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Weak solutions of unconditionally stable second-order difference schemes for nonlinear sine-Gordon systems

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Abstract. This paper presents the existence and uniqueness of the weak solution for the nonlinear system of sine-Gordon equations which describes DNA dynamics. An unconditionally stable second order difference scheme generated by the unbounded operator A^2 corresponding to the system of sine-Gordon equations is considered. Weak solutions are a more general type of solution to the system of sine-Gordon equations than classical solutions and are important in the case of low regularity conditions. The weak solvability is studied in the space of distributions using variational methods. A very efficient numerical method that combines the finite difference method and the fixed point theory is used to perform numerical experiments to verify theoretical statements.

Keywords. Existence, uniqueness, weak solutions, difference schemes, stability.

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1 Introduction

Wave phenomena are effectively described by hyperbolic partial differential equations (PDEs), but classical solutions, which require high smoothness, often fail to capture the complexities of real-world behavior. This limitation has led to weak solutions, a practical mathematical framework that reduces smoothness requirements, enabling the modeling of discontinuities such as shock waves and material failure. These phenomena are critical in physics, where sudden changes frequently occur in various systems.

Weak solutions, constructed in the space of distributions using variational or energy methods, extend the applicability of hyperbolic PDEs by providing a more adaptable and accurate approach for analyzing and representing complex wave behavior. This framework connects mathematical precision with the challenging nature of physical processes, making it a fundamental tool in modern physics (see, [1, 2, 4–6, 11, 13, 21, 22, 27, 28, 35]). In the research on the mathematical modeling of many phenomena in physics, biology, engineering in particular relativistic quantum mechanics, acoustics, biomedical engineering, and field theory are wave-type equations and are of great interest (see, [12–14, 18, 26, 30–32]). In In recent decades many mathematicians have been studying the theoretical and numerical research fields of nonlinear systems for sine-Gordon, Klein-Gordon, and coupled sine-Gordon equations (see, [6, 12–14, 26, 32]). Nowadays, these type of problems get more attention due to the existence of solitons. Nonlinear equations of the soliton type are waves that occur in proteins, DNA, and signal conduction between neurons (cf., [30]). The system

$$\begin{cases} u_{tt} - u_{xx} = -\delta^2 \sin(u - v), \\ v_{tt} - v_{xx} = \sin(u - v) \end{cases}$$

models the DNA dynamics. This type of system models the open states in DNA double helices and is studied by many scientists (see, [12, 28, 30, 32, 33] and the references given therein). In this paper, we use the proof methodology and Mathematical tools of Temam R. et al. (see, [1, 2, 4, 21, 22]). This article uses second-order hyperbolic evolution equations in the following form. Let $\Omega_T = \Omega \times (0, T]$, with $T > 0$, and $S = [0, T] \times \Gamma$ for $\Gamma = \partial\Omega$, $\overline{\Omega} = \Omega \cup \Gamma$, and $\Omega \subset \mathbb{R}^n$ be an open and bounded set. A widely known (see, [6]) initial/boundary-value problem is

$$\begin{cases} w_{tt} + w_t + Lw = f, \text{ in } \Omega_T \\ w = 0 \text{ on } S, \\ w = g, w_t = h \text{ on } \in \Omega \times \{t = 0\}, \end{cases} \tag{1}$$

with given functions $f : \Omega_T \to \mathbb{R}$, $g, h : \Omega \to \mathbb{R}$, and $w : \overline{\Omega}_T \to \mathbb{R}$ is the unknown, $w(x, t)$. Here L denotes a partial differential operator for each time t in the form

$$Lw = -\sum_{i,j=1}^{n} a^{ij}(x, t)\, w_{x_i x_j} + \sum_{i=1}^{n} b^i(x, t)\, w_{x_i} + c(x, t)\, w \tag{2}$$

for the coefficients $a^{ij}, b^i, c\ (i, j = 1, \ldots, n)$.

In this work, the second order differential operator

$$Lw = -a(x, t)\, w_{xx} + b(x, t)\, w_x + c(x, t)\, w \tag{3}$$

for the coupled system of problem (1) for Ω_T and $\Omega \subset \mathbb{R}$. Here, we will assume initially the coefficints $a, b, c \in C^1\left(\overline{\Omega}_T\right)$ and $f \in L^2(\Omega_T)$, $g \in H_0^1(\Omega)$, $h \in L^2(\Omega)$.

In the present paper the weak solutions of second-order of accuracy unconditionally stable difference scheme corresponding to the nonlinear system of coupled sine-Gordon equations

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + \alpha_{11}\frac{\partial u}{\partial t} + \alpha_{12}\frac{\partial v}{\partial t} - \beta_1 \Delta u + \gamma_1 \sin(\delta_{11}u + \delta_{12}v) = f \text{ in } \Omega_T, \\ \frac{\partial^2 v}{\partial t^2} + \alpha_{21}\frac{\partial u}{\partial t} + \alpha_{22}\frac{\partial v}{\partial t} - \beta_2 \Delta v + \gamma_2 \sin(\delta_{21}u + \delta_{22}v) = g \text{ in } \Omega_T \end{cases} \tag{4}$$

with boundary conditions

$$u = 0 \text{ and } v = 0 \text{ on } S, \tag{5}$$

and initial conditions

$$u(0,x) = \varphi_1(x) \text{ and } \frac{\partial u}{\partial t}(0,x) = \psi_1(x) \text{ in } \Omega \times \{t=0\}, \tag{6}$$

$$v(0,x) = \varphi_2(x) \text{ and } \frac{\partial v}{\partial t}(0,x) = \psi_2(x) \text{ in } \in \Omega \times \{t=0\} \tag{7}$$

is studied. Here, $\Omega \subset \mathbb{R}$ is a bounded open set and Δ is Laplacian. The coefficients $\alpha_{ij}, \beta_i, \gamma_i, \delta_{ij}, \rho_{ij}$ are nonzero real numbers for $i, j = 1, \ldots$. We denote

$$\tilde{f}(t,x,u,v,u_t,v_t) = f(t,x) - \gamma_1 \sin(\delta_{11}u + \delta_{12}v) - \alpha_{11}u_t - \alpha_{12}v_t,$$

$$\tilde{g}(t,x,u,v,u_t,v_t) = g(t,x) - \gamma_2 \sin(\delta_{21}u + \delta_{22}v) - \alpha_{21}u_t - \alpha_{22}v_t.$$

