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The positivity of the differential operator generated by hyperbolic system of equations

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Abstract. In the present paper, the initial value problem for the hyperbolic system of equations with nonlocal boundary conditions is investigated. The positivity of the space operator generated by the given problem in interpolation spaces is established. The structure of interpolation spaces of this differential operator is studied. The positivity of this space operator in Slobodeckij-Sobolev spaces is established.

Keywords. Hyperbolic system of equations, nonlocal boundary value problems, interpolation spaces, positivity of the differential operator.

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1 Introduction

Various local and nonlocal boundary value problems for partial differential equations can be considered as an abstract boundary value problem for the ordinary differential equation in a Banach space with a densely defined unbounded space operator. It is well-known that the study of the various properties of partial differential equations is based on the positivity property of the differential operators in Banach spaces. The structure of fractional spaces generated by positive differential and difference operators and its applications to partial differential equations have been investigated by many researchers [1–10]. Finally, a survey of results in fractional spaces generated by positive operators and their applications to partial differential equations are given in [11].

The method of operators, as a tool for the investigation of the solution hyperbolic differential equations in Hilbert and Banach spaces, has been systematically developed by several authors (see, e.g., [12–17]).

In the paper [14], the initial value problem for the first order partial differential equation with the nonlocal boundary condition was studied. The positivity of the space operator generated by this problem in interpolation spaces was established. The structure interpolation spaces of this space operator was investigated. The positivity of this space operator in Holder spaces was established. In applications, the stability estimates for the first order partial differential equation with the nonlocal boundary condition were obtained.

In the papers [18, 19], the following initial nonlocal boundary value problem

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} + a(x)\frac{\partial u(t,x)}{\partial x} + \delta(u(t,x) - v(t,x)) = f_1(t,x), \\ 0 < x < l, \quad 0 < t < T, \\ \frac{\partial v(t,x)}{\partial t} - a(x)\frac{\partial v(t,x)}{\partial x} + \delta v(t,x) = f_2(t,x), \\ 0 < x < l, \quad 0 < t < T, \\ u(t,0) = \gamma u(t,l),\ 0 \le \gamma \le 1, \quad \beta v(t,0) = v(t,l),\ 0 \le \beta \le 1, \\ 0 \le t \le T, \\ u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad 0 \le x \le l \end{cases} \tag{1}$$

for a hyperbolic system with nonlocal boundary conditions was investigated. Here, $a(x) \ge a > 0$, $u_0(x)$, $v_0(x)$, $x \in [0,l]$, $f_1(t,x)$, $f_2(t,x)$, $(t,x) \in [0,T] \times [0,l]$ are given smooth functions and they satisfy all compatibility conditions which guarantees that the problem (1) has a smooth solutions $u(t,x)$ and $v(t,x)$.

Let E be a Banach space and $A : D(A) \subset E \to E$ be a linear operator defined in a Banach space. We call an operator A in the Banach space E positive if the operator $(\lambda I + A)$ has a bounded in E inverse and for any $\lambda \geqslant 0$ the following estimate holds:

$$\left\|(\lambda I + A)^{-1}\right\|_{E\to E} \le \frac{M}{\lambda + 1}. \tag{2}$$

Throughout the present paper, M denotes a positive constant, which may differ in time and thus is not a subject of precision. However, we will use $M(\alpha, \beta, \ldots)$ to stress the fact that the constant M depends only on $\alpha, \beta, \ldots$

For a positive operator A in the Banach space E, let us introduce the fractional spaces $E_{\alpha,1(E,A)}$ $(0 < \alpha < 1)$ consisting of those $u \in E$ for which the norm

$$\|u\|_{E_{\alpha,1(E,A)}} = \int_0^\infty \lambda^\alpha \left\|A(\lambda I + A)^{-1}u\right\|_E \frac{d\lambda}{\lambda} + \|u\|_E$$

is finite.

Let us introduce the Slobodeckij-Sobolev space as

$$\mathbb{W}_1^\alpha[0,l] = W_1^\alpha\left([0,l], R\right) \times W_1^\alpha\left([0,l], R\right), 0 \le \alpha \le 1$$

of all strongly measurable vector functions $u = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}$ defined on $[0,l]$ for which the following norm is finite

$$\|u\|_{\mathbb{W}_1^\alpha[0,l]} = \|u\|_{\mathbb{L}_1[0,l]}$$

$$+\int_{x\in[0,l]}\int_{\substack{x+\tau\in[0,l]\\ \tau\neq 0}}\frac{|u_1(x+\tau)-u_1(x)|}{|\tau|^{1+\alpha}}d\tau dx$$
$$+\int_{x\in[0,l]}\int_{\substack{x+\tau\in[0,l]\\ \tau\neq 0}}\frac{|u_2(x+\tau)-u_2(x)|}{|\tau|^{1+\alpha}}d\tau dx.$$

