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Iterative methods for solving fractional differential equations using non-polynomial splines

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Abstract. In this work, a non-polynomial spline function is proposed to solve a linear fractional differential equation where the derivatives are in the Caputo sense. This approach transforms the fractional differential equation into a system of linear equations. The Gauss-Seidel and conjugate gradient methods are used to iteratively solve the linear system. Finally, to validate the method's accuracy, several numerical examples with known analytical solutions are tested. According to the numerical experiments, the findings are in satisfactory agreement with the exact solutions.

Keywords. Fractional differential equation, non-polynomial spline, conjugate gradient.

2020 Mathematics Subject Classification. 26A33, 34A08, 41A15, 65D07.

1 Introduction

Fractional calculus plays a vital role in a variety of areas due to its extensive applications in several fields of science and technology, including electromagnetism, diffusion processes, signal processing, materials modeling, and mathematical economics (see [2,6,8,21,22]). The general concept of fractional-order differentiation has various definitions, such as the Caputo definition [14], Riemann-Liouville definition [18], and the Grunwald-Letnikov definition [17]. Several approaches to solving fractional differential equations (FDEs) have been developed by mathematicians in recent years. These include the fractional explicit Adams method [26], the homotopy analysis method [11], the fractional finite difference method [16], the Adomain decomposition method [12], the B-polynomial basis approach to solve FDE by Muhammad I. Bhatti and Md. Habibur Rahman [3], the spectral Tau method investigated by Hari Mohan Srivastava et al. [20], the Taylor basis function presented in [13], and the matrix approach method for solving FDE discussed in [5].

The spline technique is used by many researchers to solve differential equations due to its accuracy and efficiency. For example, a sixth-order linear special case boundary value problem was solved using a septic degree spline [19]. Hamasalh

F.K. et al. proposed a non-polynomial spline to solve fractional differential equations in [9].

The conjugate gradient method is a powerful technique for solving systems of equations. The linear conjugate gradient method was proposed in the 1950s by Hestenes and Stiefel to solve linear systems of equations with positive definite matrices as an alternative to the Gauss elimination method [23]. Fletcher and Reeves discussed the nonlinear conjugate gradient method in 1964 [7]. Faraidun K.Hamasalh et al. used conjugate gradient method for solving FDEs [10]. The conjugate gradient method is applied to the optimal solution of differential equations; more details can be found in [25].

2 Mathematical Formulation for Non-Polynomial Spline

In this study, we consider the fractional differential equation of the form

$$y^{(\alpha)} + \phi(x)y'' + \psi(x)y = \tau(x), \quad x \in [a, b], \tag{1}$$

with the boundary conditions

$$y(a) = B_1, \quad y(b) = B_2 \tag{2}$$

such that $\phi(x)$, $\psi(x)$, and $\tau(x)$ are functions of x, B_1 and B_2 are constants. Then, the interval $[a, b]$ can be uniformly divided into j subintervals, the length of uniform subintervals can be defined as: $\Delta x = \frac{b-a}{j}$, $n = j - 1$. In the existing literature, we can modify the model of non-polynomial spline and the fractional continuity by using Caputo type as follows:

$$S(x) = S_i(x), \quad x \in [x_i, x_{i+1}], \quad i = 0, 1, 2, \ldots, n. \tag{3}$$

Here, the non-polynomial spline function with fractional order defined by

$$S_i(x) = \alpha_i + \beta_i(x - x_i) + \gamma_i \cos\omega(x - x_i) + \delta_i \sin\omega(x - x_i), \tag{4}$$

where α_i, β_i, γ_i, δ_i are constants for $i = 0, 1, 2, \ldots, n$ and ω is a free parameter. The function $S_i(x)$ interpolates $y(x)$ at the points x_i by depending on ω. To find the value of constants in equation (4), we impose the following conditions:

$$S_i(x_i) = y_i,\ S_i(x_{i+1}) = y_{i+1},\ S_i^{(1/2)}(x_i) = p_i,\ S_i^{(1/2)}(x_{i+1}) = p_{i+1}. \tag{5}$$