The source functions $\tilde{f}$ and $\tilde{g}$ satisfy the Lipschitz conditions

$$\left|\tilde{f}(t,x,w_1,(w_1)_t) - \tilde{f}(t,x,w_2,(w_2)_t)\right| \leq L\left[|w_1 - w_2| + |(w_1)_t - (w_2)_t|\right].$$

with $w = (u,v)$ on Ω_T, where L is a positive constant.

The neighboring atoms, and related kinetic energy are described by higher order derivatives. The sine trigonometric function which is a nonlinear terms containing stands for the potential energy. The rest of the terms are source functions and damping terms.

Let $A = -\Delta$ be a self-adjoint positive definite unbounded operator in a Hilbert space H. One can write problem (4)-(7) as follows

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + \alpha_{11}\frac{\partial u}{\partial t} + \alpha_{12}\frac{\partial v}{\partial t} + \beta_1 Au + \gamma_1 sin(\delta_{11}u + \delta_{12}v) = f, 0 < t < T, \\ \frac{\partial^2 v}{\partial t^2} + \alpha_{21}\frac{\partial u}{\partial t} + \alpha_{22}\frac{\partial v}{\partial t} + \beta_2 Av + \gamma_2 \sin(\delta_{21}u + \delta_{22}v) = g, 0 < t < T, \\ u(0) = \varphi_1 \in V, u'(0) = \psi_1 \in H, \\ v(0) = \varphi_2 \in V, v'(0) = \psi_2 \in H. \end{cases} \tag{8}$$

Here, V is the Hilbert space satisfying the relation $V \subset H$.

Existence and uniqueness of problem (8) is considered as the limit of second order of accuracy unconditionally stable difference scheme generated by A^2

$$\begin{cases} \tau^{-2}(u_{k+1} - 2u_k + u_{k-1}) + \beta_1 A u_k + \frac{\beta_1 \tau^2}{4} A^2 u_{k+1} \\ + \frac{\alpha_{11}}{2\tau}\left(u_{k+1} - u_{k-1}\right) + \frac{\alpha_{12}}{2\tau}\left(v_{k+1} - v_{k-1}\right) \\ +\gamma_1 \sin\left(\delta_{11} u_k + \delta_{12} v_k\right) = f_k, \\ f_k = f(t_k), \quad t_k = k\tau, \ 1 \le k \le N-1, N\tau = 1, \\ \tau^{-2}(v_{k+1} - 2v_k + v_{k-1}) + \beta_2 A v_k + \frac{\beta_2 \tau^2}{4} A^2 v_{k+1} \\ + \frac{\alpha_{21}}{2\tau}\left(u_{k+1} - u_{k-1}\right) + \frac{\alpha_{22}}{2\tau}\left(v_{k+1} - v_{k-1}\right) \\ +\gamma_2 \sin\left(\delta_{21} u_k + \delta_{22} v_k\right) = g_k, \\ g_k = g(t_k), \quad t_k = k\tau, \ 1 \le k \le N-1, N\tau = 1, \\ u'(0) = (I + \frac{\tau^2 A}{4})\tau^{-1}\left(u_1 - u_0\right) - \frac{\tau}{2}\left(\widetilde{f}_0 - A u_0\right) = \psi_1, \\ v'(0) = (I + \frac{\tau^2 A}{4})\tau^{-1}\left(v_1 - v_0\right) - \frac{\tau}{2}\left(\widetilde{g}_0 - A v_0\right) = \psi_2 \end{cases} \tag{9}$$

with a damping term for the nonlinear system. For the solution of (9) we consider the set of a family of grid points

$$\Omega_h = [0, T]_\tau \times [0, L]_h = \{(t_k, x_n) : t_k = k\tau, \ \ 0 \le k \le N,$$

$$N\tau = T, x_n = nh, 0 \le n \le M, Mh = L\} \tag{10}$$

with parameters τ, h, and constants T, L. Here f_k, g_k, φ_1, φ_2, ψ_1, and ψ_2 are given nonzero elements. The unconditional stability and the convergence of the linear undamped form of difference scheme (9) is presented in [18, 20].

The weak and global solutions, nonlinear dynamics of partial differential equations, finite difference, and finite element methods are extensively studied by many scientists (see, [18-28] and the references given therein).

2 Preliminaries and problem settings

In this section, several theoretical statements that are necessary for the the sequel will be presented. For the results of the elementary spectral theory, and bilinear forms we refer to [1, 2, 4, 6–8, 10, 16]. We use the Hilbert spaces $H = L^2(\Omega)$, $V = H_0^1(\Omega)$, and dual space $V' = H_0^{-1}(\Omega)$. The inner product

$$(v, w) = \int_\Omega v(x)w(x)dx, |v| = (v, v)^{1/2}, \forall w, v \in L^2(\Omega), \tag{11}$$

and the norm

$$((v,w)) = \sum_{i=1}^{n} \int_{\Omega} \frac{\partial}{\partial x_i} v(x) \frac{\partial}{\partial x_i} w(x) dx, \|v\| = ((v,v))^{1/2}, \forall w, v \in H_0^1(\Omega) \tag{12}$$

are equipped for these spaces.