Here, $\mathbb{L}_1[0,l]=L_1([0,l],R)\times L_1([0,l],R)$ is the Banach space of all strongly measurable vector functions $u=\begin{pmatrix} u_1(x)\\ u_2(x)\end{pmatrix}$ defined on $[0,l]$ for which the following norm is finite

$$\|u\|_{\mathbb{L}_1[0,l]}=\int_{x\in[0,l]}|u_1(x)|dx+\int_{x\in[0,l]}|u_2(x)|dx.$$

We consider the state operator A generated by problem (1) defined by the formula

$$Au=\begin{pmatrix} a(x)\frac{du_1(x)}{dx}+\delta u_1(x) & -\delta u_2(x)\\ 0 & -a(x)\frac{du_2(x)}{dx}+\delta u_2(x)\end{pmatrix} \tag{3}$$

with domain

$$D(A)=\left\{\begin{pmatrix} u_1(x)\\ u_2(x)\end{pmatrix}: u_m(x),\frac{du_m(x)}{dx}\in L_1[0,l], m=1,2;\right.$$
$$\left. u_1(0)=\gamma u_1(l), \beta u_2(0)=u_2(l)\right\}.$$

We will study the resolvent of the operator $-A$, i.e.

$$A\begin{pmatrix} u\\ v\end{pmatrix}+\lambda\begin{pmatrix} u\\ v\end{pmatrix}=\begin{pmatrix} \varphi\\ \psi\end{pmatrix} \tag{4}$$

or

$$\begin{cases} a(x)\frac{du(x)}{dx}+(\delta+\lambda)u(x)-\delta v(x)=\varphi(x),\ 0<x<l,\\ -a(x)\frac{dv(x)}{dx}+(\delta+\lambda)v(x)=\psi(x),\ 0<x<l,\\ u(0)=\gamma u(l),\ \beta v(0)=v(l).\end{cases} \tag{5}$$

The positivity of the operator A in the Banach space $\mathbb{L}_1[0,l]$ is established. It is proved that for any $\alpha\in(0,1)$ the norms in spaces $E_{\alpha,1}(\mathbb{L}_1[0,l],A)$ and $\overset{\circ}{\mathbb{W}}{}_1^{\alpha}[0,l]$ are equivalent. The positivity of A in the Slobodeckij-Sobolev spaces of $\overset{\circ}{\mathbb{W}}{}_1^{\alpha}[0,l]$, $\alpha\in(0,1)$ is established.

2 Positivity of operator A in $\mathbb{L}_1[0,l]$

Lemma 2.1 ([18]). *For any $\lambda \geq 0$, problem (5) is uniquely solvable and the following formula holds:*

$$\begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = (\lambda I + A)^{-1} \begin{pmatrix} \varphi(x) \\ \psi(x) \end{pmatrix} = \int_0^l G(x,s;\lambda) \begin{pmatrix} \varphi(s) \\ \psi(s) \end{pmatrix} ds, \quad (6)$$

$$G(x,s;\lambda) = \begin{pmatrix} G_{11}(x,s;\lambda) & G_{12}(x,s;\lambda) \\ 0 & G_{22}(x,s;\lambda) \end{pmatrix}.$$

Here,

$$G_{11}(x,s;\lambda) = Q \times \begin{cases} \frac{1}{a(s)} \exp\left[-(\delta+\lambda) \int_s^x \frac{dy}{a(y)}\right], \ 0 \leq s \leq x, \\ \frac{\gamma}{a(s)} \exp\left[-(\delta+\lambda)\left(\int_0^x \frac{dy}{a(y)} + \int_s^l \frac{dy}{a(y)}\right)\right], \ x \leq s \leq l, \end{cases} \quad (7)$$

$$G_{22}(x,s;\lambda) = P \times \begin{cases} \frac{\beta}{a(s)} \exp\left[-(\delta+\lambda)\left(\int_x^l \frac{dy}{a(y)} + \int_0^s \frac{dy}{a(y)}\right)\right], \ 0 \leq s \leq x, \\ \frac{1}{a(s)} \exp\left[-(\delta+\lambda) \int_x^s \frac{dy}{a(y)}\right], \ x \leq s \leq l, \end{cases} \quad (8)$$

$$G_{12}(x,s;\lambda) = \delta \times \int_0^l G_{11}(x,p;\lambda) G_{22}(p,s;\lambda) dp, \quad (9)$$

$$P = \left(1 - \beta \times \exp\left[-(\delta+\lambda) \int_0^l \frac{dy}{a(y)}\right]\right)^{-1},$$

$$Q = \left(1 - \gamma \times \exp\left[-(\delta+\lambda) \int_0^l \frac{dy}{a(y)}\right]\right)^{-1}.$$

