e-Journal of Analysis and Applied Mathematics **2024** (2024), 36–46
DOI 10.62780/ejaam/2024-004 EJAAM 2024

On p-convexification of the Banach-Kantorovich lattice

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Communicated by Yasar Sozen

Abstract. Let B be a complete Boolean algebra, $Q(B)$ the Stone compact of B, and let $C_\infty(Q(B))$ be the commutative unital algebra of all continuous functions $x : Q(B) \to [-\infty, +\infty]$, assuming possibly the values $\pm\infty$ on nowhere-dense subsets of $Q(B)$. Let $(E, \|\cdot\|_E) \subset C_\infty(Q(B))$ be a Banach-Kantorovich lattice over the algebra $L^0(\Omega)$ of equivalence classes of almost everywhere finite real-valued measurable functions on a measurable space (Ω, Σ, μ) with σ-finite measure μ. The paper defines the p-convexification of the Banach-Kantorovich lattice $(E, \|\cdot\|_E)$ and proves that it is also a Banach-Kantorovich lattice over $L^0(\Omega)$.

Keywords. p-convexification, Maharam measure, Banach-Kantorovich space.

2020 Mathematics Subject Classification. 46B04, 46B42, 46E30, 46G10.

1 Introduction

One of the important tools in studying the isomorphic properties of Banach lattices is the concepts of p-convexity and q-concavity of these lattices. For example, these concepts are actively used in the study of the uniform convexity in Banach lattices, as well as in the study of the properties of symmetric functional spaces (see [1]).

In [1], a general procedure for constructing p-convex and q-concave lattices is given, starting from an arbitrary Banach lattice, namely, for a given Banach lattice X, its p-convexification X^p is determined. In case X is a Banach lattice of functions, X^p can be identified with the space of all functions f so that $|f|^p \in X$ equipped with the norm $\|f\|_{X^p} = \||f|^p\|_X^{\frac{1}{p}}$. It is known that for a Banach lattice $(X, \|\cdot\|_X)$, its p-convexification $(X^p, \|\cdot\|_{X^p})$ is also a Banach lattice. In addition, properties such as the order continuity of the norm and the Fatou property carry over from X to X^p.

The development of the theory of Banach-Kantorovich spaces naturally involves the introduction and study of the properties of p-convexity and q-concavity of these spaces.

Let B be a complete Boolean algebra, $Q(B)$ the Stone compact of B. Denote by $L^0(B)$ the algebra $C_\infty(Q(B))$ of all continuous functions $x : Q(B) \to [-\infty, +\infty]$, assuming possibly the values $\pm\infty$ on nowhere-dense subsets of $Q(B)$.

Let $(E, \|\cdot\|_E) \subset L^0(B)$ be a lattice-normed space over the algebra $L^0(\Omega)$ of equivalence classes of almost everywhere finite real-valued measurable functions on a measurable space (Ω, Σ, μ) with σ-finite measure μ. In this paper, we define the p-convexification of a lattice-normed space $(E, \|\cdot\|_E)$ and prove that if $(E, \|\cdot\|_E)$ is a Banach-Kantorovich lattice over $L^0(\Omega)$, then its p-convexification is also a Banach-Kantorovich lattice over $L^0(\Omega)$.

We use the terminology and notation of the theory of Boolean algebras from [2], the theory of vector lattices from [3], the theory of vector integration and the theory of Banach-Kantorovich spaces from [4], as well as the terminology of the general theory of Banach lattices from [1].

2 Preliminaries

Let E be a vector lattice, i.e. an ordered vector space that is also a lattice. Thereby in a vector lattice there exist a least upper bound sup$\{x_1, \dots, x_n\} := x_1 \vee \dots \vee x_n$ and a greatest lower bound inf$\{x_1, \dots, x_n\} := x_1 \wedge \dots \wedge x_n$ for every finite set $\{x_1, \dots, x_n\} \subset E$. In particular, every element $x \in E$ has the positive part $x_+ := x \vee 0$, the negative part $x_- := (-x)_+ ; = -x \wedge 0$, and the modulus $|x| := x \vee (-x) = x_+ + x_-$. Let $E_+ = \{x \in E : x \geq 0\}$. An order interval in E is a set of the form $[a, b] := \{x \in X : a \leq x \leq b\}$, where $a, b \in E$. Two elements x and y are called disjoint if $|x| \wedge |y| = 0$.