After applying these conditions the values of constants α_i, β_i, γ_i, δ_i in (4) are obtained as follows:

$$\begin{aligned}
\alpha_i &= -\tfrac{1}{M_1}y_{i+1} + \tfrac{M_1+1}{M_1}y_i + \tfrac{\sqrt{\pi h}}{2M_1}p_{i+1} - \tfrac{M_2}{M_1}p_i,\\
\beta_i &= \tfrac{\sqrt{\pi\omega}\sin\theta}{\sqrt{2h}M_1}y_{i+1} - \tfrac{\sqrt{\pi\omega}\sin\theta}{\sqrt{2h}M_1}y_i + (\tfrac{\sqrt{\pi}}{2\sqrt{h}} - \tfrac{\pi\sqrt{\theta}\sin\theta}{2\sqrt{2h}M_1})p_{i+1}\\
&\quad +(\tfrac{\sqrt{\pi\omega}\sin\theta M_2}{\sqrt{2h}M_1} - \tfrac{\sqrt{2\pi}\sin(\theta+\frac{\pi}{4})}{2\sqrt{h}})p_i,\\
\gamma_i &= \tfrac{1}{M_1}y_{i+1} - \tfrac{M_1+1}{M_1}y_i - \tfrac{\sqrt{\pi h}}{2M_1}p_{i+1} + \tfrac{M_2}{M_1}p_i,\\
\delta_i &= -\tfrac{1}{M_1}y_{i+1} + \tfrac{1}{M_1}y_i + \tfrac{\sqrt{\pi h}}{2M_1}p_{i+1} + (\tfrac{\sqrt{2}}{\sqrt{\omega}} - \tfrac{M_2}{M_1})p_i, \quad i = 0,1,2,\ldots,n,
\end{aligned} \tag{6}$$

such that $\theta = \omega h$, $M_1 = \cos\theta + \frac{\sqrt{\pi\theta}-\sqrt{2}}{\sqrt{2}}\sin\theta - 1$, $M_2 = \frac{\sqrt{2\pi h}\sin(\theta+\frac{\pi}{4})}{2} - \frac{\sqrt{2}\sin\theta}{\sqrt{\omega}}$. Substituting these values in (4) we obtain

$$\begin{aligned}
S(x) &= -\tfrac{1}{M_1}y_{i+1} + \tfrac{M_1+1}{M_1}y_i + \tfrac{\sqrt{\pi h}}{2M_1}p_{i+1} - \tfrac{M_2}{M_1}p_i\\
&\quad +(\tfrac{\sqrt{\pi\omega}\sin\theta}{\sqrt{2h}M_1}y_{i+1} - \tfrac{\sqrt{\pi\omega}\sin\theta}{\sqrt{2h}M_1}y_i + (\tfrac{\sqrt{\pi}}{2\sqrt{h}} - \tfrac{\pi\sqrt{\theta}\sin\theta}{2\sqrt{2h}M_1})p_{i+1}\\
&\quad +(\tfrac{\sqrt{\pi\omega}\sin\theta M_2}{\sqrt{2h}M_1} - \tfrac{\sqrt{2\pi}\sin(\theta+\frac{\pi}{4})}{2\sqrt{h}})p_i)(x-x_i)\\
&\quad +(\tfrac{1}{M_1}y_{i+1} - \tfrac{M_1+1}{M_1}y_i - \tfrac{\sqrt{\pi h}}{2M_1}p_{i+1} + \tfrac{M_2}{M_1}p_i)\cos\omega(x-x_i)\\
&\quad +(-\tfrac{1}{M_1}y_{i+1} + \tfrac{1}{M_1}y_i + \tfrac{\sqrt{\pi h}}{2M_1}p_{i+1} + (\tfrac{\sqrt{2}}{\sqrt{\omega}} - \tfrac{M_2}{M_1})p_i)\sin\omega(x-x_i).
\end{aligned} \tag{7}$$

Now, applying the fractional continuity conditions of the spline function $S_i(x)$ where the splines $S_{i-1}^{(m)}(x) = S_i^{(m)}(x), m = \frac{1}{2}, 1$ are joined, we obtain the following equations:

$$\begin{aligned}
S_i'(x_i) &= (\tfrac{\sqrt{\pi}}{2\sqrt{h}} - \tfrac{\pi\sqrt{\theta}\sin\theta}{2\sqrt{2h}M_1} + \tfrac{\omega\sqrt{\pi h}}{2M_1})p_{i+1} + (\tfrac{\sqrt{\pi\omega}\sin\theta M_2}{\sqrt{2h}M_1} - \tfrac{\sqrt{2\pi}\sin(\theta+\frac{\pi}{4})}{2\sqrt{h}})\\
&\quad +\omega(\tfrac{\sqrt{2}}{\sqrt{\omega}} - \tfrac{M_2}{M_1})p_i + (\tfrac{\sqrt{\pi\omega}\sin\theta}{\sqrt{2h}M_1} - \tfrac{\omega}{M_1})y_{i+1} - (\tfrac{\sqrt{\pi\omega}\sin\theta}{\sqrt{2h}M_1} - \tfrac{\omega}{M_1})y_i,
\end{aligned} \tag{8}$$

