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An asymptotic result for linear nonhomogeneous mixed type differential equation

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Abstract. We consider a nonhomogeneous linear mixed type differential equation with variable coefficients and establish an asymptotic result for its solutions. Our result is obtained by the use of a solution of the so-called generalized characteristic equation of the corresponding homogeneous linear mixed type differential equation.

Keywords. Mixed differential equation, characteristic equation, asymptotic behavior, solution.

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1 Introduction

Mixed type differential difference equations are a special class of functional differential equations where the time derivative depends on both past and future values of the variable. There are lots of interesting articles on applications of such equations as, for example, models for economic dynamics [1, 2], nerve conduction theory [3], optimal control problems [4], and the description of traveling waves in a spatial lattice [5-7].

This paper deals with the mixed advanced-delay differential equation with variable coefficients

$$x'(t) = \sum_{i\in I} p_i(t)x(t-\tau_i) + \sum_{j\in J} q_j(t)x(t+\sigma_j) + h(t), \quad t \geq 0, \tag{1}$$

where I and J are initial segments of natural numbers, the function coefficients p_i and q_j are continuous on the interval $[0,\infty)$, the delays τ_i and advances σ_j are positive real numbers and h is a continuous real-valued function on the interval $[0,\infty)$. We assume that at least one p_i or one q_j is not identically zero on $[0,\infty)$ and we denote

$$\tau_0 = \max\{\tau_i : i \in I\}, \quad \sigma_0 = \max\{\sigma_j : j \in J\}.$$

By a *solution* x to the mixed differential equation (1), we mean a continuous real-valued function defined on $[-\tau_0, \infty)$, which is continuously differentiable on $[0, \infty)$ and satisfies (1).

Together with the mixed differential equation (1), we specify an initial condition of the form

$$x(t) = \phi(t) \quad \text{for } -\tau_0 \le t \le \sigma_0, \tag{2}$$

where the initial function ϕ is a given continuous real-valued function on the initial interval $[-\tau_0, \sigma_0]$.

In the case where the function h is identically zero on the interval $[0, \infty)$, the mixed type differential equation (1) reduces to

$$x'(t) = \sum_{i \in I} p_i(t) x(t - \tau_i) + \sum_{j \in J} q_j(t) x(t + \sigma_j), \quad t \ge 0. \tag{3}$$

Along with the mixed differential equation (3), we associate the following equation

$$\lambda(t) = \sum_{i \in I} p_i(t) \exp\left\{ -\int_{t-\tau_i}^{t} \lambda(s) ds \right\} + \sum_{j \in J} q_j(t) \exp\left\{ \int_{t}^{t+\sigma_j} \lambda(s) ds \right\} \tag{4}$$

which will be called the *generalized characteristic equation* of (3). Equation (4) is obtained from (3) by looking for solutions of the form $x(t) = \exp\left[\int_0^t \lambda(s) ds\right]$ for $t \ge -\tau_0$, where λ is a continuous real-valued function on the interval $[-\tau_0, \infty]$.

A *solution* of the generalized characteristic equation (4) is a continuous real-valued function λ defined on the interval $[-\tau_0, \infty]$ and satisfies (4) for all $t \ge 0$.

Mallet-Paret and Lunel [6] obtained Fredholm properties and exponential dichotomies for mixed differential equations (see also similar work by Härterich, Sandstede and Scheel [8]). This approach was recently used successfully by Hupkes and Verduyn Lunel to extend Lin's method to mixed differential equations [9]. Diblík and Vážanová [10] obtained upper and lower estimates for solutions of mixed-type functional differential equations using the monotonic iterative technique and the Schauder-Tychonov fixed point theorem. In the paper by Berezansky, Braverman and Pinelas [11], explicit non-oscillation conditions for mixed advanced-delay differential equations with positive and negative coefficients were obtained. Biçer [12] and Pinelas [13], in their papers dealing with a topic very close to our research, deal with the asymptotic behavior of solutions of mixed type equations using the fixed point theorem. Our aim in this paper is to apply a different method from the papers [12, 13] for the asymptotic behavior of the solutions of equation (1). That is, to obtain the asymptotic behavior of the solutions of equation (1) using a suitable solution of the generalized characteristic equation (3). Philos and Purnaras [14] studied some results on the asymptotic properties and the stability of the solutions to linear autonomous delay, and neutral delay, differential equations. Later, Dix, Philos and Purnaras [15] obtained the asymptotic properties of the solutions to first order linear nonautonomous neutral differential equations. Pinelas, Ramdani, Yeniçerioğlu and Yan [16] investigated the oscillation and stability of the mixed type difference equation. Yeniçerioğlu [17] obtained some results about the behavior of solutions of linear impulsive neutral delay differential equations with constant coefficients. The techniques used to obtain the results are a combination of the methods used in previous works [14-17]. For rudiments of advanced, delayed, and some classes of mixed (advanced-delay) differential equations, we refer the reader to the books [18-20].

