{\tau}$ and $\rho = \frac{\gamma}{\tau} - l$.

So, we get the first order of ADS for SIP (5)

$$\begin{cases} \frac{v_n^k - v_n^{k-1}}{\tau} - \frac{(1+x_n^2)\left(v_{n+1}^k - 2v_n^k + v_{n-1}^k\right)}{h^2} - \frac{x_n\left(v_{n+1}^k - v_{n-1}^k\right)}{h} + v_n^k \\ \quad = f(t_k, x_n) + p(x_n),\ k = 1,\cdots,N,\ n = 1,\cdots,M-1, \\ v_n^N = \varphi_n,\ v_n^0 - v_n^l = \psi_n,\ n = 0,\cdots,M, \\ v_0^k = v_1^k,\ v_M^k = 0,\ k = 0,\cdots,N. \end{cases} \tag{7}$$

Then, $p(x_n)$ can be obtained by

$$p(x_n) = -\frac{(1+x_n^2)\left(u_{n+1}^N - 2u_n^N + u_{n-1}^N\right)}{h^2} - \frac{x_n\left(u_{n+1}^N - u_{n-1}^N\right)}{h} + u_n^N, \tag{8}$$

where $\left\{u_n^k\right\}$ is the solution of the difference problem

$$\begin{cases} \frac{u_n^k - u_n^{k-1}}{\tau} - \frac{(1+x_n^2)\left(u_{n+1}^k - 2u_n^k + u_{n-1}^k\right)}{h^2} - \frac{x_n\left(u_{n+1}^k - u_{n-1}^k\right)}{h} + u_n^k \\ \quad = f(t_k, x_n) + \frac{(1+x_n^2)(\varphi_{n+1} - 2\varphi_n + \varphi_{n-1})}{h^2} + \frac{x_n(\varphi_{n+1} - \varphi_{n-1})}{h} - \varphi_n, \\ k = 1,\cdots,N,\ n = 1,\cdots,M-1, \\ u_n^0 - u_n^l = \psi_n,\ n = 0,\cdots,M, \\ u_0^k - u_M^k = 0,\ u_M^k = 0,\ k = 0,\cdots,N, \end{cases} \tag{9}$$

which is the first order of ADS for approximate solution of the nonlocal BVP (6).

For computational reasons it is convenient to write (9) in the following matrix form

$$A_n u_{n+1} + B_n u_n + C_n u_{n-1} = I\theta_n,\ n = 1,\cdots,M-1,\\ u_0 = u_1,\ u_M = \vec{0}. \tag{10}$$

Here, θ_n is a column vector, A_n, B_n, C_n are square matrices with $(N+1)$ rows and columns:

$$A_n = \begin{bmatrix} 0 & \dots & 0 & 0 \\ & & & 0 \\ & a_n R & & 0 \\ & & & 0 \end{bmatrix}, \quad C_n = \begin{bmatrix} 0 & \dots & 0 & 0 \\ & & & 0 \\ & c_n R & & 0 \\ & & & 0 \end{bmatrix},$$

$$B_n = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & -1 & \cdots & 0 & 0 & 0 & 0 \\ b_n & d & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & b_n & d & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & b_n & d & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & b_n & d \end{bmatrix},$$

where R is $N \times N$ identity matrix and

$$a_n = -(1+x_n^2)h^{-2} - x_n h^{-1}, d = \tfrac{1}{\tau},$$
$$b_n = 1 + d + 2(1+x_n^2)h^{-2}, c_n = -(1+x_n^2)h^{-2} + x_n h^{-1},$$

$$\theta_n = \begin{bmatrix} \theta_n^0 \\ \vdots \\ \theta_n^N \end{bmatrix}_{(N+1)\times 1}, \; u_{n\pm 1} = \begin{bmatrix} u_{n\pm 1}^0 \\ \vdots \\ u_{n\pm 1}^N \end{bmatrix}_{(N+1)\times 1}, \quad u_n = \begin{bmatrix} u_n^0 \\ \vdots \\ u_n^N \end{bmatrix}_{(N+1)\times 1},$$