$$\begin{aligned}
S_{i-1}'(x_i) &= (\tfrac{\sqrt{\pi}}{2\sqrt{h}} - \tfrac{\pi\sqrt{\theta}\sin\theta}{2\sqrt{2h}M_1} + \tfrac{\sqrt{\pi h}\omega}{2M_1}(\sin\theta+\cos\theta))p_i\\
&\quad +(\tfrac{\sqrt{\pi\omega}\sin\theta M_2}{\sqrt{2h}M_1} - \tfrac{\sqrt{2\pi}\sin(\theta+\frac{\pi}{4})}{2\sqrt{h}} + \omega(\tfrac{\sqrt{2}}{\sqrt{\omega}} - \tfrac{M_2}{M_1})\cos\theta - \tfrac{\omega M_2\sin\theta}{M_1})p_{i-1}\\
&\quad +(\tfrac{\sqrt{\pi\omega}\sin\theta}{\sqrt{2h}M_1} - \tfrac{\omega}{M_1}(\cos\theta+\sin\theta))y_i\\
&\quad -(\tfrac{\sqrt{\pi\omega}\sin\theta}{\sqrt{2h}M_1} - \tfrac{\omega}{M_1}(\cos\theta+\sin\theta))y_{i-1}.
\end{aligned} \tag{9}$$

Equating (8) and (9), we obtain:

$$C_1p_{i+1} + C_2p_i + C_3y_{i+1} + C_4y_i + C_5p_{i-1} + C_6y_{i-1} = 0. \tag{10}$$

From equation (1) we have

$$\begin{aligned}
p_{i+1} &= -\phi_{i+1}(x)y''_{i+1} - \psi_{i+1}(x)y_{i+1} + \tau_{i+1}(x),\\
p_i &= -\phi_i(x)y''_i - \psi_i(x)y_i + \tau_i(x),\\
p_{i-1} &= -\phi_{i-1}(x)y''_{i-1} - \psi_{i-1}(x)y_{i-1} + \tau_{i-1}(x).
\end{aligned} \tag{11}$$

Using backward, central, and forward difference formulas for y''_{i+1}, y''_i, and y''_{i-1}, respectively, we obtain:

$$\begin{aligned} y''_{i+1} &= \tfrac{y_{i+1}-2y_i+y_{i-1}}{h^2},\\ y''_i &= \tfrac{y_{i+1}-2y_i+y_{i-1}}{h^2},\\ y''_{i-1} &= \tfrac{y_{i+1}-2y_i+y_{i-1}}{h^2}. \end{aligned} \tag{12}$$

Substituting (11) and (12) in equation (10), we obtain:

$$a_i y_{i-1} + b_i y_i + c_i y_{i+1} = F_i. \tag{13}$$

Then, a system of linear equations is formulated using equation (13) as follows:

$$Ay = F \tag{14}$$

such that

$$A = \begin{bmatrix} b_1 & c_1 & & & & \\ a_2 & b_2 & c_2 & & & \\ & a_3 & b_3 & c_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & a_{n-1} & b_{n-1} & c_{n-1} \\ & & & & a_n & b_n \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix}$$

and $F = [F_1 - a_1 y_0 \;\; F_2 \;\; F_3 \;\; \cdots \;\; F_{n-1} \;\; F_n - c_n y_{n+1}]^T$, where

$$\begin{aligned} a_i &= C_6 + \tfrac{C_1\phi_{i+1}}{h^2} - \tfrac{C_2\phi_i}{h^2} - \tfrac{C_5\phi_{i-1}}{h^2} - C_5\psi_{i-1},\\ b_i &= C_4 + \tfrac{2C_1\phi_{i+1}}{h^2} - \tfrac{2C_2\phi_i}{h^2} - \tfrac{2C_5\phi_{i-1}}{h^2} - C_2\psi_i,\\ c_i &= C_3 + \tfrac{C_1\phi_{i+1}}{h^2} - \tfrac{C_2\phi_i}{h^2} - \tfrac{C_5\phi_{i-1}}{h^2} - C_1\psi_{i+1},\\ F_i &= -C_1\tau_{i+1} - C_2\tau_i - C_5\tau_{i-1}, \quad i = 1, 2, \ldots, n \end{aligned}$$