We have the embedding $V \hookrightarrow H \equiv H' \hookrightarrow V'$ with $V' = H^{-1}(\Omega)$, and the pair (V, H) is a Gelfand triple space. We consider continuous, dense, and compact embedding $V \subset H$ and $H \subset V'$. The weak solutions are studied in the establishments of the triple space. In the variational formulation, a bilinear form

$$a(w,v) = \int_{\Omega} \nabla w . \nabla v dx = ((w,v)), \forall w, v \in V = H_0^1(\Omega), \tag{13}$$

will be used which is symmetric on $V \times V = H_0^1(\Omega)^2$ and bounded. Moreover, the bilinear form is coercive and satisfies

$$a(w,w) \geq \mu_i \|w\|^2, i = 1, \ldots, \forall w \in V. \tag{14}$$

Let the operator A that is an isomorphism from V onto V' and $A = -\Delta$ with relation

$$(Aw, v) = a((w,v)). \tag{15}$$

Here A is self-adjoint, unbounded operator in H with dense domain $D(A) = \{w \in V | Aw \in H\}$ in H and in V. When the bilinear form a is symmetric, it

follows that A is self-adjoint (from V into V' and as an unbounded operator in H)

$$\langle Av, w \rangle = \langle Aw, v \rangle = a((v,w)), \ \forall v, w \in V \tag{16}$$

and moreover A^{-1} the inverse of A is also self-adjoint in H. Thus, A^{-1} is a compact operator in H, self-adjoint.

Let A be an unbounded, strictly positive self-adjoint operator in H. Then one can employ the spectral theory (see, [8, 10]) and can define the powers A^s of A, for $s \in \mathbb{R}$. Here we consider the case of the compact injection of V in H.

In the present study, A^{-1} is considered as a compact self-adjoint operator in H, and we use the spectral theory of compact self-adjoint operators in a Hilbert space (see, for example, [9]). There exists a orthonormal complete family of H, $\{w_j\}_{j \in \mathbb{N}}$ with eigenvectors of A

$$A^{-1} w_j = \mu_j w_j, \qquad \forall j \in \mathbb{N} \tag{17}$$

and the sequence μ_j is monotonic and tends to 0. Having $w_j \in D(A)$, $\forall j$ and setting $\lambda_j = \mu_j^{-1}$, we have

$$Aw_j = \lambda_j w_j, j = 1,...,0 < \lambda_1 \leq \lambda_2, ..., \quad \lambda_j \to \infty \text{ as } j \to \infty. \tag{18}$$

The family w_j is orthonormal in H, and orthogonal for a in V,

$(w_j, w_k) = \delta_{jk} =$ the Kronecker symbol, $a(w_j, w_k) = \langle Aw_j, w_k \rangle = \lambda_j \delta_{jk}, \quad \forall j, k.$

In particular, if $a(u, v) = ((u, v))$ is the inner product of V

$$((w_j, w_k)) = \lambda_j \delta_{jk}, ((w_j, w_k))_* = ((A^{-1} w_j, w_k)) = \frac{1}{\lambda_j} \delta_{jk}, \quad \forall j, k.$$

Here, A^s is a self-adjoint unbounded operator in H with a dense domain $D(A^s) \subset H$, for every $s > 0$. The operator A^s is injective and strictly positive. The space $D(A^s)$ is equipped with the inner product and the norm

$$(u, v)_{D(A^s)} = (A^s u, A^s v), |u|_{D(A^s)} = \left\{ (u, u)_{D(A^s)} \right\}^{1/2} \tag{19}$$

which makes it a Hilbert space and A^s is an isomorphism from $D(A^s)$ onto H. In particular, letting $s = 1$, we have $D(A)$ and for $s = 1/2$, $D(A^{1/2}) = V$.

Defining $D(A^{-s})$ be the dual of $D(A^s)$, $(s > 0)$ then A^s is extended as an isomorphism from H onto $D(A^{-s})$. Moreover, $D(A^{-s})$ can be equipped with the inner product and the norm in (19) with s is replaced by $-s$. Finally, one can denote an increasing family of spaces $D(A^s)$, $s \in \mathbb{R}$,

$$D(A^{s_1}) \subset D(A^{s_2}), \qquad \forall s_1, s_2 \in \mathbb{R}, \quad s_1 \geq s_2.$$

The space is dense in the next one, the injection is continuous, and $A^{s_1 - s_2}$ is an isomorphism of $D(A^{s_1})$ into $D(A^{s_2})$, $\forall s_1, s_2 \in \mathbb{R}, \; s_1 > s_2$. In the compact self-adjoint case these operators and spaces can be characterized by making use of the spectral basis of A. Here, for real and positive s, we have

$$D(A^s) = \left\{ u \in H, \sum_{j=1}^{\infty} \lambda_j^{2s} (u, w_j)^2 < \infty \right\}, \tag{20}$$

and for negative s, $D(A^s)$ is the completion of H for the norm

$$\left\{ \sum_{j=1}^{\infty} \lambda_j^{2s} (u, w_j)^2 \right\}^{1/2}. \tag{21}$$

The inner product and the norm of $D(A^s)$ in (11), for $s \in \mathbb{R}$, can also be written as

$$(u,v)_{D(A^s)} = \sum_{j=1}^{\infty} \lambda_j^{2s}(u,w_j)(v,w_j), \tag{22}$$

$$|u|_{D(A^s)} = \left\{ \sum_{j=1}^{\infty} \lambda_j^{2s}(u,w_j)^2 \right\}^{1/2}, \tag{23}$$

and for $u \in D(A^s)$ we have:

$$A^s u = \sum_{j=1}^{\infty} \lambda_j^{s}(u,w_j)w_j. \tag{24}$$

The detailed theory of the function spaces $D(A), V$, and H, as well as the operators A and a are given in the references, e.g., [1] and [4].