I is the $(N+1) \times (N+1)$ identity matrix, as well as

$$\theta_n^0 = \psi_n, \; n = 1, \cdots, M-1,$$
$$\theta_n^k = f(t_k, x_n) - \tfrac{(1+x_n^2)(\varphi_{n+1} - 2\varphi_n + \varphi_{n-1})}{h^2} - \tfrac{x_n(\varphi_{n+1} - \varphi_{n-1})}{h} + \varphi_n,$$
$$k = 1, \cdots, N, \; n = 1, \cdots, M-1.$$

We search the solution of (10) by the recurrent formula ([17])

$$u_n = \alpha_{n+1} u_{n+1} + \beta_{n+1}, \; n = M-1, \cdots, 1,$$

where α_n are $(N+1) \times (N+1)$ square matrices and β_n are column vectors with $(N+1)$ elements. For the solution of difference equation (10) we use the following formulas for α_n and β_n

$$\begin{aligned} \alpha_n &= -(B_n + C_n \alpha_{n-1})^{-1} A_n, \\ \beta_n &= (B_n + C_n \alpha_{n-1})^{-1} (\theta_n - C_n \beta_{n-1}), \; n = 1, \dots, M-1, \end{aligned}$$

where $\alpha_1 = I$ and β_1 is a zero vector.

Second, applying appropriate approximation formulas for derivatives in the nonlocal BVP (5), we get the second order of ADS in t and x

$$\begin{cases} \frac{v_n^k - v_n^{k-1}}{\tau} + \frac{q_2\left(v_{n+1}^k - v_{n-1}^k\right)}{2h} + \frac{q_3\left(v_{n+1}^k - 2v_n^k + v_{n-1}^k\right)}{h^2} \\ + \frac{\tau q_0}{2}\frac{\left(v_{n+2}^k - 3v_{n+1}^k + 3v_n^k - v_{n-1}^k\right)}{h^3} + \frac{\tau q_1}{2}\frac{\left(v_{n+2}^k - 4v_{n+1}^k + 6v_n^k - 4v_{n-1}^k + v_{n-2}^k\right)}{h^4} \\ = \theta_n^k + p(x_n) - \frac{\tau}{2}\cdot\frac{(1+x_n^2)(p(x_{n+1}) - 2p(x_n) + p(x_{n-1}))}{h^2} \\ - \frac{\tau}{2}\cdot\frac{x_n(p(x_{n+1}) - p(x_{n-1}))}{h} + \frac{\tau p(x_n)}{2}, \\ k = 1, \cdots, N,\ n = 2, \cdots, M-2, \\ -3v_0^k + 4v_1^k - v_2^k = 0,\ v_M^k = 0, \\ 10v_0^k - 15v_1^k + 6v_2^k - v_3^k = 0,\ k = 0, \cdots, N, \\ v_n^N = \varphi_n,\ v_n^0 - (1-\rho)v_n^l - \rho v_n^{l+1} = \psi(x_n),\ n = 0, \cdots, M, \end{cases} \tag{11}$$

where

$$q_1^n = (1+x_n^2)^2,\ q_0^n = (1+x_n^2)(2x_n - 2)$$
$$q_3^n = -(1+x_n^2) + \frac{\tau}{2}\left(-6x_n^2 + 4x_n - 3\right),\ q_2^n = 2x_n + 5x_n\tau.$$

Then, we calculate $p(x_n)$ by using (8), with $\left\{u_n^k\right\}$ being the solution of the difference problem