A linear subspace J of a vector lattice E is called an order ideal if the inequality $|x| \leq |y|$ implies $x \in J$ for arbitrary $x \in E$ and $y \in J$. Every order ideal of a vector lattice is a vector lattice. A set in E is called order bounded if it is included in some order interval.

A vector lattice is called Dedekind complete if every non-empty order bounded set in it has least upper and greatest lower bounds. If, in a vector lattice, least upper and greatest lower bounds exist only for countably bounded sets, then it is called σ-Dedekind complete.

We say that a Dedekind complete (σ-Dedekind complete) vector lattice is extended if its every subset (countably subset) of pairwise disjoint elements is bounded.

Let (Ω, Σ, μ) be a σ-finite measure space, and let $L^0(\Omega) = L^0(\Omega, \Sigma, \mu)$ be the algebra of equivalence classes of almost everywhere finite real-valued measurable functions on (Ω, Σ, μ). With respect to the partial order $f \leq g \Leftrightarrow g - f \geq 0$ (almost everywhere), the algebra $L^0(\Omega)$ is a Dedekind complete vector lattice with a weak unit $\mathbf{1}(\omega) \equiv 1$, and the set $B(\Omega)$ of all idempotents in $L^0(\Omega)$ is a complete

Boolean algebra with respect to the partial order induced from $L^0(\Omega)$.

Let X be a vector space over the field $\mathbb{R}$ of real numbers. A mapping $\|\cdot\| : X \to L^0(\Omega)$ is called an $L^0(\Omega)$-valued norm on X if the following relations hold for any $x, y \in X$ and $\lambda \in \mathbb{R}$:

(1) $\|x\| \geq 0$, $\|x\| = 0 \Leftrightarrow x = 0$;

(2) $\|\lambda x\| = |\lambda|\, \|x\|$;

(3) $\|x + y\| \leq \|x\| + \|y\|$.

The pair $(X, \|\cdot\|)$ is called a lattice-normed space (LNS for short) over $L^0(\Omega)$. A lattice-normed space X is said to be decomposable (d-decomposable) if for any $x \in X$ and any decomposition $\|x\| = f_1 + f_2$ into a sum of nonnegative (respectively, disjunct) elements $f_1, f_2 \in L^0(\Omega)$, there exist $x_1, x_2 \in X$ such that $x = x_1 + x_2,$ and $\|x_k\| = f_k,\ k = 1, 2.$

Suppose that X is a vector lattice. The norm $\|\cdot\|$ is monotone if $|x| \leq |y|$ implies $\|x\| \leq \|y\|$ for $x, y \in X$. If a lattice-normed space $(X, \|\cdot\|)$ over $L^0(\Omega)$ is a vector lattice with a monotone norm, then it is called a lattice-normed vector lattice over $L^0(\Omega)$.

A net $\{x_\alpha\}_{\alpha \in A}$ of elements of $(X, \|\cdot\|)$ is said to (bo)-converge to $x \in X$ if the net $\{\|x - x_\alpha\|\}_{\alpha \in A}$ (o)-converges to zero in the lattice $L^0(\Omega)$ (recall that the o-convergence of a net in $L^0(\Omega)$ is equivalent to its convergence almost everywhere). A net $\{x_\alpha\}_{\alpha \in A} \subset X$ is called (bo)-fundamental if the net $\{x_\alpha - x_\beta\}_{(\alpha,\beta) \in A \times A}$ (bo)-converges to zero. A lattice-normed space is called (bo)-complete if every (bo)-fundamental net in it (bo)-converges to an element of this space.

Every d-decomposable (bo)-complete lattice-normed space is called a Banach-Kantorovich space (BKS for short). If a Banach Kantorovich space is in addition a vector lattice and the norm is monotone, then it is called a Banach-Kantorovich lattice. Every BKS is a decomposable LNS (see [4],[5]).

The theory of the integral of elements of an extended σ-Dedekind complete vector lattice by a σ-additive measure with values in a bo-complete lattice-normed space has proved to be very effective for constructing useful examples of Banach-Kantorovich spaces. Let us recall some basic notions of the theory of vector integration (see [4],[6]).