The solution space of distributions can be expressed in the following form

$$W(0,T) = \left\{ g | g \in L^2\left(0,T;H_0^1\right), g' \in L^2\left(0,T;H\right), \right.$$

$$\left. g'' \in L^2\left(0,T;H_0^{-1}\right)\right\}. \tag{25}$$

In this article we assume $f, g \in L^{\infty}\left(\mathbb{R}_+; L^2\left(\Omega\right)^2\right)$ with

$$|f|_{\infty} := |f|_{L^{\infty}\left(\mathbb{R}_+;L^2(\Omega)^2\right)},$$

$$|g|_{\infty} := |g|_{L^{\infty}\left(\mathbb{R}_+;L^2(\Omega)^2\right)}.$$

The definition of the weak solutions for (9) can be stated in the the following lemma.

Lemma 2.1 ([1]). *Consider X be a given Banach space, X' is the dual and let v and g be functions in $L^1\left(a,b;X\right)$. The following conditions are equivalent:*

(i) v is a.e. equal to the primitive function of g, i.e., $\exists\, w \in X$ with

$$v\left(t\right) = w + \int_0^t g\left(s\right) ds, \text{ for a.e. } t \in \left[a,b\right]. \tag{26}$$

(ii) For each test function $\varphi \in B\left(]a,b[\right)$,

$$\int\limits_a^b v(t)\,\varphi^{'}(t)\,dt = -\int\limits_a^b g(t)\,\varphi(t)\,dt \quad \left(\varphi^{'}(t) = \frac{d\varphi}{dt}\right). \tag{27}$$

(iii) For every $\eta \in X^{'}$,

$$\frac{d}{dt}\langle v,\eta\rangle = \langle g,\eta\rangle \tag{28}$$

on $]a,b[$ *in the scalar distribution sense.*

When the conditions (i)-(iii) are satisfied then g *is said to be the* $(X - valued)$ *distribution derivative of* v, *and* v *is a.e. equal to a function from* $[a,b]$ *into* X *which is continuous*.

Now we present a discrete Gronwall lemma which will be used in the sequel.

Lemma 2.2 ([2,21]). *Let* $\tau > 0$, n_1, n_2, n_* *be positive integers with* $n_1 < n_*, n_1 + n_2 + 1 \leq n_*$, *and* γ_n, η_n, *and* ϑ_n *are positive sequences with*

$$w_n \leq w_{n-1}\left(1 + \tau\eta_{n-1}\right) + \tau\vartheta_n \textit{ for } n = n_1, \ldots, n_*, \tag{29}$$

and with upper bounds

$$\sum\nolimits_{n=n^{'}}^{n^{'}+n_2} \tau\eta_n \leq a_1\left(n_1,n_*\right), \sum\nolimits_{n=n^{'}}^{n^{'}+n_2} \tau\vartheta_n \leq a_2\left(n_1,n_*\right),$$
$$\sum\nolimits_{n=n^{'}}^{n^{'}+n_2} \tau w_n \leq a_3\left(n_1,n_*\right) \tag{30}$$

for any $n^{'}$ *satisfying* $n_1 \leq n^{'} \leq n_* - n_2$, *then we have*

$$w_n \leq \left(\frac{a_3\left(n_1,n_*\right)}{\tau n_2} + a_2\left(n_1,n_*\right)\right)\exp\left(a_1\left(n_1,n_*\right)\right) \tag{31}$$

for any n *such that* $n_1 + n_2 + 1 \leq n \leq n_*$.

In the next section, we present the weak solutions by establishing the variational formulation for a nonlinear coupled system of difference equation (9). Some results on the strong convergence of the sequences will be derived using the theorems for compactness.

3 Weak solution of the second order of accuracy difference scheme generated by A^2

In the present section, the theoretical results on the approximate solution of (8) will be obtained for difference scheme (9) in the weak sense. Applying the variational approach in the settings of the space of distributions, we will show that system (9) converges to a weak solution and is unique. Note that, throughout this paper, $K_i, \widetilde{K}_i, c_i, d_i, \mu_i$ represent generic constants, which may have different values at different places.

Definition 3.1. The mesh function sets $\left\{v_k^h\right\}$ and $\left\{w_k^h\right\}$ are called the approximate weak solutions of (9) if $v_k^h, w_k^h \in V^h$ satisfy the weak formulation of (9). The family of grid points (10) are used to present Hilbert space

$$L_{2h}\left(\Omega\right) = L_{2h}\left(\Omega_h\right).$$

The following norm

$$\left\|v_k\right\|_{L_{2h}(\Omega)} = \left(\sum_{j=1}^{N} \left|v_k^j\right|^2 h\right)^{\frac{1}{2}} \tag{32}$$

is used in this space.

Now we present the main theorem on the weak solvability of (9). In the following theorem solutions u_k^h and v_k^h of (9) are proved to be bounded.

Theorem 3.2. *Consider a bounded open set $\Omega \subset \mathbb{R}^n$ with $f, g \in L^\infty\left(\mathbb{R}_+; L^2\left(\Omega\right)^2\right)$ and piecewise smooth boundary, $v_0 \in L^2\left(\Omega\right)$, $v_0^{'} \in H_0^1(\Omega)$, then the solutions u_k^h and v_k^h of (9) are bounded in the following sense:*

$$\left|u_k^h\right|^2 \leq C, \left|v_k^h\right|^2 \leq C, k = 0, \ldots, N, \tag{33}$$

$$\sum_{k=1}^{N-1} \left|u_{k+1}^h - u_k^h\right|_h^2 \leq C, \sum_{k=1}^{N-1} \left|v_{k+1}^h - v_k^h\right|_h^2 \leq C, \tag{34}$$

$$\tau^2 \sum_{k=1}^{N-1} \left\|u_k^h\right\|_h^2 \leq \frac{C}{2\beta_1\mu_1}, \tau^2 \sum_{k=1}^{N-1} \left\|v_k^h\right\|_h^2 \leq \frac{C}{2\beta_2\mu_2}, \tag{35}$$

with

$$C = \left|u_0^h\right|_h^2 + \left|v_0^h\right|_h^2 + d_5 \left|u_{k-1}^h\right|_h^2 + d_6 \left|v_{k-1}^h\right|_h^2$$

$$+d_7 \int_0^T |f(s)|^2 \, ds + d_8 \int_0^T |g(s)|^2 \, ds, \tag{36}$$

and $\beta_1, \beta_2, \mu_1, \mu_2, d_5, d_6, d_7$*, and* d_8 *are positive constants.*