Lemma 2.2 ([18]). *The following pointwise estimates hold:*

$$|P|, |Q| \leq \frac{1}{1 - \exp\left[-\frac{(\delta+\lambda)l}{a}\right]}, \quad (10)$$

$$|G_{11}(x,s;\lambda)| \leq \frac{1}{a\left(1-\exp\left[-\frac{(\delta+\lambda)l}{a}\right]\right)}$$

$$\times\begin{cases} \exp\left[-\frac{(\delta+\lambda)(x-s)}{a}\right], \; 0\leq s\leq x, \\ \exp\left[-\frac{(\delta+\lambda)(l+x-s)}{a}\right], \; x\leq s\leq l, \end{cases} \tag{11}$$

$$|G_{22}(x,s;\lambda)| \leq \frac{1}{a\left(1-\exp\left[-\frac{(\delta+\lambda)l}{a}\right]\right)}$$

$$\times\begin{cases} \exp\left[-\frac{(\delta+\lambda)(l-x+s)}{a}\right], \; 0\leq s\leq x, \\ \exp\left[-\frac{(\delta+\lambda)(s-x)}{a}\right], \; x\leq s\leq l, \end{cases} \tag{12}$$

$$|G_{12}(x,s;\lambda)| \leq \frac{1}{a\left(1-\exp\left[-\frac{(\delta+\lambda)l}{a}\right]\right)}$$

$$\times\begin{cases} \exp\left[-\frac{(\delta+\lambda)(x-s)}{a}\right], \; 0\leq s\leq x, \\ \exp\left[-\frac{(\delta+\lambda)(s-x)}{a}\right], \; x\leq s\leq l. \end{cases} \tag{13}$$

From Lemmas 2.1 and 2.2, we derive the following result.

Theorem 2.3. *The operator* $(\lambda I + A)$ *has a bounded in* $\mathbb{L}_1[0,l]$ *inverse for any* $\lambda \geq 0$ *and the following estimate holds:*

$$\left\|(\lambda I + A)^{-1}\right\|_{\mathbb{L}_1[0,l]\to\mathbb{L}_1[0,l]} \leq \frac{M_1}{1+\lambda}. \tag{14}$$

The proof of Theorem 2.3 is based on the formula (6), triangle inequality, and estimates (11), (12), and (13).

3 The structure of fractional spaces generated by A and positivity of A in Slobodeckij-Sobolev spaces

Clearly, the operator A and its resolvent $(\lambda I + A)^{-1}$ commute. By the definition of the norm in the fractional space $E_{\alpha,1} = E_{\alpha,1}(\mathbb{L}_1[0,l], A)$, we get

$$\left\|(\lambda I + A)^{-1}\right\|_{E_{\alpha,1}\to E_{\alpha,1}} \leq \left\|(\lambda I + A)^{-1}\right\|_{\mathbb{L}_1[0,l]\to\mathbb{L}_1[0,l]}.$$

Thus, from Theorem 2.3 it follows that A is a positive operator in the fractional spaces $E_{\alpha,1}(\mathbb{L}_1[0,l], A)$. Moreover, we have the following result.

Theorem 3.1. *For* $\alpha \in (0,1)$, *the norms of the spaces* $E_{\alpha,1}(\mathbb{L}_1[0,l], A)$ *and the Slobodeckij-Sobolev spaces* $\overset{\circ}{\mathbb{W}}{}_1^{\alpha}[0,l]$ *are equivalent. Here,*

$$\overset{\circ}{\mathbb{W}}{}_1^{\alpha}[0,l] = \left\{ \begin{pmatrix} \varphi(x) \\ \psi(x) \end{pmatrix} \in \overset{\circ}{\mathbb{W}}{}_1^{\alpha}[0,l] : \varphi(0) = \gamma\varphi(l),\ 0 \le \gamma \le 1, \right.$$

$$\left. \beta\psi(0) = \psi(l),\ 0 \le \beta \le 1 \right\}.$$

Proof. For any $\lambda \ge 0$ we have the obvious equality

$$A(\lambda I + A)^{-1} \begin{pmatrix} \varphi(x) \\ \psi(x) \end{pmatrix} = \begin{pmatrix} \varphi(x) \\ \psi(x) \end{pmatrix} - \lambda(\lambda I + A)^{-1} \begin{pmatrix} \varphi(x) \\ \psi(x) \end{pmatrix}.$$