2 Statement of results

Theorem 2.1. *Let λ be a solution of the generalized characteristic equation* (4). *Assume that*

$$\sup_{t\geq\tau_0}\left\{\sum_{i\in I}|p_i(t)|\tau_i\exp\left\{-\int_{t-\tau_i}^{t}\lambda(s)ds\right\}+\sum_{j\in J}|q_j(t)|\sigma_j\right.$$
$$\left.\times\exp\left\{\int_{t}^{t+\sigma_j}\lambda(s)ds\right\}+|h(t)|\exp\left[-\int_0^t\lambda(s)ds\right]\right\}<1. \tag{5}$$

Then, for each solution x of (1)-(2) *there exists a constant $K(\lambda;\phi)$ such that*

$$\lim_{t\to\infty}x(t)\exp\left[-\int_0^t\lambda(s)ds\right]=K(\lambda;\phi) \tag{6}$$

and

$$\lim_{t\to\infty}\left\{x(t)\exp\left[-\int_0^t\lambda(s)ds\right]\right\}'=0. \tag{7}$$

Before moving on to the proof of Theorem 2.1, we will present the important consequences of this theorem below. It is clear that $\lambda=0$ is a solution of (4) according to the property (5) if and only if the following conditions are true:

$$\sum_{i\in I}p_i(t)+\sum_{j\in J}q_j(t)=0 \tag{8}$$

and

$$\sup_{t\geq\tau_0}\left\{\sum_{i\in I}|p_i(t)|\tau_i+\sum_{j\in J}|q_j(t)|\sigma_j+|h(t)|\right\}<1. \tag{9}$$

Thus, a special form of Theorem 2.1 is produced, as shown below.

Remark 2.2. Let (8) and (9) be satisfied. Then, each solution x of (1)-(2) satisfies

$$\lim_{t\to\infty} x(t) = K(0;\phi)$$

and

$$\lim_{t\to\infty} x'(t) = 0.$$

Remark 2.3. Under the condition of Theorem 2.1, a solution to (1) cannot grow faster than the exponential function determined by the characteristic equation; i.e., there exists some positive real constant $L(\lambda;\phi)$ such that

$$|x(t)| \leq L(\lambda;\phi) \exp\left[\int_0^t \lambda(s)ds\right] \quad \text{for all } \ t \geq -\tau_0.$$

Moreover, for the solution x of (1)-(2), we have:
(i) x is bounded if

$$\limsup_{t\to\infty} \int_0^t \lambda(s)ds < \infty;$$

(ii) x tends to zero at ∞ if

$$\lim_{t\to\infty} \int_0^t \lambda(s)ds = -\infty.$$

Proof of Theorem 2.1. Let x be the solution of (1)-(2), and consider the function z defined by

$$z(t) = x(t) \exp\left[-\int_0^t \lambda(s)ds\right] \quad \text{for } \ t \geq -\tau_0. \tag{10}$$

We shall prove that $\lim_{t\to\infty} z(t)$ exists and that $\lim_{t\to\infty} z'(t) = 0$. By taking into account the fact that λ is a solution of the generalized characteristic equation (4) and using (1), we obtain

$$\begin{aligned} z'(t) &= \big(x'(t) - x(t)\lambda(t)\big) \exp\left[-\int_0^t \lambda(s)ds\right] \\ &= \Bigg(\sum_{i\in I} p_i(t)x(t-\tau_i) + \sum_{j\in J} q_j(t)x(t+\sigma_j) + h(t) \\ &\quad - x(t)\sum_{i\in I} p_i(t) \exp\left\{-\int_{t-\tau_i}^t \lambda(s)ds\right\} \\ &\quad - x(t)\sum_{j\in J} q_j(t) \exp\left\{\int_t^{t+\sigma_j} \lambda(s)ds\right\}\Bigg) \exp\left[-\int_0^t \lambda(s)ds\right] \end{aligned}$$