Proof. We need to prove that $\left|u_k^h\right|_h, \left\|u_k^h\right\|_h$ are uniformly bounded. For that purpose, we take the inner product of the equations in system (9) with $2u_{k+1}$ and $2v_{k+1}$, respectively and obtain

$$\begin{cases} \left(u_{k+1}^h - u_k^h, 2u_{k+1}^h\right) + \left(u_{k-1}^h - u_k^h, 2u_{k+1}^h\right) \\ -\beta_1\tau^2(\nabla^2 u_k^h, 2u_{k+1}^h) + \frac{\beta_1}{4}\tau^4(\nabla^4 u_{k+1}^h, 2u_{k+1}^h) \\ +\tau^2\gamma_1(\sin\left(\delta_{11}u_k^h + \delta_{12}v_k^h\right), 2u_{k+1}^h) \\ +\frac{\alpha_{11}\tau}{2}[(u_{k+1}^h, 2u_{k+1}^h) - (u_{k-1}^h, 2u_{k+1}^h)] \\ +\frac{\tau^2\alpha_{12}}{2}[(v_{k+1}^h, 2u_{k+1}^h) - (v_{k-1}^h, 2u_{k+1}^h)] = (\tau^2 f_k, 2u_{k+1}^h), \\ \left(v_{k+1}^h - v_k^h, 2v_{k+1}^h\right) + \left(v_{k-1}^h - v_k^h, 2v_{k+1}^h\right) \\ -\beta_2\tau^2(\nabla^2 v_k^h, 2v_{k+1}^h) + \frac{\beta_1}{4}\tau^4(\nabla^4 v_{k+1}^h, 2v_{k+1}^h) \\ +\tau^2\gamma_2(\sin\left(\delta_{21}u_k^h + \delta_{22}v_k^h\right), 2v_{k+1}^h) \\ +\frac{\alpha_{21}\tau}{2}[(u_{k+1}^h, 2v_{k+1}^h) - (u_{k-1}^h, 2v_{k+1}^h)] \\ +\frac{\alpha_{22}\tau}{2}[(v_{k+1}^h, 2v_{k+1}^h) - (v_{k-1}^h, 2v_{k+1}^h)] = (\tau^2 g_k, 2v_{k+1}^h). \end{cases} \tag{37}$$

Using relations

$$2(\varphi - \psi, \varphi)_h = |\varphi|_h^2 - |\psi|_h^2 + |\varphi - \psi|_h^2, \forall \varphi, \psi \in v_h^h, \tag{38}$$

$$2(\varphi - \psi, \psi)_h = |\varphi|_h^2 - |\psi|_h^2 - |\varphi - \psi|_h^2, \forall \varphi, \psi \in v_h^h, \tag{39}$$

and denoting $\Delta = -A$, we obtain system

$$\begin{cases} \left|u_{k+1}^h\right|_h^2 - \left|u_k^h\right|_h^2 + \left|u_{k+1}^h - u_k^h\right|_h^2 + \left(u_{k-1}^h, 2u_{k+1}^h\right) - \left(u_k^h, 2u_{k+1}^h\right) \\ +\beta_1\tau^2(Au_k^h, 2u_{k+1}^h) + \frac{\beta_1}{4}\tau^4(A^2 u_{k+1}^h, 2u_{k+1}^h) \\ +\tau^2\gamma_1(\sin\left(\delta_{11}u_k^h + \delta_{12}v_k^h\right), 2u_{k+1}^h) \\ +\frac{\alpha_{11}\tau}{2}[(u_{k+1}^h, 2u_{k+1}^h) - (u_{k-1}^h, 2u_{k+1}^h)] \\ +\frac{\tau^2\alpha_{12}}{2}[(v_{k+1}^h, 2u_{k+1}^h) - (v_{k-1}^h, 2u_{k+1}^h)] = (\tau^2 f_k, 2u_{k+1}^h), \\ \left|v_{k+1}^h\right|_h^2 - \left|v_k^h\right|_h^2 + \left|v_{k+1}^h - v_k^h\right|_h^2 + \left(v_{k-1}^h, 2v_{k+1}^h\right) - \left(v_k^h, 2v_{k+1}^h\right) \\ +\beta_2\tau^2(Av_k^h, 2v_{k+1}^h) + \frac{\beta_1}{4}\tau^4\left(A^2 v_{k+1}^h, 2v_{k+1}^h\right) \\ +\tau^2\gamma_2(\sin\left(\delta_{21}u_k^h + \delta_{22}v_k^h\right), 2v_{k+1}^h) \\ +\frac{\alpha_{21}\tau}{2}[(u_{k+1}^h, 2v_{k+1}^h) - (u_{k-1}^h, 2v_{k+1}^h)] \\ +\frac{\alpha_{22}\tau}{2}[(v_{k+1}^h, 2v_{k+1}^h) - (v_{k-1}^h, 2v_{k+1}^h)] = (\tau^2 g_k, 2v_{k+1}^h). \end{cases} \tag{40}$$