$$\begin{cases} \frac{u_n^k - u_n^{k-1}}{\tau} + \frac{q_2^n\left(u_{n+1}^k - u_{n-1}^k\right)}{2h} + \frac{q_3^n\left(u_{n+1}^k - 2u_n^k + u_{n-1}^k\right)}{h^2} \\ + \frac{\tau}{2}\frac{q_0^n\left(u_{n+2}^k - 2u_{n+1}^k + 2u_{n-1}^k - u_{n-2}^k\right)}{2h^3} + \frac{\tau}{2}\frac{q_1^n\left(u_{n+2}^k - 4u_{n+1}^k + 6u_n^k - 4u_{n-1}^k + u_{n-2}^k\right)}{h^4} \\ = \theta_n^k,\ k = 1, \cdots, N,\ n = 2, \cdots, M-2, \\ -3u_0^k + 4u_1^k - u_2^k = 0,\ u_M^k = 0, \\ 10u_0^k - 15u_1^k + 6u_2^k - u_3^k = 0,\ k = 0, \cdots, N, \\ u_n^0 - (1-\rho)u_n^l - \rho u_n^{l+1} = \psi(x_n),\ n = 0, \cdots, M, \end{cases} \tag{12}$$

which is the second order of ADS for the approximate solution of the nonlocal BVP (6). For computational reasons it is convenient to rewrite this system in the matrix form

$$\begin{aligned} &A_n u_{n+2} + B_n u_{n+1} + C_n u_n + D_n u_{n-1} + E_n u_{n-2} = I\theta_n,\ n = 2, \cdots, M-2, \\ &-3u_0 + 4u_1 - u_2 = \overrightarrow{0},\ 10u_0 - 15u_1 + 6u_2 - u_3 = \overrightarrow{0},\ u_M = \overrightarrow{0}, \end{aligned} \tag{13}$$

where θ_n is a column vector, A_n, B_n, C_n, D_n, E_n are $(N+1)\times(N+1)$ square matrices, R is $N\times N$ identity matrix,

$$A_n = \begin{bmatrix} 0 & \dots & 0 & 0 \\ & & & 0 \\ & e_nR & & \vdots \\ & & & 0 \end{bmatrix},\ B_n = \begin{bmatrix} 0 & \dots & 0 & 0 \\ & & & 0 \\ & y_nR & & \vdots \\ & & & 0 \end{bmatrix},$$

$$D_n = \begin{bmatrix} 0 & \dots & 0 & 0 \\ & & & 0 \\ & z_nR & & \vdots \\ & & & 0 \end{bmatrix},\ E_n = \begin{bmatrix} 0 & \dots & 0 & 0 \\ & & & 0 \\ & w_nR & & \vdots \\ & & & 0 \end{bmatrix},$$

$$C_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & -(1-\rho) & \rho & \cdots & 0 & 0 & 0 \\ r_n & d & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & r_n & d & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & r_n & d & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & r_n & d \end{bmatrix}, \theta_n = \begin{bmatrix} \theta_n^0 \\ \vdots \\ \theta_n^N \end{bmatrix},$$

$$\begin{aligned}
e_n &= \tfrac{\tau q_0}{4h^3} + \tfrac{\tau q_1}{2h^4}, \\
y_n &= \tfrac{q_2}{2h} + \tfrac{1}{h^2}q_3 - \tfrac{\tau q_0}{2h^3} - \tfrac{2\tau q_1}{h^4}, \\
r_n &= 1 + \tfrac{1}{\tau} + \tfrac{\tau}{2} - \tfrac{2}{h^2}q_3 + \tfrac{3\tau q_1}{h^4}, \\
z_n &= -\tfrac{q_2}{2h} + \tfrac{1}{h^2}q_3 + \tfrac{\tau q_0}{h^3} - \tfrac{2\tau q_1}{h^4}, \\
w_n &= -\tfrac{\tau q_0}{4h^3} + \tfrac{\tau q_1}{2h^4},\ n = 2, \cdots, M-2.
\end{aligned}$$

