e-Journal of Analysis and Applied Mathematics **2023** (2023), 71–80
DOI 10.62780/ejaam/2023-007 EJAAM 2023

On the Schrödinger-parabolic equation with multipoint nonlocal boundary condition

Yıldırım Özdemir

Communicated by Charyyar Ashyralyyev

Abstract. The nonlocal boundary value problem for a Schrödinger-parabolic equation with multipoint nonlocal boundary conditions is examined. Stability estimates for the solution of this problem are established. Additionally, these stability estimates are applied to a mixed-type boundary value problem for the Schrödinger-parabolic equation with multipoint nonlocal boundary conditions.

Keywords. Partial differential equation, nonlocal boundary value problem, stability.

2010 Mathematics Subject Classification. 65L10, 34B10, 65M12.

1 Introduction

Specific problems of modern physics and technology can be effectively illustrated in terms of nonlocal boundary value problems for partial differential equations with nonlocal boundary conditions. These conditions occur when the data on the boundary cannot be measured directly. Methods of solutions of nonlocal boundary value problems for partial differential equations and partial differential equations of mixed types have been studied extensively by many researchers (see [1–10] and references therein).

In the papers [11, 12], a two point nonlocal boundary value problem

$$\begin{cases} \frac{du(t)}{dt} + Au(t) = f(t), \quad 0 \le t \le 1, \\ i\frac{du(t)}{dt} + Au(t) = g(t), \quad -1 \le t \le 0, \\ u(-1) = u(1) + \varphi \end{cases}$$

for a Schrödinger-parabolic equation in a Hilbert space H with a self-adjoint positive definite operator A was studied. The stability of this problem was established. The first and second order accuracy difference schemes for the approximate solutions of this nonlocal boundary value problem were presented. The stability of these difference schemes was established. In applications, stability inequalities for the solutions of difference schemes for Schrödinger-parabolic equations were obtained. The Matlab implementation of these difference schemes for the Schrödinger-parabolic equation was presented. Additionally, extensive numerical experiments were conducted to verify the theoretical results. These experiments demonstrated the effectiveness and reliability of the proposed schemes in practical scenarios. The findings contribute significantly to the existing literature, providing a solid foundation for future research in this area.

In the present paper, the nonlocal boundary value problem for the following Schrödinger-parabolic equation with the multipoint nonlocal boundary condition

$$\begin{cases} \frac{du(t)}{dt} + Au(t) = f(t), \quad 0 \leq t \leq 1, \\ i\frac{du(t)}{dt} + Au(t) = g(t), \quad -1 \leq t \leq 0, \\ u(-1) = \sum\limits_{j=1}^{N} \alpha_j u\left(\mu_j\right) + \varphi, \quad 0 < \mu_j \leq 1 \end{cases} \tag{1}$$

in a Hilbert space H with a self-adjoint positive definite operator A is considered.

A function $u(t)$ is called a solution of the problem (1) if the following conditions are satisfied:

- $u(t)$ is continuously differentiable on the segment $[-1, 1]$, the derivative at the endpoints of the segment are understood as the appropriate unilateral derivatives;
- the element $u(t)$ belongs to $D(A)$ for all $t \in [-1, 1]$, and the function $Au(t)$ is continuous on the segment $[-1, 1]$;
- $u(t)$ satisfies the equations and multipoint nonlocal boundary condition (1).

The main purpose of this work is to obtain the stability of a nonlocal boundary value problem (1) for a Schrödinger-parabolic equation with multipoint nonlocal boundary conditions. We establish rigorously the conditions under which the stability of the solution can be guaranteed. The work involves a comprehensive mathematical analysis to derive and prove the stability criteria, ensuring that the solutions remain stable under the specified nonlocal boundary conditions.

2 Stability Analysis

First of all, let us give two auxiliary lemmas that will be needed below.