Replacing the operator A with bilinear form (15), and rewriting the equations of system (40) separately, we get

$$\begin{cases} \left|u_{k+1}^h\right|_h^2 - \left|u_k^h\right|_h^2 + \left|u_{k+1}^h - u_k^h\right|_h^2 + 2\beta_1 a\left(\left(u_k^h, u_{k+1}^h\right)\right) \\ +\frac{\beta_1}{2}\tau^4(A^2 u_{k+1}^h, u_{k+1}^h) = \left(\tau^2 f_k, 2u_{k+1}^h\right) - \alpha_{12}\tau\left(v_{k+1}^h, u_{k+1}^h\right) \\ +\alpha_{12}\tau\left(v_{k-1}^h, u_{k+1}^h\right) - \alpha_{11}\tau\left(u_{k+1}^h, u_{k+1}^h\right) + (\alpha_{11}\tau - 2)\left(u_{k-1}^h, u_{k+1}^h\right) \\ +2(u_k^h, u_{k+1}^h) - \tau^2\gamma_1\left(\sin\left(\delta_{11}u_k^h + \delta_{12}v_k^h\right), 2u_{k+1}^h\right), \end{cases} \tag{41}$$

and

$$\begin{cases} \left|v_{k+1}^h\right|_h^2 - \left|v_k^h\right|_h^2 + \left|v_{k+1}^h - v_k^h\right|_h^2 + 2\beta_2 a\left(\left(v_k^h, v_{k+1}^h\right)\right) \\ +\frac{\beta_2}{2}\tau^4\left(A^2 v_{k+1}^h, v_{k+1}^h\right) = \left(\tau^2 g_k, 2v_{k+1}^h\right) - \alpha_{21}\tau\left(u_{k+1}^h, v_{k+1}^h\right) \\ +\alpha_{21}\tau\left(u_{k-1}^h, v_{k+1}^h\right) - \alpha_{22}\tau\left(v_{k+1}^h, v_{k+1}^h\right) + (\alpha_{22}\tau - 2)\left(v_{k-1}^h, v_{k+1}^h\right) \\ +2\left(v_k^h, v_{k+1}^h\right) - \tau^2\gamma_2\left(\sin\left(\delta_{21}u_k^h + \delta_{22}v_k^h\right), 2v_{k+1}^h\right). \end{cases} \tag{42}$$

In this step, since the operator A^2 does not have coercivity property, we need to obtain an estimate for $\left(A^2 u_{k+1}^h, u_{k+1}^h\right)$. We will consider the spectral theory that is presented in the last section above. We use the complete orthonormal family of H, $\{w_j\}_{j\in\mathbb{N}}$ consist of eigenvectors of A for equation (18). The family w_j is orthonormal in H, and orthogonal for a in V. For every $s = 2$, A^2 is a self-adjoint unbounded operator in H in a dense domain $D(A^2) \subset H$. The operator A^2 is injective and strictly positive. The space

$$D(A^2) = \left\{u \in H, \sum_{j=1}^{\infty}\lambda_j^4(u, w_j)^2 < \infty\right\}$$

is equipped with the inner product and the norm as

$$(u, v)_{D(A^2)} = \sum_{j=1}^{\infty}\lambda_j^4(u, w_j)(v, w_j),$$

$$|u|_{D(A^2)} = \left\{\sum_{j=1}^{\infty}\lambda_j^4(u, w_j)^2\right\}^{1/2},$$

and for $u \in D(A^2)$ we can write

$$A^2 u = \sum_{j=1}^{\infty}\lambda_j^2(u, w_j)w_j.$$

Using these definitions and results of the spectral theory of the operators one can write

$$\left\langle A^2 u_{k+1}^h, u_{k+1}^h \right\rangle = \left\langle A u_{k+1}^h, A u_{k+1}^h \right\rangle = \left\langle \lambda_j u_{k+1}^h, \lambda_j u_{k+1}^h \right\rangle$$
$$= \lambda_j^2 a\left(\left(u_{k+1}^h, u_{k+1}^h \right) \right), \ \forall j, k,$$

and it follows

$$a\left(\left(u_{k+1}^h, u_{k+1}^h \right) \right) \geq \eta_i \left\| u_{k+1}^h \right\|^2, \eta_i = \frac{\sigma}{\lambda_1^2}, i = 1, 2, \forall u_{k+1}^h \in V^h. \tag{43}$$

Here, λ_1 is the minimum of the eigenvalues and σ is a positive constant.

Now we obtain a priori estimates that prove the nonnegativity and boundedness of the components of equations (41),(42) of the system (37). We infer from Poincaré inequality, the existence of a constant $c_i > 0$, such that

$$c_i \left| u_k^h \right|_h \leq \left\| u_k^h \right\|_h, i = 1, \ldots, 9, \forall u_k^h \in V^h, \tag{44}$$

(see, for instance, [1]) that will used in the sequel. We obtain the estimate

$$\left| \left(\sin\left(\delta_{11} u_k^h + \delta_{12} v_k^h \right), u_{k+1}^h \right) \right| \leq \left| \sin\left(\delta_{11} u_k^h + \delta_{12} v_k^h \right) \right| \left| u_{k+1}^h \right|$$
$$\leq \frac{c_1}{2} \left(M_1 \left| \delta_{11} \right| \left(\left| u_k^h \right|^2 + \left| u_{k+1}^h \right|^2 \right) + M_1 \left| \delta_{12} \right| \left(\left| v_k^h \right|^2 + \left| u_{k+1}^h \right|^2 \right) \right). \tag{45}$$