for every $t \geq 0$. Using

$$x(t-\tau_i) = z(t-\tau_i)\exp\left[\int_0^{t-\tau_i}\lambda(s)ds\right]$$

and

$$x(t+\sigma_j) = z(t+\sigma_j)\exp\left[\int_0^{t+\sigma_j}\lambda(s)ds\right],$$

the above equality yields

$$\begin{aligned} z'(t) = &-\sum_{i\in I} p_i(t)\left[z(t)-z(t-\tau_i)\right]\exp\left\{-\int_{t-\tau_i}^{t}\lambda(s)ds\right\} \\ &+\sum_{j\in J} q_j(t)\left[z(t+\sigma_j)-z(t)\right]\exp\left\{\int_{t}^{t+\sigma_j}\lambda(s)ds\right\} \\ &+h(t)\exp\left[-\int_0^t\lambda(s)ds\right] \qquad \text{for } t\geq 0. \end{aligned} \tag{11}$$

From (11) it follows immediately that

$$\begin{aligned} z'(t) = &-\sum_{i\in I} p_i(t)\left[\int_{t-\tau_i}^{t} z'(s)ds\right]\exp\left\{-\int_{t-\tau_i}^{t}\lambda(s)ds\right\} \\ &+\sum_{j\in J} q_j(t)\left[\int_{t}^{t+\sigma_j} z'(s)ds\right]\exp\left\{\int_{t}^{t+\sigma_j}\lambda(s)ds\right\} \\ &+h(t)\exp\left[-\int_0^t\lambda(s)ds\right] \qquad \text{for } t\geq \tau_0. \end{aligned} \tag{12}$$

Let

$$\begin{aligned} \eta(\lambda) = \sup_{t\geq\tau_0}\Bigg\{&\sum_{i\in I}|p_i(t)|\tau_i\exp\left\{-\int_{t-\tau_i}^{t}\lambda(s)ds\right\} \\ &+\sum_{j\in J}|q_j(t)|\sigma_j\exp\left\{\int_{t}^{t+\sigma_j}\lambda(s)ds\right\}+|h(t)|\exp\left[-\int_0^t\lambda(s)ds\right]\Bigg\}. \end{aligned}$$

Then, by (5),

$$0 < \eta(\lambda) < 1. \tag{13}$$

Furthermore, we observe that the maximum of $|z'(t)|$ on $[0, \tau_0]$ depends on x and λ; hence, on the initial function ϕ and λ. Set

$$F(\lambda; \phi) = \max \left\{ 1, \max_{0 \leq t \leq \tau_0} |z'(t)| \right\}. \tag{14}$$

We shall show that $F(\lambda; \phi)$ is also a bound of $|z'(t)|$ on the whole interval $[0, \infty)$; i.e.,

$$|z'(t)| \leq F(\lambda; \phi) \quad \text{for all} \;\; t \geq 0. \tag{15}$$

To this end, consider an arbitrary number $\epsilon > 0$. We claim that

$$|z'(t)| < F(\lambda; \phi) + \epsilon \quad \text{for} \quad t \geq 0. \tag{16}$$

Let us assume that inequality (15) is not satisfied. In this case, because of (14), by the continuity of z' , there exist a point $t^* > \tau_0$ such that

$$|z'(t)| < F(\lambda; \phi) + \epsilon, \quad t \in [0, t^*) \cup (t^*, t^* + \sigma_0] \quad \text{and} \quad |z'(t^*)| = F(\lambda; \phi) + \epsilon.$$

Then, by taking into account the definition of $\eta(\lambda)$ and by using (13), from (12) we get

$$\begin{aligned}
F(\lambda; \phi) + \epsilon &= |z'(t^*)| \\
&\leq \sum_{i \in I} |p_i(t^*)| \left[\int_{t^* - \tau_i}^{t^*} |z'(s)| ds \right] \exp \left\{ - \int_{t^* - \tau_i}^{t^*} \lambda(s) ds \right\} \\
&+ \sum_{j \in J} |q_j(t^*)| \left[\int_{t^*}^{t^* + \sigma_j} |z'(s)| ds \right] \exp \left\{ \int_{t^*}^{t^* + \sigma_j} \lambda(s) ds \right\} \\
&+ |h(t^*)| \exp \left[- \int_0^{t^*} \lambda(s) ds \right] \\
&\leq (F(\lambda; \phi) + \epsilon) \left\{ \sum_{i \in I} |p_i(t^*)| \tau_i \exp \left\{ - \int_{t^* - \tau_i}^{t^*} \lambda(s) ds \right\} \right. \\
&+ \sum_{j \in J} |q_j(t^*)| \sigma_j(t^*) \exp \left\{ \int_{t^*}^{t^* + \sigma_j} \lambda(s) ds \right\} \\
&\left. + |h(t^*)| \exp \left[- \int_0^{t^*} \lambda(s) ds \right] \right\} \\
&\leq (F(\lambda; \phi) + \epsilon) \left(\eta(\lambda) \right) < F(\lambda; \phi) + \epsilon,
\end{aligned}$$