Lemma 2.1. *Let H be a Hilbert space and A be a self-adjoint positive definite operator with $A \geq \delta I$, where $\delta > 0$. The following estimates hold:*

$$\left\|e^{-tA}\right\|_{H \to H} \leq 1, \quad t \geq 0, \tag{2}$$

$$\left\|e^{\pm itA}\right\|_{H \to H} \leq 1. \tag{3}$$

Lemma 2.2. *Assume that* $\sum\limits_{j=1}^{N} |\alpha_j| e^{-\mu_j \delta} < 1$. *Then, the operator*

$$I - \sum_{j=1}^{N} \alpha_j e^{iA} e^{-\mu_j A}$$

has an inverse

$$T = \left(I - \sum_{j=1}^{N} \alpha_j e^{iA} e^{-\mu_j A} \right)^{-1}$$

and the estimate holds:

$$\|T\|_{H \to H} \leq \frac{1}{1 - \sum\limits_{j=1}^{N} |\alpha_j| e^{-\mu_j \delta}}. \tag{4}$$

Proof. The proof of the estimate (4) is based on the inequality

$$\left\| \sum_{j=1}^{N} \alpha_j e^{-(\mu_j + i)A} \right\|_{H \to H} \leq \sum_{j=1}^{N} |\alpha_j| \left| e^{-\mu_j \delta} \right| \left| e^{-i\delta} \right| \leq \sum_{j=1}^{N} |\alpha_j| e^{-\mu_j \delta} < 1.$$

Then,

$$\|T\|_{H \to H} = \left\| \left(I - \sum_{j=1}^{N} \alpha_j e^{iA} e^{-\mu_j A} \right)^{-1} \right\|_{H \to H} \leq \frac{1}{1 - \sum\limits_{j=1}^{N} |\alpha_j| e^{-\mu_j \delta}}.$$

That is, the estimate (4) has been proven.

We have the following main theorem on the stability of the problem (1).

Theorem 2.3. *Assume that all assumptions of Lemmas 2.1 and 2.2 are satisfied. Suppose that* $\varphi \in D(A)$. *Let* $f(t)$ *and* $g(t)$ *be continuously differentiable functions on intervals* $[0, 1]$ *and* $[-1, 0]$, *respectively. Then, there is a unique solution of the problem (*1*) and the following stability inequalities*

$$\max_{-1 \leq t \leq 1} \|u(t)\|_H \leq M_1 \left[\|\varphi\|_H + \max_{-1 \leq t \leq 0} \|g(t)\|_H + \max_{0 \leq t \leq 1} \|f(t)\|_H \right], \tag{5}$$

$$\max_{-1 \leq t \leq 1} \|Au(t)\|_H \leq M_1 \left[\|A\varphi\|_H + \|g(0)\|_H + \max_{-1 \leq t \leq 0} \left\|g'(t)\right\|_H \right.$$

$$+\|f(0)\|_H + \max_{0\le t\le 1}\left\|f'(t)\right\|_H\Big] \quad (6)$$

hold, where M_1 *is independent of* $f(t),\ t\in[0,1]$, $g(t),\ t\in[-1,0]$ *and* φ.

Proof. First, we will derive a formula for solving the problem (1). It is known that the initial value problems

$$\frac{du(t)}{dt} + Au(t) = f(t),\ 0\le t\le 1,\quad u(0)=u_0 \quad (7)$$

$$i\frac{du(t)}{dt} + Au(t) = g(t),\ -1\le t\le 0,\quad u(-1)=u_{-1} \quad (8)$$

have unique solutions

$$u(t) = e^{-tA}u(0) + \int_0^t e^{-(t-s)A}f(s)ds,\ 0\le t\le 1, \quad (9)$$

$$u(t) = e^{i(t+1)A}u_{-1} - i\int_{-1}^t e^{i(t-s)A}g(s)ds,\ -1\le t\le 0, \quad (10)$$

respectively. Using the formula (10) we have

$$u(0) = e^{iA}u_{-1} - i\int_{-1}^0 e^{-isA}g(s)ds, \quad (11)$$

with the help of which we obtain from (9)

$$u(t) = e^{-tA}\left[e^{iA}u_{-1} - i\int_{-1}^0 e^{-isA}g(s)ds\right] + \int_0^t e^{-(t-s)A}f(s)ds,\ 0\le t\le 1. \quad (12)$$