Let B be a complete Boolean algebra with zero $\mathbf{0}$ and unit $\mathbf{1}$. The exact upper and lower bounds of a set $\{e, q\} \subset B$ are denoted by $e \vee q$ and $e \wedge q$. A Boolean subalgebra A in B is called a regular if $\sup E \in A$, and $\inf E \in A$ for any subset $E \subset A$. Every regular Boolean subalgebra in B is a complete Boolean algebra.

A mapping $m : B \to L^0(\Omega)$ is called a $L^0(\Omega)$-*valued measure* if it satisfies the following conditions: $m(e) \geq 0$ for all $e \in B$; $m(e \vee g) = m(e) + m(g)$ for any $e, g \in B$ with $e \wedge g = \mathbf{0}$; $m(e_\alpha) \downarrow 0$ for any net $e_\alpha \downarrow \mathbf{0},\ \{e_\alpha\} \subset B$.

A measure m is said to be *strictly positive*, if $m(e) = 0$ implies $e = \mathbf{0}$. A strictly positive $L^0(\Omega)$-valued measure m is said to be *decomposable*, if for any $e \in B$ and a decomposition $m(e) = f_1 + f_2$, $f_1, f_2 \in L^0(\Omega)_+$ there exist $e_1, e_2 \in B$, such that $e = e_1 \vee e_2$, $m(e_1) = f_1$ and $m(e_2) = f_2$. A measure m is decomposable if and only if it is a Maharam measure, that is, the measure m is strictly positive and for any $e \in B$, $0 \leq f \leq m(e)$, $f \in L^0(\Omega)$, there exists $q \in B$, $q \leq e$ such that $m(q) = f$ [7].

The following statement shows that, in the case of the Maharam measure m, there is a natural embedding of the Boolean algebra $B(\Omega)$ into the Boolean algebra B.

Proposition 2.1. [8, Proposition 3.2] *For each $L^0(\Omega)$-valued Maharam measure $m : B \to L^0(\Omega)$ there exists a unique injective completely additive Boolean homomorphism $\varphi : B(\Omega) \to B$ such that $\varphi(B(\Omega))$ is a regular Boolean subalgebra of B, and $m(\varphi(q)e) = qm(e)$ for all $q \in B(\Omega)$, $e \in B$.*

Let $Q(B)$ be the Stone compact of a complete Boolean algebra B, and let $L^0(B) := C_\infty(Q(B))$ be the algebra of all continuous functions $x : Q(B) \to [-\infty, +\infty]$, assuming possibly the values $\pm\infty$ on nowhere-dense subsets of $Q(B)$. With respect to the partial order $x \leq y \Leftrightarrow y(t) - x(t) \geq 0$ for all $t \in Q(B) \setminus \big(x^{-1}(\pm\infty) \cup y^{-1}(\pm\infty)\big)$, the algebra $L^0(B)$ is an extended σ-Dedekind complete vector lattice (see [[4, 1.4.2]]).

We identify B with the complete Boolean algebra of all idempotents in $L^0(B)$, i.e., we assume $B \subset L^0(B)$. According to Proposition 2.1, for the Maharam measure $m : B \to L^0(\Omega)$, there exists a regular Boolean subalgebra $\nabla(m)$ in B and a Boolean isomorphism φ from $B(\Omega)$ onto $\nabla(m)$ such that $m(\varphi(q)e) = qm(e)$ for all $q \in B(\Omega)$, $e \in B$. In this case, the algebra $L^0(\Omega)$ is identified with the algebra $L^0(\nabla(m)) = C_\infty(Q(\nabla(m)))$ (the corresponding isomorphism will also be denoted by φ), and the algebra $C_\infty(Q(\nabla(m)))$ itself can be considered as a subalgebra and as a regular vector sublattice in $L^0(B)$ (this means that the exact upper and lower bounds for bounded subsets of $L^0(\nabla(m))$ are the same in $L^0(B)$ and in $L^0(\nabla(m))$).

