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The spacewise dependent source identification problem for the elliptic-telegraph differential equation with involution and Robin condition

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Abstract. In the present paper, the spacewise dependent source identification problem (SIP) for the elliptic-telegraph differential equation with involution and Robin condition is investigated. The theorem on stability estimates for the solution of this space-wise dependent SIP is established.

Keywords. Elliptic-telegraph differential equations, spacewise dependent source identification problem, stability, differential operator with involution.

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1 Introduction

It is known that the telegraph equation is important for modeling several relevant problems such as signal analysis [1], wave propagation [2], vibrational systems [3], random walk theory [4], mechanical systems [5] and etc. In paper [6] studied the development, analysis, and implementation of stable methods for the numerical solutions of second order hyperbolic equations.

Local and nonlocal problems for the elliptic-telegraph differential and difference equations have been studied extensively by many researches (see [7–10] and the references therein).

In the papers [7, 8], the stability of the local and nonlocal problems for the elliptic-telegraph differential equations were investigated. The first and the second order of accuracy difference schemes approximately solving these problems were presented. The stability estimates for the solution of these difference schemes were established. Numerical results for the first and the second order of accuracy difference schemes were given. In the paper [9], the equation of mixed elliptic-hyperbolic type in rectangular area with the conditions of periodicity and the nonlocal problem of A. A. Desin was studied. Theorems of convergence of the constructed series in the class of regular solutions and the stability of the solution were proved. In the paper [10], the existence of traveling wave solutions for a hyperbolic-elliptic system of partial differential equations was established. The geometric theory of singular perturbations was employed.

The spacewise dependent SIPs for the elliptic-telegraph differential equations have been studied by many authors (see [11–14] and the references given therein). In the paper [11], the inverse problem for an equation of elliptic-hyperbolic type with a nonlocal boundary condition was studied. Theorems of the uniqueness criterion and the stability of solutions with respect to the boundary value problem were proved.

In the paper [12], the boundary value problem for the differential equation with parameter in a Hilbert space with self-adjoint definite operator was investigated. The well-posedness of this problem was established. The stability inequalities for the solution of spacewise dependent SIPs for elliptic-hyperbolic equations were given.

The spacewise SIP for the elliptic-telegraph differential equation in Hilbert spaces with the self-adjoint positive definite operator was investigated in the paper [13]. The main theorem on the stability of the spacewise dependent SIP for the elliptic-telegraph differential equation was proved. In applications, theorems on the stability of three spacewise SIPs for one dimensional with nonlocal conditions and multidimensional elliptic-telegraph differential equations were established. Finally, in the paper [14], the first order of accuracy absolute stable difference scheme for the approximate solution of the spacewise dependent SIP for the elliptic-telegraph equation in Hilbert spaces with the self-adjoint positive definite operator was presented. The main theorem on the stability of the difference scheme was established. In applications, theorems on the stability of difference schemes for two types of the spacewise SIPs for multidimensional elliptic-telegraph partial differential equations were proved. Numerical analysis was provided.

Our goal in this paper is to investigate the stability of the spacewise dependent SIP for the elliptic-telegraph differential equation (ETDE) with involution and Robin condition

$$\begin{cases} u_{tt}(t,x)+\alpha u_t(t,x)-(a(x)u_x(t,x))_x \\ +\beta\left(a(-x)u_x\left(t,-x\right)\right)_x+\sigma u(t,x) \\ =b\left(-\left(a(x)u_x(-t,x)\right)_x+\beta\left(a(-x)u_x\left(-t,-x\right)\right)_x+\sigma u(-t,x)\right) \\ +p(x)+f(t,x), t\in(0,T), -l<x<l, \\ -u_{tt}(t,x)-(a(x)u_x(t,x))_x+\beta\left(a(-x)u_x\left(t,-x\right)\right)_x+\sigma u(t,x) \\ =b\left(-\left(a(x)u_x(-t,x)\right)_x+\beta\left(a(-x)u_x\left(-t,-x\right)\right)_x+\sigma u(-t,x)\right) \\ +p(x)+g(t,x), \quad -T<t<0, -l<x<l, \\ u(0,x)=\xi(x), u_t(0^+,x)=u_t(0^-,x), \\ u(-T,x)=\varphi(x), u(T,x)=\psi(x), -l\le x\le l, \\ \varphi\left(-l\right)=\gamma\varphi_x\left(-l\right), -\varphi\left(l\right)=\eta\varphi_x\left(l\right), \\ \psi\left(-l\right)=\gamma\psi_x\left(-l\right), -\psi\left(l\right)=\eta\psi_x\left(l\right), \\ u(t,-l)=\gamma u_x(t,-l), -u(t,l)=\eta u_x\left(t,l\right),\ t\in[-T,T] \end{cases} \tag{1}$$