Using the multipoint nonlocal boundary condition

$$u(-1) = \sum_{j=1}^N \alpha_j u(\mu_j) + \varphi,$$

we get the following operator equation

$$\left(I - \sum_{j=1}^N \alpha_j e^{iA}e^{-\mu_j A}\right)u_{-1}$$

$$= \sum_{j=1}^{N} \alpha_j \left(-ie^{-\mu_j A} \int_{-1}^{0} e^{-isA} g(s)ds + \int_{0}^{\mu_j} e^{-(\mu_j - s)A} f(s)ds \right) + \varphi. \quad (13)$$

Thus, from the operator equation (13), we obtain

$$u_{-1} = T \left[\sum_{j=1}^{N} \alpha_j \left(-ie^{-\mu_j A} \int_{-1}^{0} e^{-isA} g(s)ds + \int_{0}^{\mu_j} e^{-(\mu_j - s)A} f(s)ds \right) + \varphi \right]. \quad (14)$$

Therefore, for the solution of problem (1), the formulas (10), (12), and (14) are obtained.

Now, the proof of estimates (5) and (6) will be obtained. These inequalities are the stability estimates of the solution and the first derivative of the solution of the problem (1), respectively.

First, the inequality (5) will be proved. Using the formula (14) and estimates (2) and (3),

$$\|u_{-1}\|_H \leq \|T\|_{H\to H} \left[\sum_{j=1}^{N} |\alpha_j| \left\|e^{-\mu_j A}\right\|_{H\to H} \int_{-1}^{0} \left\|e^{-iAs}\right\|_{H\to H} \|g(s)\|_H \, ds \right.$$

$$\left. + \sum_{j=1}^{N} |\alpha_j| \int_{0}^{\mu_j} \left\|e^{-(\mu_j - s)A}\right\|_{H\to H} \|f(s)\|_H \, ds + \|\varphi\|_H \right]$$

$$\leq M_1 \left[\int_{-1}^{0} \|g(s)\|_H \, ds + \int_{0}^{1} \|f(s)\|_H \, ds + \|\varphi\|_H \right].$$

Thus, we have

$$\|u_{-1}\|_H \leq M_1 \left[\|\varphi\|_H + \max_{-1\leq t\leq 0} \|g(t)\|_H + \max_{0\leq t\leq 1} \|f(t)\|_H \right]. \quad (15)$$

Using the formula (12), estimates (2), (3), (15) and triangle inequality, we get

$$\|u(t)\|_H \leq \left\|e^{-tA}\right\|_{H\to H} \left(\left\|e^{iA}\right\|_{H\to H} \|u_{-1}\|_H \right.$$

$$\left. + \int_{-1}^{0} \left\|e^{-isA}\right\|_{H\to H} \|g(s)\|_H \, ds + \int_{0}^{t} \left\|e^{-(t-s)A}\right\|_{H\to H} \|f(s)\|_H \, ds \right)$$

$$\leq \|u_{-1}\|_H + \int\limits_{-1}^{0} \|g(s)\|_H \, ds + \int\limits_{0}^{1} \|f(s)\|_H \, ds, \quad 0 \leq t \leq 1.$$

Thus,

$$\|u(t)\|_H \leq M_1 \left[\|\varphi\|_H + \max_{-1 \leq t \leq 0} \|g(t)\|_H + \max_{0 \leq t \leq 1} \|f(t)\|_H \right], \; 0 \leq t \leq 1. \quad (16)$$

Using the formula (10), estimates (2), (3), (15) and triangle inequality, we obtain

$$\|u(t)\|_H \leq \left\| e^{i(t+1)A} \right\|_{H \to H} \|u_{-1}\|_H + \int\limits_{-1}^{t} \left\| e^{i(t-s)A} \right\|_{H \to H} \|g(s)\|_H \, ds$$

$$\leq \|u_{-1}\|_H + \int\limits_{-1}^{0} \|g(s)\|_H \, ds, \; -1 \leq t \leq 0.$$