Denote by $\mathbb{N}$ the set of all natural numbers, and for each element $x \in L^0(B)$ we define its carrier $s(x) := \sup_{n\in\mathbb{N}}\{|x| > n^{-1}\}$, where $\{|x| > \lambda\} \in B$ is the characteristic function χ_{E_λ} of the set E_λ which is the closure in $Q(B)$ of the set $\{t \in Q(B) : |x(t)| > \lambda\}$, $\lambda \in \mathbb{R}$.

We now specify the vector integral of the [4] for elements of some abstract σ-Dedekind complete vector lattice. Take as an extended σ-Dedekind complete vector lattice the algebra $L^0(B)$. Consider in $L^0(B)$ the vector sublattice $\mathcal{S}(B)$ of all B-simple elements of $x = \sum_{i=1}^{n} \alpha_i e_i$, where $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and $e_1, \ldots, e_n \in B$ are pairwise disjoint. Let $m : B \to L^0(\Omega)$ be a $L^0(\Omega)$-valued measure on B. If $x \in \mathcal{S}(B)$ then we put by definition

$$I_m(x) := \int x\, dm := \sum_{k=1}^{n} \alpha_k m(e_k).$$

As it was described in [4], the integral I_m can be extended to the spaces of m-integrable elements $\mathcal{L}^1(B, m)$. On identifying equivalent elements, we obtain the K_σ-space $L^1(B, m)$. For each $x \in L^1(B, m)$ (the entry $x \in L^1(B, m)$ means that an equivalence class with a representative of x is considered) the formula

$$\|x\|_{1,m} := \int |x| dm$$

defines an $L^0(\Omega)$-valued norm, that is $(L^1(B, m), \|x\|_{1,m})$ is a lattice-normed space over $L^0(\Omega)$ (see [4, 6.1.3]). Moreover, in the case when $m : B \to L^0(\Omega)$ is a Maharam measure, the pair $(L^1(B, m), \|x\|_{1,m})$ is a Banach-Kantorovich space. In addition, $L^0(\nabla(m)) \cdot L^1(B, m) \subset L^1(B, m)$, $\int (\varphi(\alpha)x) dm = \alpha \int x dm$ for all $x \in L^1(B, m)$, $\alpha \in L^0(\Omega)$ [4, Theorem 6.1.10].

Let $p \in [1, \infty)$, and let

$$L^p(B, m) = \{x \in L^0(B) : |x|^p \in L^1(B, m)\},$$

$$\|x\|_{p,m} := \Big[\int |x|^p dm \Big]^{\frac{1}{p}}, \quad x \in L^p(B, m).$$

It is known that for the Maharam measure m the pair $(L^p(B, m), \|x\|_{p,m})$ is the Banach-Kantorovich space [6, 4.2.2]. In addition,

$$\varphi(\alpha)x \in L^p(B, m) \ \forall \ x \in L^p(B, m), \ \alpha \in L^0(\Omega), \ 1 \le p < \infty,$$

and $\|\varphi(\alpha)x\|_{p,m} = |\alpha| \|x\|_{p,m}$.

3 On p-convexification of Banach-Kantorovich lattices over the ring of measurable functions

Let m be an $L^0(\Omega)$-valued Maharam measure on a complete Boolean algebra B. In the rest of this section we assume that $m(\mathbf{1}) = \mathbf{1}$.

Let E be a nonzero linear subspace in $L^0(B)$, and let $\| \cdot \|_E$ be an $L^0(\Omega)$-valued norm on E, which endows E with the structure of a lattice-normed vector lattice over $L^0(\Omega)$. Let $0 < p, q < \infty$. The lattice-normed vector lattice E is said to be *p-convex* (respectively, *q-concave*) if and only if there exists a constant $M > 0$ such that, for any finite sequence $\{x_k\}_{k=1}^n \subset E$,

$$\Big\|\Big(\sum_{k=1}^n |x_k|^p\Big)^{1/p}\Big\|_E \leq M\Big(\sum_{k=1}^n \|x_k\|_E^p\Big)^{1/p}, \tag{1}$$

respectively,

$$\Big(\sum_{k=1}^n \|x_k\|_E^q\Big)^{1/q} \leq M\Big\|\Big(\sum_{k=1}^n |x_k|^q\Big)^{1/q}\Big\|_E. \tag{2}$$