By formula (6), we can write

$$A(\lambda I + A)^{-1} \begin{pmatrix} \varphi(x) \\ \psi(x) \end{pmatrix} = \begin{pmatrix} \varphi(x) \\ \psi(x) \end{pmatrix} - \lambda \int_0^l G(x,s;\lambda) \begin{pmatrix} \varphi(s) \\ \psi(s) \end{pmatrix} ds \quad (15)$$

$$= \begin{pmatrix} \left(1 - \lambda \int_0^x G_{11}(x,s;\lambda)ds\right) \varphi(x) - \lambda \int_x^l G_{11}(x,s;\lambda)ds\varphi(l) \\ 0 \end{pmatrix}$$

$$+ \begin{pmatrix} -\lambda \int_0^x G_{22}(x,s;\lambda)ds\psi(0) + \left(1 - \lambda \int_x^l G_{22}(x,s;\lambda)ds\right) \psi(x) \\ -\lambda \int_0^l G_{12}(x,s;\lambda)ds\psi(x) \end{pmatrix}$$

$$+ \begin{pmatrix} \lambda \int_0^x G_{11}(x,s;\lambda)\left(\varphi(x) - \varphi(s)\right) ds + \lambda \int_x^l G_{11}(x,s;\lambda)\left(\varphi(l) - \varphi(s)\right) ds \\ 0 \end{pmatrix}$$

$$+ \begin{pmatrix} \lambda \int_0^l G_{12}(x,s;\lambda)\left(\psi(x) - \psi(s)\right) ds \\ \lambda \int_0^x G_{22}(x,s;\lambda)\left(\psi(0) - \psi(s)\right) ds + \lambda \int_x^l G_{22}(x,s;\lambda)\left(\psi(x) - \psi(s)\right) ds \end{pmatrix}.$$

Applying formula (15) and nonlocal boundary conditions

$$\varphi(0) = \gamma\varphi(l),\ \beta\psi(0) = \psi(l),$$

we get

$$A(\lambda I+A)^{-1}\begin{pmatrix}\varphi(x)\\ \psi(x)\end{pmatrix}$$

$$=\begin{pmatrix}\left(\frac{\delta}{\delta+\lambda}Q-\gamma Q\exp\left[-(\delta+\lambda)\int\limits_0^l\frac{dy}{a(y)}\right]\right)\varphi(x)\\ -\frac{\lambda\delta}{2(\delta+\lambda)^2}QP\left(\left(1-\exp\left[-(\delta+\lambda)\int\limits_0^l\frac{dy}{a(y)}\right]\right)^2\right.\end{pmatrix}$$

$$+\begin{pmatrix}\frac{\lambda}{\delta+\lambda}Q\exp\left[-(\delta+\lambda)\int\limits_0^x\frac{dy}{a(y)}\right](\varphi(x)-\varphi(0))\\ \beta\exp\left[-(\delta+\lambda)\int\limits_x^l\frac{dy}{a(y)}\right]\left(1-\exp\left[-(\delta+\lambda)\int\limits_0^x\frac{dy}{a(y)}\right]\right)^2\end{pmatrix}$$

$$+\begin{pmatrix}\frac{\gamma\lambda}{\delta+\lambda}Q\exp\left[-(\delta+\lambda)\int\limits_0^l\frac{dy}{a(y)}\right]\varphi(l)\\ \beta\gamma\left(1-\exp\left[-(\delta+\lambda)\int\limits_0^x\frac{dy}{a(y)}\right]\right)\end{pmatrix}$$

$$+\begin{pmatrix}\left(\frac{\delta}{\delta+\lambda}P-\beta P\exp\left[-(\delta+\lambda)\int\limits_0^l\frac{dy}{a(y)}\right]\right)(\psi(x)-\psi(0))\\ \exp\left[-(\delta+\lambda)\int\limits_0^x\frac{dy}{a(y)}\right]\left(1-\exp\left[-2(\delta+\lambda)\int\limits_x^l\frac{dy}{a(y)}\right]\right)\end{pmatrix}$$

$$+\begin{pmatrix}-\exp\left[-(\delta+\lambda)\int\limits_0^l\frac{dy}{a(y)}\right]\psi(0)\\ \left(1-\exp\left[-(\delta+\lambda)\int\limits_x^l\frac{dy}{a(y)}\right]\right)\left(1-\exp\left[-2(\delta+\lambda)\int\limits_0^x\frac{dy}{a(y)}\right]\right)\end{pmatrix}$$