Thus, the following estimate

$$\|u(t)\|_H \leq M_1 \left[\|\varphi\|_H + \max_{-1 \leq t \leq 0} \|g(t)\|_H + \max_{0 \leq t \leq 1} \|f(t)\|_H \right], \; -1 \leq t \leq 0 \quad (17)$$

is obtained. Therefore, using the inequalities (16) and (17), the inequality (5) is proven.

Secondly, the inequality (6) will be proven. Acting the operator A to the formulas (14), (10), (12) and using integration by parts and estimates (2), (3), we get

$$\|Au(-1)\|_H \leq M_1 \left[\|A\varphi\|_H + \|g(0)\|_H + \max_{-1 \leq t \leq 0} \left\| g'(t) \right\|_H \right.$$

$$\left. + \|f(0)\|_H + \max_{0 \leq t \leq 1} \left\| f'(t) \right\|_H \right], \quad (18)$$

$$\|Au(t)\|_H \leq M_2 \left[\|Au(-1)\|_H + \|f(0)\|_H + \max_{0 \leq t \leq 1} \left\| f'(t) \right\|_H \right.$$

$$\left. + \|g(0)\|_H + \max_{-1 \leq t \leq 0} \left\| g'(t) \right\|_H \right]$$

$$\leq M_2 \left[\|A\varphi\|_H + \|g(0)\|_H + \max_{-1 \leq t \leq 0} \left\| g'(t) \right\|_H \right.$$

$$\left. + \|f(0)\|_H + \max_{0 \leq t \leq 1} \left\| f'(t) \right\|_H \right], \; 0 \leq t \leq 1, \quad (19)$$

$$\|Au(t)\|_H \leq \|Au(-1)\|_H + \|g(0)\|_H + \max_{-1\leq t\leq 0} \left\|g'(t)\right\|_H$$

$$\leq M_3 \left[\|A\varphi\|_H + \|g(0)\|_H + \max_{-1\leq t\leq 0} \left\|g'(t)\right\|_H \right.$$

$$\left. + \|f(0)\|_H + \max_{0\leq t\leq 1} \left\|f'(t)\right\|_H \right], \ -1 \leq t \leq 0. \tag{20}$$

From estimates (18), (19), (20) it follows the inequality (6). Thus, the proof of Theorem 2.3 is completed.

3 An Application

Now, we consider the application of Theorem 2.3. In $[-1, 1] \times \Omega$, we consider the nonlocal boundary value problem

$$\begin{cases} v_t - \sum\limits_{r=1}^{m} (a_r(x)v_{x_r})_{x_r} = f(t, x), \ 0 \leq t \leq 1, \ x = (x_1, \cdots, x_m) \in \Omega, \\ iv_t - \sum\limits_{r=1}^{m} (a_r(x)v_{x_r})_{x_r} = g(t, x), \ -1 \leq t \leq 0, \ x = (x_1, \cdots, x_m) \in \Omega, \\ v(-1, x) = \sum\limits_{j=1}^{N} \alpha_j v(\mu_j, x) + \varphi(x), \ x \in \overline{\Omega}, \ 0 < \mu_j \leq 1, \\ v(t, x) = 0, \ x \in S, \ -1 \leq t \leq 1 \end{cases} \tag{21}$$

for multi-dimensional Schrödinger-parabolic equation. Here, Ω is the unit open cube in the m-dimensional Euclidean space $\mathbb{R}^m$

$$\left\{x \ : \ x = (x_1, \cdots, x_m) \in \mathbb{R}^m, \ 0 < x_k < 1, \ 1 \leq k \leq m\right\}$$

with the boundary S and $\overline{\Omega} = \Omega \cup S$.