The least constant $M > 0$ satisfying (1) (respectively (2)) is called the *p-convexity* (respectively, *q-concavity*) constant of E and will be denoted by $M^{(p)}(E)$ (respectively, $M_{(q)}(E)$). It is clear that lattice-normed vector lattice E over $L^0(\Omega)$ is 1-convex with convexity constant 1. If $E = L^p(B, m)$, $1 \leq p < \infty$, then for all finite sequences $\{x_k\}_{k=1}^n \subset L^p(B, m)$,

$$\Big\|\Big(\sum_{k=1}^n |x_k|^p\Big)^{1/p}\Big\|_{p,m} = \Big(\sum_{k=1}^n \|x_k\|_{p,m}^p\Big)^{1/p}.$$

Consequently, $L^p(B, m)$ is p-convex and q-concave and moreover

$$M^{(p)}(L^p(B, m)) = M_{(p)}(L^p(B, m)) = 1.$$

Let $E \subseteq L^0(B)$ is a lattice-normed vector lattice over $L^0(\Omega)$. Since $L^0(B)$ is a σ-Dedekind complete vector lattice, then E is an order ideal. Hence, E is also a σ-Dedekind complete vector lattice. In addition, it is possible to define the structure of the $L^0(\Omega)$-module on E as follows: $\lambda \cdot x = \varphi(\lambda)x$, $x \in E$, $\lambda \in L^0(\Omega)$, where φ is an isomorphism from $L^0(\Omega)$ to $L^0(\nabla(m))$ (see the remark made after Proposition 2.1).

Following the work [1, chapter 1, section d], we consider the construction of p-convexification for lattice-normed vector lattices $(E, \|\cdot\|_E)$ over $L^0(\Omega)$.

First of all, let us focus on the concept of the p-th degree of an element of space $L^0(B)$. Let's take an arbitrary $x \in L^0(B), x \geq 0$. The set $G = \{t \in Q(B) : -\infty < x(t) < +\infty\}$ is dense and open in $Q(B)$. For the number $p \geq 0$, put $y(t) = (x(t))^p, t \in G$. Since $y = y(t)$ is a continuous function on G, there exists a unique continuous extension of $y(t)$ onto the whole of $Q(B)$ (see [3], Lemma V.2.1). Denote this extension by x^p. Note that $x^p(t) = +\infty$ if and only if $x(t) = +\infty$.

For $1 \leq p < \infty$, the set $E^{(p)} \subseteq L^0(B)$ is defined by setting

$$E^{(p)} = \{x \in L^0(B) : |x|^p \in E\},$$

and set

$$\|x\|_{E^{(p)}} = \||x|^p\|_E^{\frac{1}{p}}, \ x \in E^{(p)}.$$

It is clear that in the case of $E = (L^1(B,m), \|\cdot\|_{1,m})$ the equalities hold

$$L^1(B,m)^{(p)} = L^p(B,m) := \{x \in L^0(B) : |x|^p \in L^1(B,m)\},$$

$$\|x\|_{L^1(B,m)^{(p)}} = \||x|^p\|_{1,m}^{\frac{1}{p}} = \|x\|_{p,m}.$$

Let us first show that $(E^{(p)}, \|\cdot\|_{E^{(p)}})$ is a lattice-normed vector lattice over $L^0(\Omega)$.

Lemma 3.1. *$E^{(p)}$ is an order ideal in $L^0(B)$, that is, a linear subspace in $L^0(B)$ with the property of ideality.*

Proof. It is clear that $\alpha x \in E^{(p)}$ whenever $x \in E^{(p)}$ and $\alpha \in \mathbb{R}$. Let $x, y \in E^{(p)}$, $e_1 = \{|x| \geq |y|\} := s((|x| - |y|)_+)$, $e_2 = \mathbf{1} - e_1$. Then it follows from the estimate

$$|x+y|^p e_1 \leq (|x| + |y|)^p e_1 = (|x|e_1 + |y|e_1)^p \leq 2^p |x|^p e_1 \leq 2^p |x|^p$$

and from the fact that E is an order ideal in $L^0(B)$, that $|x+y|^p e_1 \in E$. Similarly, $|x+y|^p e_2 \in E$. Consequently, $|x+y|^p = |x+y|^p e_1 + |x+y|^p e_2 \in E$, and $x + y \in E^p$. Thus, $E^{(p)}$ is a linear subspace in $L^0(B)$. Furthermore, it is clear that $x \in E^{(p)}$ whenever $x \in L^0(B)$ satisfies $|x| \leq |y|$ for some $y \in E^{(p)}$. □