We search the solution of linear system equation (13) in the form

$$u_n = \alpha_{n+1}u_{n+1} + \beta_{n+1}u_{n+2} + \gamma_{n+1},\ n = M-2, \cdots, 0,$$

where

$$\gamma_0 = \gamma_1 = \overrightarrow{0},\ \alpha_0 = \tfrac{4}{3}R,\ \beta_0 = -\tfrac{1}{3}R,\ \alpha_1 = \tfrac{8}{5}R,\ \beta_1 = -\tfrac{3}{5}R,$$

and $\alpha_n, \beta_n, \gamma_n$ are defined by recurrent formulas

$$\begin{aligned}
&F_n = \left(C_n + D_n\alpha_{n-1} + E_n\beta_{n-2} + E_n\alpha_{n-2}\alpha_{n-1}\right),\ n = 2, \cdots, M-1, \\
&\alpha_n = -F_n^{-1}\left(B_n + D_n\beta_{n-1} + E_n\alpha_{n-2}\beta_{n-1}\right),\ \beta_n = -F_n^{-1}A_n, \\
&\gamma_n = -F_n^{-1}\left(R\varphi_n - D_n\gamma_{n-1} - E_n\alpha_{n-2}\gamma_{n-1} - E_n\gamma_{n-2}\right).
\end{aligned}$$

For u_M and u_{M-1}, we have formulas

$$\begin{aligned}
&u_M = \overrightarrow{0},\ u_{M-1} = (Q_1+Q_2)^{-1}(Q_3+Q_4),\\
&Q_1 = B_{M-2} + C_{M-2}\alpha_{M-1} + D_{M-2}\left(\alpha_{M-2}\alpha_{M-1} + \beta_{M-2}\alpha_{M-1}\right),\\
&Q_2 = E_{M-2}\left(\alpha_{M-3}\left(\alpha_{M-2}\alpha_{M-1} + \beta_{M-2}\right) + \beta_{M-3}\right),\\
&Q_3 = I\theta_{M-2} - C_{M-2}\gamma_{M-1} - D_{M-2}\left(\alpha_{M-2}\gamma_{M-1} + \gamma_{M-2}\right),\\
&Q_4 = -E_{M-2}\left(\alpha_{M-3}\left(\alpha_{M-2}\gamma_{M-1} + \gamma_{M-2}\right) + \beta_{M-3}\gamma_{M-1} + \gamma_{M-3}\right).
\end{aligned}$$

Numerical illustration is carried out by using MATLAB program. Solutions of DSs are computed for different values of (N, M). v_n^k and u_n^k correspond to the corresponding numerical values of $v(t,x)$ and $u(t,x)$ at $(t,x) = (t_k, x_n)$ and p_n represents the numerical value of $p(x)$ at point $x = x_n$. The errors are computed by

$$\begin{aligned}
Ev_M^N &= \max_{0\le k\le N}\left(\sum_{n=1}^{M-1}\left|v(t_k,x_n) - v_n^k\right|^2 h\right)^{\frac{1}{2}},\\
Eu_M^N &= \max_{0\le k\le N}\left(\sum_{n=1}^{M-1}\left|u(t_k,x_n) - u_n^k\right|^2 h\right)^{\frac{1}{2}},\\
Ep_M &= \left(\sum_{n=1}^{M-1}\left|p(x_n) - p_n\right|^2 h\right)^{\frac{1}{2}}.
\end{aligned}$$

Table 1. Errors in the numerical solutions of the first order of ADS for different values of (N, M).

$N = M$	20	40	80	160
Ev_M^N	0.20745	0.09636	0.04654	0.02288
Ep_M	0.07210	0.03214	0.01518	0.00737
Ev_M^N	0.14042	0.06396	0.03061	0.01498

Tables 1 and 2 illustrate the errors between the exact and approximate solutions of DSs for various values of N and M, respectively. It can be seen from output results that the second order of ADS is more accurate than the first order of ADS. The error analysis shown in Tables 1 and 2 indicate that both DSs have correct convergence rates.

Table 2. Errors in the numerical solutions of the second order of ADS for different values of (N, M).

$N = M$	20	40	80	160
Ev_M^N	0.01902	0.00423	0.00091	0.00021
Ep_M	0.09052	0.02323	0.00678	0.00181
Ev_M^N	0.09078	0.02181	0.00417	0.00095

4 Conclusion

In this work, SIP for a multi-dimensional parabolic partial differential equation with multi-point nonlocal and mixed boundary conditions is studied. Stability estimates for solutions of inverse problem and its approximations are established. Numerical illustration is given for the simple test problem.

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