and

$$\begin{aligned} C_1 &= \tfrac{\sqrt{\pi}}{2\sqrt{h}} - \tfrac{\pi\sqrt{\theta}\sin\theta}{2\sqrt{2h}M_1} + \tfrac{\omega\sqrt{\pi h}}{2M_1},\\ C_2 &= \tfrac{\sqrt{\pi\omega}\sin\theta}{\sqrt{2h}M_1} - \tfrac{\sqrt{2\pi}\sin(\theta+\frac{\pi}{4})}{2\sqrt{h}} + \omega\Big(\tfrac{\sqrt{2}}{\sqrt{\omega}} - \tfrac{M_2}{M_1}\Big) + \tfrac{\pi\sqrt{\theta}\sin\theta}{2\sqrt{2h}M_1} + \tfrac{\sqrt{\pi h}\omega}{2M_1}(\sin\theta + \cos\theta) - \tfrac{\sqrt{\pi}}{2\sqrt{h}},\\ C_3 &= \tfrac{\sqrt{\pi\omega}\sin\theta}{\sqrt{2h}M_1} - \tfrac{\omega}{M_1},\\ C_4 &= \tfrac{\omega}{M_1}(1 + \sin\theta + \cos\theta) - 2\tfrac{\sqrt{\pi\omega}\sin\theta}{\sqrt{2h}M_1},\\ C_5 &= \tfrac{\sqrt{\pi\omega}M_2\sin\theta}{\sqrt{2h}M_1} - \tfrac{\sqrt{2\pi}\sin(\theta+\frac{\pi}{4})}{2\sqrt{h}} + \omega\Big(\tfrac{\sqrt{2}}{\sqrt{\omega}} - \tfrac{M_2}{M_1}\Big)\cos\theta - \tfrac{\omega M_2\sin\theta}{M_1},\\ C_6 &= \tfrac{\omega}{M_1}(\sin\theta + \cos\theta) - \tfrac{\sqrt{\pi h}\omega}{2M_1}\sin\theta. \end{aligned}$$

3 Numerical Experiments

In this section, the method is applied to several numerical examples of boundary value problems for fractional differential equations and the results are compared with exact analytical solutions to show the method's efficiency. The computational programs were written in MATLAB. We first present here the algorithms of Gauss-Seidel and the conjugate gradient methods.

Algorithm 1. [4] Suppose that we have the linear system (14), where A is symmetric positive definite matrix. First, matrix A is decomposed as $A = D + L + U$ such that D is a diagonal matrix, L is strictly lower triangular matrix and U is strictly upper triangular matrix. Then, the linear system (13) can be written as:

$$(D + L + U)y = F$$

or

$$(D + L)y = -Uy + F.$$

Since $|D + L| \neq 0$, we obtain

$$y = (D + L)^{-1}Uy + (D + L)^{-1}F.$$

Then, the Gauss-Seidel (GS) algorithm can be written as:

- start with an initial guess $y^{(0)} \in \mathbb{R}^n$;
- compute $y^{(i+1)} = -(D + L)^{-1}Uy^{(i)} + (D + L)^{-1}F$ for $i = 0, 1, 2, \ldots$

Algorithm 2. [23] The conjugate gradient (CG) algorithm is expressed as:

- choose $y_0 \in \mathbb{R}^n$ and put $d_0 = r_0 = F - Ay_0$;
- for $k = 0, 1, 2, \ldots$
  - If $d_k = 0$, stop and y_k is a solution of $Ay = F$.
  - Otherwise, compute
    * $\alpha_k = \frac{r_k^T r_k}{d_k^T A d_k}, \quad y_{k+1} = y_k + \alpha_k d_k,$
    * $r_{k+1} = r_k - \alpha_k A d_k, \quad \beta_k = \frac{r_{k+1}^T r_{k+1}}{r_k^T r_k},$
    * $d_{k+1} = r_{k+1} + \beta_k d_k.$

Example 1. [1] Consider the fractional differential equation

$$D^{(\lambda)}y(x) + y(x) = x^4 - \frac{1}{2}x^3 + \frac{24}{\Gamma(5-\lambda)}x^{4-\lambda} - \frac{3}{\Gamma(4-\lambda)}x^{3-\lambda}, \ 0 < \lambda \leq 1 \quad (15)$$

with the boundary conditions $y(0) = 0$, $y(1) = \frac{1}{2}$ and the exact solution given by $y(x) = x^4 - \frac{1}{2}x^3$.