Lemma 3.2. *Suppose that $1 < p, q, r < \infty$ satisfy $1/p + 1/q = 1/r$. If $x \in E^{(p)}$ and $y \in E^{(q)}$, then $xy \in E^{(r)}$ and $\|xy\|_{E^{(r)}} \leq \|x\|_{E^{(p)}} \|y\|_{E^{(q)}}$.*

Proof. Consider first the case that $r = 1$. For a proof, it may be assumed that $\|x\|_{E^{(p)}} = \|y\|_{E^{(q)}} = 1$. Young's inequality

$$|xy| \leq \frac{1}{p}|x|^p + \frac{1}{q}|y|^q,$$

implies that $xy \in E$ and that

$$\|xy\|_E \leq \frac{1}{p}\||x|^p\|_E + \frac{1}{q}\||y|^q\|_E = \frac{1}{p} + \frac{1}{q} = 1.$$

This proves the case $r = 1$.

To prove the general case, suppose that $x \in E^{(p)}$ and $y \in E^{(q)}$. Observing that $|x|^r \in E^{(p/r)}$ and $|y|^r \in E^{(q/r)}$, it follows from the first part of the proof that $|xy|^r \in E$ and that

$$\|xy\|_{E^{(r)}} = \||xy|^r\|_E^{1/r} \leq \||x|^r\|_{E^{(p/r)}}^{1/r} \||y|^r\|_{E^{(q/r)}}^{1/r} = \|x\|_{E^{(p)}} \|y\|_{E^{(q)}}.$$

□

Lemma 3.3. *$\|\cdot\|_{E^{(p)}}$ is a monotone $L^0(\Omega)$-valued norm on $E^{(p)}$.*

Proof. It is clear that $\|\alpha x\|_{E^{(p)}} = |\alpha|\|x\|_{E^{(p)}}$ for all $x \in E^{(p)}$ and $\alpha \in \mathbb{R}$. Moreover, $\|x\|_{E^{(p)}} \geq 0$ and $\||x|^p\|_E = 0$ if and only if $x = 0$.

For the proof of the triangle inequality it may be assumed that $p > 1$. Let $x, y \in E^{(p)}$, $f = \|x\|^p_{E^{(p)}}$, $g = \|y\|^p_{E^{(p)}}$, $h = \|x+y\|^p_{E^{(p)}}$. It must be shown that $h^{1/p} \leq f^{1/p} + g^{1/p}$. It is clear that $|x+y|^{p-1} \in E^{(q)}$ with $\||x+y|^{p-1}\|_{E^{(q)}} = \|x+y\|^{p/q}_{E^{(p)}}$ (where $1/p + 1/q = 1$). Therefore, via Lemma 3.2 (with $r = 1$), it follows that

$$\|x+y\|^p_{E^{(p)}} = \||x+y|^p\|_E = \||x+y||x+y|^{p-1}\|_E \leq$$

$$\||x||x+y|^{p-1}\|_E + \||y||x+y|^{p-1}\|_E \leq (\|x\|_{E^{(p)}} + \|y\|_{E^{(q)}})\|x+y\|^{p/q}_{E^{(p)}}.$$

Thus, we have the following inequality $h \leq h^{1/q}(f^{1/p} + g^{1/p})$. Therefore, $h^{1/q}(f^{1/p} + g^{1/p} - h^{1/p}) \geq 0$, which implies that $f^{1/p} + g^{1/p} - h^{1/p} \geq 0$. Indeed, if we assume that there is an idempotent $e \in B(\Omega)$ and a number $\varepsilon > 0$ such that $(f^{1/p} + g^{1/p} - h^{1/p})e < -\varepsilon e$, then $he \neq 0$, and multiplying both parts of the latter inequality by $h^{1/q}e$, we get $h^{1/q}(f^{1/p} + g^{1/p} - h^{1/p})e < 0$, which contradicts to what has already been proven.