Next we will use the coercivity estimate (14), spectral properties of the operator A and estimates (44), (45) in the proof. By the given estimates and classical inequalities such as Cauchy Schwarz inequality, Young's inequality, and triangle inequality, and some simple identities together, inequalities

$$\left\{ \begin{array}{l} \left| u_{k+1}^h \right|_h^2 - \left| u_k^h \right|_h^2 + \left| u_{k+1}^h - u_k^h \right|_h^2 \\ +2\beta_1 \mu_1 \tau^2 \left\| u_k^h \right\|^2 + \frac{\beta_1}{2} \tau^4 \eta_1 \left\| u_{k+1}^h \right\|^2 \\ \leq 2\tau^2 \left| f_k \right|_\infty \left| u_{k+1}^h \right|_h + \frac{1}{2} \left| \alpha_{12} \tau \right| M_1 \left(\left| v_{k+1}^h \right|_h^2 + \left| u_{k+1}^h \right|_h^2 \right) \\ +\frac{1}{2} \left| \alpha_{12} \tau \right| M_2 \left(\left| v_{k-1}^h \right|_h^2 + \left| u_{k+1}^h \right|_h^2 \right) \\ + \left| \alpha_{11} \tau \right| \left| u_{k+1}^h \right|_h^2 + \frac{1}{2} \left| \alpha_{11} \tau - 2 \right| M_3 \left(\left| u_{k-1}^h \right|_h^2 + \left| u_{k+1}^h \right|_h^2 \right) \\ +M_4 \left(\left| u_k^h \right|_h^2 + \left| u_{k+1}^h \right|_h^2 \right) \\ +\tau^2 \left| \gamma_1 \right| \left| c_2 \right| \left| \delta_{11} \right| \left| M_5 \right| \left(\left| u_k^h \right|_h^2 + \left| u_{k+1}^h \right|_h^2 \right) \\ +\tau^2 \left| \gamma_1 \right| \left| c_2 \right| \left| \delta_{12} \right| M_5 \left(\left| v_k^h \right|_h^2 + \left| u_{k+1}^h \right|_h^2 \right) \end{array} \right. \tag{46}$$

and

$$
\begin{cases}
\left|v_{k+1}^{h}\right|_{h}^{2}-\left|v_{k}^{h}\right|_{h}^{2}+\left|v_{k+1}^{h}-v_{k}^{h}\right|_{h}^{2} \\
+2\beta_{2}\mu_{1}\tau^{2}\left\|v_{k}^{h}\right\|^{2}+\frac{\beta_{2}}{2}\tau^{4}\eta_{2}\left\|v_{k+1}^{h}\right\|^{2} \\
\leq 2\tau^{2}\left|g_{k}\right|_{\infty}\left|v_{k+1}^{h}\right|_{h}+\frac{1}{2}\left|\alpha_{21}\tau\right|\widetilde{M}_{6}\left(\left|u_{k+1}^{h}\right|_{h}^{2}+\left|v_{k+1}^{h}\right|_{h}^{2}\right) \\
+\frac{1}{2}\left|\alpha_{21}\tau\right|\widetilde{M}_{7}\left(\left|u_{k-1}^{h}\right|_{h}^{2}+\left|v_{k+1}^{h}\right|_{h}^{2}\right) \\
+\left|\alpha_{22}\tau\right|\left|v_{k+1}^{h}\right|_{h}^{2}+\frac{1}{2}\left|\alpha_{22}\tau-2\right|\widetilde{M}_{8}\left(\left|v_{k-1}^{h}\right|_{h}^{2}+\left|v_{k+1}^{h}\right|_{h}^{2}\right) \\
+\widetilde{M}_{9}\left(\left|v_{k}^{h}\right|_{h}^{2}+\left|v_{k+1}^{h}\right|_{h}^{2}\right) \\
+\tau^{2}\left|\gamma_{2}\right|\left|c_{4}\right|\left|\delta_{21}\right|\widetilde{M}_{10}\left(\left|u_{k}^{h}\right|_{h}^{2}+\left|v_{k+1}^{h}\right|_{h}^{2}\right) \\
+\tau^{2}\left|\gamma_{2}\right|\left|c_{4}\right|\left|\delta_{22}\right|\widetilde{M}_{10}\left(\left|v_{k}^{h}\right|_{h}^{2}+\left|v_{k+1}^{h}\right|_{h}^{2}\right)
\end{cases}
\tag{47}
$$

hold. By estimate 44 (see, [1]) inequalities

$$
\begin{cases}
\left|u_{k+1}^{h}\right|_{h}^{2}-\left|u_{k}^{h}\right|_{h}^{2}+\left|u_{k+1}^{h}-u_{k}^{h}\right|_{h}^{2} \\
+2\beta_{1}\mu_{1}\tau^{2}\left\|u_{k}^{h}\right\|^{2}+\frac{\beta_{1}}{2}\tau^{4}\eta_{1}\left\|u_{k+1}^{h}\right\|^{2} \\
\leq 2\tau^{2}c_{1}\left|f_{k}\right|_{\infty}\left\|u_{k+1}^{h}\right\|_{h}+\frac{1}{2}\left|\alpha_{12}\tau\right|M_{1}\left(\left|v_{k+1}^{h}\right|_{h}^{2}+\left|u_{k+1}^{h}\right|_{h}^{2}\right) \\
+\frac{1}{2}\left|\alpha_{12}\tau\right|M_{2}\left(\left|v_{k-1}^{h}\right|_{h}^{2}+\left|u_{k+1}^{h}\right|_{h}^{2}\right)+\left|\alpha_{11}\tau\right|\left|u_{k+1}^{h}\right|_{h}^{2} \\
+\frac{1}{2}\left|\alpha_{11}\tau-2\right|M_{3}\left(\left|u_{k-1}^{h}\right|_{h}^{2}+\left|u_{k+1}^{h}\right|_{h}^{2}\right) \\
+M_{4}\left(\left|u_{k}^{h}\right|_{h}^{2}+\left|u_{k+1}^{h}\right|_{h}^{2}\right) \\
+\tau^{2}\left|\gamma_{1}\right|\left|c_{2}\right|\left|\delta_{11}\right|M_{5}\left(\left|u_{k}^{h}\right|_{h}^{2}+\left|u_{k+1}^{h}\right|_{h}^{2}\right) \\
+\tau^{2}\left|\gamma_{1}\right|\left|c_{2}\right|\left|\delta_{12}\right|M_{5}\left(\left|v_{k}^{h}\right|_{h}^{2}+\left|u_{k+1}^{h}\right|_{h}^{2}\right)
\end{cases}
\tag{48}
$$