which is a contradiction, which establishes our claim, i.e., (16) holds true. As (16) is valid for all real numbers $\epsilon > 0$, it follows that (15) is always satisfied.

By using (15), from (12) we obtain

$$\begin{aligned}
|z'(t)| &\le \sum_{i\in I} |p_i(t)| \left[\int_{t-\tau_i}^{t} |z'(s)|ds\right] \exp\left\{-\int_{t-\tau_i}^{t} \lambda(s)ds\right\} \\
&+ \sum_{j\in J} |q_j(t)| \left[\int_{t}^{t+\sigma_j} |z'(s)|ds\right] \exp\left\{\int_{t}^{t+\sigma_j} \lambda(s)ds\right\} \\
&+ |h(t)| \exp\left[-\int_{0}^{t} \lambda(s)ds\right] \\
&\le F(\lambda;\phi) \left\{\sum_{i\in I} |p_i(t)|\tau_i \exp\left\{-\int_{t-\tau_i}^{t} \lambda(s)ds\right\}\right. \\
&\left. + \sum_{j\in J} |q_j(t)|\sigma_j \exp\left\{\int_{t}^{t+\sigma_j} \lambda(s)ds\right\} + |h(t)| \exp\left[-\int_{0}^{t} \lambda(s)ds\right]\right\} \\
&\le F(\lambda;\phi)\,(\eta(\lambda))
\end{aligned}$$

for all $t \ge \tau_0$. Thus, we have

$$|z'(t)| \le F(\lambda;\phi)\,(\eta(\lambda)) \quad \text{for all} \quad t \ge \tau_0. \tag{17}$$

By using (12) and taking into account the definition of $\eta(\lambda)$ as well as having in mind (15) and (17), one can prove, by an easy induction, that

$$|z'(t)| \le F(\lambda;\phi)\,(\eta(\lambda))^r \quad \text{for all} \quad t \ge r\tau_0 \quad (r = 0, 1, 2, ...). \tag{18}$$

For this purpose, let us consider an arbitrary point $t \ge 0$. Set $r = \left[\left|\frac{t}{\tau_0}\right|\right]$ (the greatest integer less than or equal to $\frac{t}{\tau_0}$). Then $t \ge r\tau_0$ and $\frac{t}{\tau_0} - 1 < r$. So, by applying (18) and using (13), we obtain

$$|z'(t)| \le F(\lambda;\phi)\,(\eta(\lambda))^r \le F(\lambda;\phi)\,(\eta(\lambda))^{\frac{t}{\tau_0}-1}. \tag{19}$$

As $t \to \infty$, we have $r \to \infty$, and by (13), $(\eta(\lambda))^r \to 0$. Thus, by (19),

$$\lim_{t\to\infty} z'(t) = 0$$

which proves the second limit in Theorem 2.1.

We use the Cauchy convergence criterion to prove that $\lim_{t\to\infty} z(t)$ exists. To this end, by taking into account (13), from (19) we have, for $t \geq u \geq 0$

$$\begin{aligned}|z(t)-z(u)| &\leq \int_u^t |z'(s)|ds \leq F(\lambda;\phi)\int_u^t (\eta(\lambda))^{\frac{s}{\tau_0}-1}\,ds \\ &= F(\lambda;\phi)\frac{\tau_0}{\ln(\eta(\lambda))}\left[(\eta(\lambda))^{\frac{s}{\tau_0}-1}\right]_u^t \\ &= F(\lambda;\phi)\frac{\tau_0}{\ln(\eta(\lambda))}\left[(\eta(\lambda))^{\frac{t}{\tau_0}-1}-(\eta(\lambda))^{\frac{u}{\tau_0}-1}\right].\end{aligned}$$