Now, if $x, y \in E^{(p)}$ and $|x| \leq |y|$, then $|x|^p \leq |y|^p$, and therefore

$$\|x\|_{E^{(p)}} = \||x|^p\|^{1/p}_E \leq \||y|^p\|^{1/p}_E = \|y\|_{E^{(p)}}.$$

Thus, $\|\cdot\|_{E^{(p)}}$ is a monotone $L^0(\Omega)$-valued norm on $E^{(p)}$. □

The space $(E^{(p)}, \|\cdot\|_{E^{(p)}})$ is called the *p-convexification* of the lattice-normed vector lattice $(E, \|\cdot\|_E)$. It has thus been shown that the following holds.

Proposition 3.4. *If $(E, \|\cdot\|_E)$ is a lattice-normed vector lattice over $L^0(\Omega)$, then its p-convexification is also a lattice-normed vector lattice over $L^0(\Omega)$.*

Remark 3.5. The p-convexification $E^{(p)}$ of any lattice-normed vector lattice E is p-convex with a convexity constant equal to 1, i.e. $M^{(p)}(E^{(p)}) = 1$.

Indeed, if $x_1, \ldots, x_n \in E$, then

$$\Big\|\Big(\sum_{k=1}^n |x_k|^p\Big)^{\frac{1}{p}}\Big\|_{E^{(p)}} = \Big\|\sum_{k=1}^n |x_k|^p\Big\|_E^{\frac{1}{p}} \leq \Big(\sum_{k=1}^n \||x_k|^p\|_E\Big)^{\frac{1}{p}} = \Big(\sum_{k=1}^n \|x_k\|^p_{E^{(p)}}\Big)^{\frac{1}{p}}.$$

To prove the completeness of the lattice-normalized space $(E^{(p)}, \|\cdot\|_{E^{(p)}})$, we need an analogue of the well-known Amemiya's theorem (see, for example, [9, Chapter X,§3, Theorem 2]) for lattice-normed lattices over $L^0(\Omega)$. The following is a variant of the Amemiya's theorem, established in [10, Theorem 3.2].

Theorem 3.6. *Let* $(E, \|\cdot\|)$ *be a lattice-normed vector lattice over* $L^0(\Omega)$*. The following statements are equivalent:*

(i) $(E, \|\cdot\|)$ *is a* (bo)*-complete lattice-normed space;*

(ii) *any positive increasing* (bo)*-fundamental sequence in* E (bo)*-converges;*

(iii) *for any positive increasing* (bo)*-fundamental sequence* $\{x_n\}_{n\in\mathbb{N}}$ *there exists* $x = \sup\limits_{n\in\mathbb{N}} x_n \in E$.

We also need the following result.

Lemma 3.7. *Let* $x, y \in L^0(B)$ *and* $x \geq y > 0$*. Then for any* $p \geq 1$ *the following inequality is valid:*

$$x^p - y^p \leq px^{p-1}(x - y).$$

Proof. To prove, we can assume that $p > 1$ and $x > y$. Since $x > 0$ and $y > 0$, then there are $x^{-1}, y^{-1} \in L^0(B)$. Put $z = xy^{-1}$. Then since $z > \mathbf{1}$ and $p > 1$, we have $zp > z$. Hence, $z + p < zp + z^{1-p}$, that is $zp + z^{1-p} - z - p > 0$, or $xy^{-1}p + x^{1-p}y^{p-1} - xy^{-1} - p > 0$. Further multiplying both parts of the latter inequality by $yx^{p-1} > 0$, we get $x^p p + y^p - x^p - pyx^{p-1} > 0$. From here, we finally get $x^p - y^p \leq px^{p-1}(x - y)$. □

The following theorem is a version of Proposition 3.4 for Banach-Kantorovich lattices over $L^0(\Omega)$.