We introduce the Hilbert space $L_2(\overline{\Omega})$ of all square integrable functions defined on $\overline{\Omega}$, equipped with the norm

$$\|f\|_{L_2(\overline{\Omega})} = \left\{ \int \cdots \int_{x\in\overline{\Omega}} |f(x)|^2 dx_1 \cdots dx_m \right\}^{1/2}$$

and Hilbert spaces $W_2^1\left(\overline{\Omega}\right)$ and $W_2^2\left(\overline{\Omega}\right)$ defined on $\overline{\Omega}$, equipped with norms

$$\|\varphi\|_{W_2^1(\overline{\Omega})} = \|\varphi\|_{L_2(\overline{\Omega})} + \left\{ \int \cdots \int_{x\in\Omega} \sum_{r=1}^{m} |\varphi_{x_r}|^2 dx_1 \cdots dx_m \right\}^{1/2},$$

$$\|\varphi\|_{W_2^2(\overline{\Omega})} = \|\varphi\|_{L_2(\overline{\Omega})} + \left\{ \int \cdots \int_{x\in\Omega} \sum_{r=1}^{m} |\varphi_{x_r x_r}|^2 dx_1 \cdots dx_m \right\}^{1/2},$$

respectively. The problem (21) has a unique smooth solution $v(t,x)$ for smooth functions $a_r(x), x \in \Omega$, $\varphi(x), x \in \overline{\Omega}$, $f(t,x), (t,x) \in [0,1] \times \Omega$, $g(t,x), (t,x) \in [-1,0] \times \Omega$. Here, $a_r(x) \geq a > 0, x \in \Omega$. This allows us to reduce problem (21) to boundary value problem (1) in Hilbert space $H = L_2(\overline{\Omega})$ with a self-adjoint positive definite operator A^x defined by formula

$$A^x v(x) = -\sum_{r=1}^{m} (a_r(x) v_{x_r})_{x_r} \tag{22}$$

with domain

$$D(A^x) = \Big\{ v(x) : v(x), (a_r(x) v_{x_r})_{x_r} \in L_2(\Omega), 1 \leq r \leq m,\ v(x) = 0, x \in S \Big\}.$$

Theorem 3.1. *Assume that* $\sum_{j=1}^{N} |\alpha_j| \, e^{-\mu_j \delta} \leq 1$. *Then, for the solution of problem* (21)*, the following stability inequalities hold:*

$$\max_{-1\leq t\leq 1} \|v(t,\cdot)\|_{L_2(\overline{\Omega})} \leq M_2 \Bigg[\|\varphi\|_{L_2(\overline{\Omega})} + \max_{-1\leq t\leq 0} \|g(t,\cdot)\|_{L_2(\overline{\Omega})} + \max_{0\leq t\leq 1} \|f(t,\cdot)\|_{L_2(\overline{\Omega})} \Bigg],$$

$$\max_{-1\leq t\leq 1} \|v(t,\cdot)\|_{W_2^2(\overline{\Omega})} \leq M_2 \Bigg[\|\varphi\|_{W_2^1(\overline{\Omega})} + \|g(0,\cdot)\|_{L_2(\overline{\Omega})} + \max_{-1\leq t\leq 0} \|g_t(t,\cdot)\|_{L_2(\overline{\Omega})} + \|f(0,\cdot)\|_{L_2(\overline{\Omega})} + \max_{0\leq t\leq 1} \|f_t(t,\cdot)\|_{L_2(\overline{\Omega})} \Bigg],$$

where M_2 *is independent of* $f(t,x), (t,x) \in [0,1]\times\Omega$, $g(t,x), (t,x) \in [-1,0]\times\Omega$ *and* $\varphi(x), x \in \overline{\Omega}$.

The proof of Theorem 3.1 is based on the abstract Theorem 2.3, symmetry properties of the operator A^x defined by formula (22) and the following theorem on the coercivity inequality for the solution of the elliptic differential problem in $L_2\left(\overline{\Omega}\right)$.