for the elliptic-telegraph partial differential equation. Under compatibility conditions problem (1) has a unique solution $(u(t,x),p(x))$ for the smooth functions $a\ge a(x)=a\left(-x\right)\ge\delta>0$ and $\sigma>0$ is a sufficiently large number, $\varphi(x),\xi(x),\psi(x)\ \ (x\in[-l,l]),\ f(t,x)\left(t\in(0,T)\right), g(t,x)(t\in\left(-T,0\right), x\in\left(-l,l\right)$, and $\alpha,\gamma,\beta,\eta\ge 0$. The theorem on stability estimates for the solution of this spacewise dependent SIP (1) is established.

2 The theorem on stability of the spacewise dependent SIP (1)

We can reduce the spacewise dependent SIP (1) to the spacewise dependent SIP for the ETDE

$$\begin{cases} \frac{d^2u(t)}{dt^2}+\alpha\frac{du(t)}{dt}+Au(t)=p+f(t), 0<t<T, \\ -\frac{d^2u(t)}{dt^2}+Au(t)=p+g(t), -T<t<0, \\ u(0)=\xi, u_t(0^+)=u_t(0^-), u(-T)=\varphi, u(T)=\psi \end{cases} \tag{2}$$

in a Hilbert space $H=L_2[-l,l]$ with a self-adjoint positive definite operator $A=A^x$ defined by formula (see [15],[16],[17])

$$A^x u(x)=-(a(x)u_x)_x+\beta\left(a(-x)u_x\left(t,-x\right)\right)_x+\sigma u(t,x) \tag{3}$$

with domain

$$D(A^x) = \{u(x) : u(x), u_x(x), (a(x)u_x)_x \in L_2[-l,l],$$

$$u(-l) = \gamma u_x(-l), -u(l) = \eta u_x(l)\}.$$

Applying the symmetry property of the space operator A^x with the domain $D(A^x) \subset W_2^2[-l,l]$, we can obtain the following theorem on stability of problem (1). Here and in the rest of this paper, $C(H) = C([-T,T],H)$ stands for the Banach space of continuous H-valued functions $u(t)$ defined on $[-T,T]$, equipped with the norm

$$||u||_{C(H)} = \max_{t\in[-T,T]} ||u(t)||_H . \tag{4}$$

Theorem 2.1. *Suppose that $\varphi, \psi, \xi \in W_2^2[-l,l]$, and $\alpha \geq 4$, $\left(\frac{\alpha}{2}+1\right)^2 \geq \delta \geq \left(\frac{\alpha}{2}\right)^2 + 1$. Let $f(t,x)$ be continuously differentiable in t on $[0,T] \times [-l,l]$ and $g(t,x)$ be continuously differentiable functions in t on $[-T,0] \times [-l,l]$. Then the solutions of the spacewise dependent SIP (2) has a unique solution $u \in C(L_2[-l,l]) = C([-T,T], L_2[-l,l])$, $p \in C[-l,l]$ and for the solution of the time dependent SIP (2) the following stability estimates hold*

$$\|u\|_{C([-T,T],L_2[-l,l])} + \left\|(A^x)^{-1}p\right\|_{L_2[-l,l]}$$

$$\leq M_1(\alpha,\delta)\Big[\|\varphi\|_{L_2[-l,l]} + \|\psi\|_{L_2[-l,l]} + \|\xi\|_{L_2[-l,l]}$$