$$+\begin{pmatrix}0\\ \beta\gamma\exp\left[-2(\delta+\lambda)\int\limits_0^x\frac{dy}{a(y)}\right]\left(1-\exp\left[-(\delta+\lambda)\int\limits_x^l\frac{dy}{a(y)}\right]\right)^2\end{pmatrix}$$

$$+\begin{pmatrix}\lambda\int\limits_0^x G_{11}(x,s;\lambda)(\varphi(x)-\varphi(s))\,ds+\lambda\int\limits_x^l G_{11}(x,s;\lambda)(\varphi(l)-\varphi(s))\,ds\\ \gamma\exp\left[-(\delta+\lambda)\int\limits_0^x\frac{dy}{a(y)}\right]\left(1-\exp\left[-(\delta+\lambda)\int\limits_x^l\frac{dy}{a(y)}\right]\right)^2\end{pmatrix}$$

$$+\begin{pmatrix} \lambda\int\limits_0^l G_{12}(x,s;\lambda)\left(\psi(x)-\psi(s)\right)ds \\ \lambda\int\limits_0^x G_{22}(x,s;\lambda)\left(\psi(l)-\psi(s)\right)ds+\lambda\int\limits_x^l G_{22}(x,s;\lambda)\left(\psi(x)-\psi(s)\right)ds \end{pmatrix}.$$

Using this formula, the triangle inequality, estimates (10), (11), (12) and (13), the definition of spaces $E_{\alpha,1}(\mathbb{L}_1[0,l],A)$ and $\overset{\circ}{\mathbb{W}}{}_1^{\alpha}[0,l]$, we get

$$\left\|\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right\|_{E_{\alpha,1}(\mathbb{L}_1[0,l],A)} \le M(a,\delta)\left\|\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right\|_{\overset{\circ}{\mathbb{W}}{}_1^{\alpha}[0,l]}.$$

Let us prove the opposite inequality. For any positive operator A we can write

$$f=\int_0^\infty A(\lambda I+A)^{-2}f d\lambda.$$

From this relation and formula (6) it follows that

$$\begin{pmatrix}\varphi(x)\\ \psi(x)\end{pmatrix}=\int_0^\infty(\lambda I+A)^{-1}A(\lambda I+A)^{-1}\begin{pmatrix}\varphi(x)\\ \psi(x)\end{pmatrix}d\lambda$$

$$=\int_0^\infty\int\limits_0^l G(x,s;\lambda)A(\lambda I+A)^{-1}\begin{pmatrix}\varphi(s)\\ \psi(s)\end{pmatrix}dsd\lambda.$$

Consequently,

$$\begin{pmatrix}\varphi(x+\tau)\\ \psi(x+\tau)\end{pmatrix}-\begin{pmatrix}\varphi(x)\\ \psi(x)\end{pmatrix}$$

$$=\int_0^\infty\int\limits_0^l\left(G(x+\tau,s;\lambda)-G(x,s;\lambda)\right)A(\lambda I+A)^{-1}\begin{pmatrix}\varphi(s)\\ \psi(s)\end{pmatrix}dsd\lambda.$$

Using this formula, the triangle inequality, estimates (10), (11), (12) and (13), the definition of spaces $E_{\alpha,1}(\mathbb{L}_1[0,l],A)$ and $\overset{\circ}{\mathbb{W}}{}_1^{\alpha}[0,l]$, we get

$$\left\|\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right\|_{\overset{\circ}{\mathbb{W}}{}_1^{\alpha}[0,l]} \le \frac{M_2(a,\delta)}{\alpha(1-\alpha)}\left\|\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right\|_{E_{\alpha,1}(\mathbb{L}_1[0,l],A)}.$$

Theorem 3.1. is proved.

Since A is a positive operator in the fractional spaces $E_{\alpha,1}(\mathbb{L}_1[0,l],A)$, from the result of Theorem 3.1 it follows that it is also positive in the Hölder space $\overset{\circ}{\mathbb{W}}{}_1^{\alpha}[0,l]$. Namely, we have the following result.

Theorem 3.2. *The operator* $(\lambda I + A)$ *has a bounded in* $\overset{\circ}{\mathbb{W}}{}_1^{\alpha}[0,l]$ *inverse for any* $\lambda \geq 0$ *and the following estimate holds:*

$$\| (\lambda I + A)^{-1} \|_{\overset{\circ}{\mathbb{W}}{}_1^{\alpha}[0,l] \to \overset{\circ}{\mathbb{W}}{}_1^{\alpha}[0,l]} \leq \frac{M_2(a,\delta)}{\alpha(1-\alpha)} \frac{M_1}{1+\lambda}.$$

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