As $u \to \infty$, we have $t \to \infty$, and by (13), the two right-most terms above approach zero. Thus, $\lim_{u\to\infty} |z(t)-z(u)| = 0$ which by the Cauchy convergence criterion implies the existence of $\lim_{t\to\infty} z(t)$. We call this limit $K(\lambda;\phi)$ because it depends on z which in turn depends on the initial function ϕ and λ. This shows the first limit in Theorem 2.1 and completes the proof. □

Example 2.4. Consider the mixed advanced-delay differential equation

$$x'(t) = \frac{1}{2}\frac{(t+1)}{(t+2)^2}\,x(t-1) - \frac{3}{2}\frac{(t+3)}{(t+2)^2}\,x(t+1) + \frac{1}{2}\frac{1}{(t+2)} \quad \text{for } t \geq 0\,. \tag{20}$$

Note that $\tau_0 = 1$ and $\sigma_0 = 1$ in this example. In this case, the generalized characteristic equation (4) becomes

$$\lambda(t) = \frac{1}{2}\frac{(t+1)}{(t+2)^2}\exp\left\{-\int_{t-1}^t \lambda(s)ds\right\} - \frac{3}{2}\frac{(t+3)}{(t+2)^2}\exp\left\{\int_t^{t+1}\lambda(s)ds\right\} \tag{21}$$

for $t \geq 0$. We immediately see that (21) has the solution

$$\lambda(t) = \frac{-1}{t+2} \quad \text{for} \quad t \geq -1.$$

We see that assumption (5) takes the form

$$\begin{aligned}&\sup_{t\geq 1}\left\{\left|\frac{1}{2}\frac{(t+1)}{(t+2)^2}\right|\exp\left\{-\int_{t-1}^t \frac{-1}{s+2}ds\right\} + \left|-\frac{3}{2}\frac{(t+3)}{(t+2)^2}\right|\exp\left\{\int_t^{t+1}\frac{-1}{s+2}ds\right\}\right. \\ &\qquad \left. + \left|\frac{1}{2}\frac{1}{(t+2)}\right|\exp\left\{-\int_0^t \frac{-1}{s+2}ds\right\}\right\} \\ &= \sup_{t\geq 1}\left\{\frac{1}{2}\frac{1}{(t+2)} + \frac{3}{2}\frac{1}{(t+2)} + \frac{1}{4}\right\} = \frac{11}{12} < 1.\end{aligned}$$

This inequality holds true, i.e., condition (5) is always satisfied. Now, with the mixed advanced-delay differential equation (20), we associate the initial condition

$$x(t) = \phi(t) \quad \text{for } -1 \leq t \leq 1, \tag{22}$$

where ϕ is a given continuous real-valued function on $[-1, 1]$. By applying Theorem 2.1 , we conclude that the solution x of the mixed advanced-delay differential equation (20) and (22) satisfies

$$\lim_{t\to\infty} \left[\frac{(t+2)x(t)}{2}\right] = K(\phi)$$

where $K(\phi)$ is some real number determined by ϕ, and

$$\lim_{t\to\infty} \left[\frac{(t+2)x(t)}{2}\right]' = 0.$$

Also, if Remark 2.3 is taken into account, we get

$$|x(t)| \leq L(\phi)\frac{2}{t+2} \quad \text{for all } \ t \geq -1$$

where $L(\phi)$ is some real number determined by ϕ. Furthermore, since

$$\lim_{t\to\infty} \int_0^t \frac{-1}{s+2} ds = -\infty,$$

by Remark 2.3-(ii) this solution satisfies

$$\lim_{t\to\infty} x(t) = 0.$$

3 Conclusions

In this paper, we proved a basic asymptotic result for solutions of a non-homogeneous linear mixed type differential equation. This result is obtained by using a solution of the so-called generalized characteristic equation of the corresponding homogeneous linear mixed type differential equation. An example of this result is given. A solution of the so-called generalized characteristic equation used in this paper plays an important role in determining the result.

We will be particularly concerned with the possibility of generalizing our results in the case of first-order linear mixed (advanced-delay) differential equations with variable delays and coefficients. This seems easy to achieve if the delays are variable and bounded. However, the general situation with variable delays seems somewhat more difficult. It will also be interesting to generalize our results for first-order linear nonhomogeneous mixed (advanced-delay) differential equations with periodic coefficients.

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