The number of iterations and numerical results compared to exact values for $\lambda = 0.5$, $x \in [0, 1]$, $h = 0.01$ are tabulated in Tables 1 and 2, respectively.

Number of iterations	
GS	450
CG	65

Table 1. The number of iterations for both methods applied to problem (15).

x	Exact solution	Numerical solution	Absolute error	Absolute error [1]
0.1	−0.0004	−0.00096	5.61×10^{-4}	4.00×10^{-4}
0.2	−0.0024	−0.00324	8.37×10^{-4}	1.1668×10^{-3}
0.3	−0.0054	−0.00498	4.19×10^{-4}	2.9299×10^{-4}
0.4	−0.0064	−0.00259	3.8×10^{-3}	4.5080×10^{-4}
0.5	0	0.008925	8.92×10^{-3}	2.2930×10^{-3}
0.6	0.0216	0.035829	1.42×10^{-2}	6.5464×10^{-3}
0.7	0.0686	0.097402	2.88×10^{-2}	1.5311×10^{-2}
0.8	0.1536	0.166854	1.32×10^{-4}	2.7885×10^{-2}
0.9	0.2916	0.290152	1.44×10^{-3}	4.5954×10^{-2}

Table 2. Exact solution, its numerical approximation, and absolute error in the numerical solution of problem (15).

Example 2. [24] Consider the fractional differential equation

$$D^{(\lambda)}y(x) + y(x) = x^2 + \frac{2x^{2-\lambda}}{\Gamma(3-\lambda)}, \quad 0 < \lambda \le 1 \tag{16}$$

with the boundary conditions $y(0) = 0$ and $y(1) = 1$. The exact solution of (16) is given by $y(x) = x^2$.

The numerical result obtained for $\lambda = 0.5$, $x \in [0, 1]$, $h = \frac{1}{160}$ is tabulated in Table 3.

x	Exact solution	Numerical solution	Absolute error
0.1	0.01	0.029064	1.9064×10^{-2}
0.2	0.04	0.075703	3.5703×10^{-2}
0.3	0.09	0.138792	4.8792×10^{-2}
0.4	0.16	0.216701	5.6701×10^{-2}
0.5	0.25	0.308459	5.8459×10^{-2}
0.6	0.36	0.413401	5.3401×10^{-2}
0.7	0.49	0.531033	4.1033×10^{-2}
0.8	0.64	0.660969	2.6909×10^{-2}
0.9	0.81	0.808386	1.6140×10^{-3}

Table 3. Exact and numerical solutions, and absolute error for problem (16).

Example 3. [15] Consider the fractional differential equation

$$y''(x) + D^{(\lambda)}y(x) + y(x) = 8, \quad 0 < \lambda < 1 \tag{17}$$

with the boundary conditions $y(0) = 0$ and $y(1) = 3.101906$.

The numerical solution for $\lambda = 0.5$, obtained by using CG method, is presented in Table 4.

x	Exact solution	Numerical solution	Absolute error	Absolute error [15]
0.1	0.03975	0.025277	1.4473×10^{-2}	1.24×10^{-4}
0.2	0.157036	0.138583	1.8453×10^{-2}	1.476×10^{-3}
0.3	0.347370	0.333777	4.8792×10^{-2}	6.255×10^{-3}
0.4	0.604695	602914	5.6701×10^{-2}	1.73×10^{-2}
0.5	0.921768	0.935951	5.8459×10^{-2}	3.82×10^{-2}
0.6	1.290457	1.320995	5.3401×10^{-2}	7.26×10^{-2}
0.7	1.702008	1.744681	4.1033×10^{-2}	1.2424×10^{-1}
0.8	2.147287	2.19264	2.0969×10^{-2}	1.9693×10^{-1}
0.9	2.617001	2.649994	1.614×10^{-3}	2.9427×10^{-1}

Table 4. Exact and numerical solutions, and absolute error for problem (17).

4 Conclusion

This paper develops a trigonometric spline method for solving fractional differential equations in conjunction with the conjugate gradient method. The findings related to non-polynomial spline functions are particularly interesting. The numerical examples illustrate that the non-polynomial spline and conjugate gradient approaches are more adaptive in approximating functions.

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