Theorem 3.8. *If* $(E, \|\cdot\|_E)$ *is a Banach-Kantorovich lattice over* $L^0(\Omega)$*, then its* p*-convexification* $(E^{(p)}, \|\cdot\|_{E^{(p)}})$ *is also a Banach-Kantorovich lattice over* $L^0(\Omega)$*.*

Proof. We show that the norm $\|\cdot\|_{E^{(p)}}$ is d-decomposable. Since $\|\cdot\|_E$ is a d-decomposable $L^0(\Omega)$-valued norm on E, then for any element $x \in E$ and any decomposition $\|x\|_E = f_1 + f_2$, where $f_1, f_2 \in L^0_+(\Omega)$, $f_1 f_2 = 0$, there are $x_1, x_2 \in E$ such that $x = x_1 + x_2$ and $\|x_k\|_E = f_k$, $k = 1, 2$. Let $e_i = s(f_i)$, $i = 1, 2$. Then, $e_1 e_2 = 0$ and $x_i = x \cdot e_i$, $i = 1, 2$.

Let $y \in E^{(p)}$, $\|y\|_{E^{(p)}} = g_1 + g_2$, where $g_1, g_2 \in L^0_+(\Omega)$, $g_1 g_2 = 0$, i.e. $\|y^p\|_E = \|y\|^p_{E^{(p)}} = g_1^p + g_2^p$. Put $q_i = s(g_i^p)$ and $y_i = y \cdot q_i$, $i = 1, 2$. Then, $y_i^p = y^p \cdot q_i$ and, using the d-decomposability of the norm $\|\cdot\|_E$ for $x = y^p$, $f_i = g_i^p$, $i = 1, 2$, we obtain that $y^p \cdot q_1 + y^p \cdot q_2 = y^p$ and $\|y \cdot q_i\|_{E^{(p)}} = g_i$, $i = 1, 2$. Since $q_1 q_2 = 0$, $y \cdot q_1 + y \cdot q_2 = y$.

We now show that the lattice-normed vector lattice $(E^{(p)}, \|\cdot\|_{E^{(p)}})$ is (bo)-complete. To prove this, we can assume that $p > 1$ and let $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $\{x_n\}$ be positive increasing (bo)-fundamental sequence of $E^{(p)}$.

By Theorem 3.6, it is sufficient to show that there exists $x = \sup\limits_{n\in\mathbb{N}} x_n \in E^{(p)}$. Let $\lambda = \sup\limits_{n\in\mathbb{N}} \|x_n\|_{E^{(p)}}$. For $n \geq m$ according to Lemma 3.7 we have the following estimate

$$0 \leq x_n^p - x_m^p \leq p x_n^{p-1}(x_n - x_m).$$

Applying Lemma 3.3, we get

$$\|x_n^p - x_m^p\|_E \leq p\|x_n^{p-1}\|_{E^{(q)}}\|(x_n - x_m)\|_{E^{(p)}}$$

$$= p\|x_n\|_{E^{(p)}}^{p/q}\|(x_n - x_m)\|_{E^{(p)}} \leq p\lambda^{p/q}\|(x_n - x_m)\|_{E^{(p)}}.$$

This implies that $\{x_n^p\}$ is positive increasing (bo)-fundamental sequence in E. Since E is (bo)-complete, it follows that there exists $0 \leq y \in E$ such that $y = \sup\limits_{n\in\mathbb{N}} x_n^p$ (Theorem 3.6). Consequently, $0 \leq x = y^{1/p} \in E^{(p)}$ and $x = \sup\limits_{n\in\mathbb{N}} x_n$. It remains to use Theorem 3.6 again, by virtue of which $(E^{(p)}, \|\cdot\|_{E^{(p)}})$ is (bo)-complete. Thus, $(E^{(p)}, \|\cdot\|_{E^{(p)}})$ is a (bo)-complete decomposable lattice-normalized vector lattice. Therefore $(E^{(p)}, \|\cdot\|_{E^{(p)}})$ is a Banach-Kantorovich lattice over $L^0(\Omega)$. □

Conclusion

In this paper, the concepts of p-convexity and q-concavity in lattice-normed vector lattices $(E, \|\cdot\|_E)$ over the ring of measurable functions are introduced and properties associated with these concepts are considered. A principal result of the paper is that, if $(E, \|\cdot\|_E)$ is a Banach-Kantorovich lattice, then its p-convexification $(E^{(p)}, \|\cdot\|_{E^{(p)}})$ is also a Banach-Kantorovich lattice.

Acknowledgments. The author would like to thank Professor Vladimir Chilin for helpful discussions and thanks the referee for valuable remarks.

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Received November 24, 2024; revised December 16, 2024; accepted December 30, 2024.

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