$$+ \|f\|_{C([0,T],L_2[-l,l])} + \|g\|_{C([-T,0],L_2[-l,l])}\Big], \tag{5}$$

$$\|u\|_{C^{(2)}([-T,T],L_2[-l,l])} + \|u\|_{C\left([-T,T],W_2^2[-l,l]\right)} + \|p\|_{L_2[-l,l]}$$

$$\leq M_2(\alpha,\delta)\Big[\|\varphi\|_{W_2^2[-l,l]} + \|\psi\|_{W_2^2[-l,l]}$$

$$+ \|\xi\|_{W_2^2[-l,l]} + \|f\|_{C^{(1)}([0,T],L_2[-l,l])} + \|g\|_{C^{(1)}([-T,0],L_2[-l,l])}\Big], \tag{6}$$

where $M_1(\alpha,\delta)$ and $M_2(\alpha,\delta)$ do not depend on $\varphi(x), \psi(x), \xi(x), f(t,x)$ and $g(t,x)$. Here, the Sobolev space $W_2^2[-l,l]$ is defined as the set of all functions $u(x)$ defined on $[-l,l]$ such that $u(x)$ and the second order derivative function $u''(x)$ are all locally integrable in $L_2[-l,l]$, equipped the norm

$$\|u\|_{W_2^2[-l,l]} = \left(\int_{-l}^{l} |u(x)|^2\, dx\right)^{\frac{1}{2}} + \left(\int_{-l}^{l} \left|u''(x)\right|^2 dx\right)^{\frac{1}{2}}.$$

Proof. The proof of Theorem 2.1 is based on the self-adjointness and positivity operator $A = A^x$ defined by the formula (3) and on the the following abstract stability result.

Theorem 2.2 ([13]). *Suppose that $\varphi, \psi, \xi \in D(A)$, and $\alpha \geq 4$, $\left(\frac{\alpha}{2}+1\right)^2 \geq \delta \geq \left(\frac{\alpha}{2}\right)^2 + 1$. Let $f(t)$ be continuously differentiable on $[0,T]$ and $g(t)$ be continuously differentiable on $[-T,0]$ functions. Then there is a unique solution of the problem* (2) *and the stability inequalities*

$$\max_{-T\leq t\leq T}\|u(t)\|_H + \left\|A^{-1}p\right\|_H \leq M(\alpha,\delta)\left[\|\varphi\|_H + \|\psi\|_H + \|\xi\|_H\right.$$
$$\left. + \max_{-T\leq t\leq 0}\left\|A^{-1/2}g(t)\right\|_H + \max_{0\leq t\leq T}\left\|A^{-1/2}f(t)\right\|\right]_H,$$
$$\max_{-T\leq t\leq T}\left\|\frac{d^2u(t)}{dt^2}\right\|_H + \max_{-T\leq t\leq T}\|Au(t)\|_H + \|p\|_H$$
$$\leq M(\alpha,\delta)\left[\|A\varphi\|_H + \|A\psi\|_H + \|A\xi\|_H + \|g(0)\|_H\right.$$
$$\left. + \max_{-T\leq t\leq 0}\left\|g'(t)\right\|_H + \|f(0)\|_H + \max_{0\leq t\leq T}\left\|f'(t)\right\|_H\right]$$

hold, where $M(\alpha,\delta)$ does not depend on $f(t), t \in [0,T], g(t), t \in [-T,0]$ and φ, ψ, ξ.

3 Conclusion

In the present paper, the spacewise dependent SIP for the ETDE with involution and Robin condition is studied. The theorem on stability estimates for the solution of this spacewise dependent SIP is established. Moreover, applying the result of the monographs [18, 19], the two-step difference schemes for the numerical solution of the spacewise dependent SIPs for the ETDE can be presented. Of course, the stability estimates for the solution of these difference schemes can be investigated.

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