Theorem 3.2. *For the solutions of the elliptic differential problem*

$$A^x v(x) = \omega(x),\ x \in \Omega, \quad v(x) = 0,\ x \in S,$$

the following coercivity inequality holds:

$$\sum_{r=1}^{m} \left\| v_{x_r x_r} \right\|_{L_2(\overline{\Omega})} \leq M_3 \left\| \omega \right\|_{L_2(\overline{\Omega})},$$

where M_3 is independent of ω [13].

4 Conclusion

This study is focused on determining the stability of a nonlocal boundary value problem for a Schrödinger-parabolic equation with multipoint nonlocal boundary condition. A comprehensive stability theorem was developed, clearly defining the condition required to ensure the stability of the solutions. Through meticulous mathematical analysis, we derived and validated the stability criteria, confirming that the solutions remain stable under the multipoint nonlocal boundary condition.

Acknowledgments. I gratefully acknowledge the invaluable contributions of Professor Allaberen Ashyralyev from Bahcesehir University, whose guidance and expertise significantly improved this study.

Bibliography

[1] B. Hicdurmaz and A. Ashyralyev, On the stability of time-fractional Schrödinger differential, *Numerical Functional Analysis and Optimization* **38** (2017) 1215-1225.

[2] A. Ashyralyev and Y. Ozdemir, On nonlocal boundary value problems for hyperbolic-parabolic equations, *Taiwanese Journal of Mathematics* **11(4)** (2007) 1075-1089.

[3] A. Ashyralyev and O.Yildirim, On multipoint nonlocal boundary value problems for hyperbolic differential and difference equations, *Taiwanese Journal of Mathematics* **14(1)** (2010) 165-194.

[4] A. S. Berdyshev and E. T. Karimov, Some non-local problems for the parabolic-hyperbolic type equation with non-characteristic line of changing type, *Central European Journal of Mathematics* **4(2)** (2006) 183-193.

[5] A. Ashyralyev and O. Gercek, Nonlocal boundary value problems for elliptic-parabolic differential and difference equations, *Discrete Dynamics in Nature and Society* **2008** (2008) 1-16.

[6] I. Orazov and M. A. Sadybekov, One nonlocal problem of determination of the temperature and density of heat sources, *Russian Mathematics* **56** (2012) 60-64.

[7] M. A. Sadybekov, B. Turmetov and B. T. Torebek, Solvability of nonlocal boundary-value problems for the Laplace equation in the ball, *Electronic Journal of Differential Equations* **2014(157)** (2014) 1-14.

[8] S. Z. Dzhamalov, S. R. Umarov and R. R. Ashurov, On unique solvability of a nonlocal boundary-value problem for a loaded multidimensional Chaplygin's equation in the Sobolev space, *Lobachevskii Journal of Mathematics* **14** (2020) 7-14.

[9] V. Karachik and B. Turmetov, On solvability of some nonlocal boundary value problems for biharmonic equation, *Mathematica Slovaca* **20(2)** (2020).

[10] T. K. Yuldshev, On a nonlocal boundary value problem for a partial integro-differential equations with degenerate kernel, *Vladikavkazskii Matematicheskii* **24(2)** (2022) 130-141.

[11] Y. Ozdemir and M. Alp, Numerical solution of the parabolic-Schrödinger equation with the nonlocal boundary value condition, *AIP Conference Proceedings* **1611** (2014) 221-224.

[12] M. Alp, Nonlocal Boundary Value Problems for Parabolic-Schrödinger Differential and Difference Equations, Master Thesis, Duzce University, Duzce, 2014, 82 pages, (in Turkish).

[13] P. E. Sobolevskii, Difference Methods for Approximate Solution of Differential Equations, Izdatelstvo Voronezhskogo Gosud Universiteta, Voronezh, 1975, (in Russian).

Received September 22, 2023; accepted December 28, 2023.

Author information

Yıldırım Özdemir, Gebze Technical University, Kocaeli, Türkiye.
E-mail: yozdemir28@gmail.com