and

$$
\begin{cases}
\left|v_{k+1}^{h}\right|_{h}^{2}-\left|v_{k}^{h}\right|_{h}^{2}+\left|v_{k+1}^{h}-v_{k}^{h}\right|_{h}^{2} \\
+2\beta_{2}\mu_{2}\tau^{2}\left\|v_{k}^{h}\right\|^{2}+\frac{\beta_{2}}{2}\tau^{4}\eta_{2}\left\|v_{k+1}^{h}\right\|^{2} \\
\leq 2\tau^{2}c_{3}\left|g_{k}\right|_{\infty}\left\|v_{k+1}^{h}\right\|_{h}+\frac{1}{2}\left|\alpha_{21}\tau\right|\widetilde{M}_{6}\left(\left|u_{k+1}^{h}\right|_{h}^{2}+\left|v_{k+1}^{h}\right|_{h}^{2}\right) \\
+\frac{1}{2}\left|\alpha_{21}\tau\right|\widetilde{M}_{7}\left(\left|u_{k-1}^{h}\right|_{h}^{2}+\left|v_{k+1}^{h}\right|_{h}^{2}\right)+\left|\alpha_{22}\tau\right|\left|v_{k+1}^{h}\right|_{h}^{2} \\
+\frac{1}{2}\left|\alpha_{22}\tau-2\right|\widetilde{M}_{8}\left(\left|v_{k-1}^{h}\right|_{h}^{2}+\left|v_{k+1}^{h}\right|_{h}^{2}\right) \\
+\widetilde{M}_{9}\left(\left|v_{k}^{h}\right|_{h}^{2}+\left|v_{k+1}^{h}\right|_{h}^{2}\right) \\
+\tau^{2}\left|\gamma_{2}\right|\left|c_{4}\right|\left|\delta_{21}\right|\widetilde{M}_{10}\left(\left|u_{k}^{h}\right|_{h}^{2}+\left|v_{k+1}^{h}\right|_{h}^{2}\right) \\
+\tau^{2}\left|\gamma_{2}\right|\left|c_{4}\right|\left|\delta_{22}\right|\widetilde{M}_{10}\left(\left|v_{k}^{h}\right|_{h}^{2}+\left|v_{k+1}^{h}\right|_{h}^{2}\right)
\end{cases}
\tag{49}
$$

hold. Using the Hölder inequality, we get

$$\left\{\begin{array}{l}
\left|u_{k+1}^h\right|_h^2-\left|u_k^h\right|_h^2+\left|u_{k+1}^h-u_k^h\right|_h^2+2\beta_1\mu_1\tau^2\left\|u_k^h\right\|^2 \\
+\frac{\beta_1}{2}\tau^4\eta_1\left\|u_{k+1}^h\right\|^2\leq\frac{\tau^4\beta_1\eta_1}{2}\left\|u_{k+1}^h\right\|_h^2+\frac{8c_1^2}{\beta_1\eta_1}\left|f_k\right|_\infty^2 \\
+\frac{1}{2}\left|\alpha_{12}\tau\right|M_1\left(\left|v_{k+1}^h\right|_h^2+\left|u_{k+1}^h\right|_h^2\right) \\
+\frac{1}{2}\left|\alpha_{12}\tau\right|M_2\left(\left|v_{k-1}^h\right|_h^2+\left|u_{k+1}^h\right|_h^2\right) \\
+\left|\alpha_{11}\tau\right|\left|u_{k+1}^h\right|_h^2+\frac{1}{2}\left|\alpha_{11}\tau-2\right|M_3 \\
\times\left(\left|u_{k-1}^h\right|_h^2+\left|u_{k+1}^h\right|_h^2\right)+M_4\left(\left|u_k^h\right|_h^2+\left|u_{k+1}^h\right|_h^2\right) \\
+\tau^2\left|\gamma_1\right|\left|c_2\right|\left|\delta_{11}\right|M_5\left(\left|u_k^h\right|_h^2+\left|u_{k+1}^h\right|_h^2\right) \\
+\tau^2\left|\gamma_1\right|\left|c_2\right|\left|\delta_{12}\right|M_5\left(\left|v_k^h\right|_h^2+\left|u_{k+1}^h\right|_h^2\right)
\end{array}\right. \tag{50}$$

and

$$\left\{\begin{array}{l}
\left|v_{k+1}^h\right|_h^2-\left|v_k^h\right|_h^2+\left|v_{k+1}^h-v_k^h\right|_h^2+2\beta_2\mu_2\tau^2\left\|v_k^h\right\|^2 \\
+\frac{\beta_2}{2}\tau^4\eta_2\left\|v_{k+1}^h\right\|^2\leq\frac{\tau^4\beta_2\eta_2}{2}\left\|v_{k+1}^h\right\|_h^2+\frac{8c_3^2}{\beta_2\eta_2}\left|g_k\right|_\infty^2 \\
+\frac{1}{2}\left|\alpha_{21}\tau\right|\widetilde{M_6}\left(\left|u_{k+1}^h\right|_h^2+\left|v_{k+1}^h\right|_h^2\right) \\
+\frac{1}{2}\left|\alpha_{21}\tau\right|\widetilde{M_7}\left(\left|u_{k-1}^h\right|_h^2+\left|v_{k+1}^h\right|_h^2\right)+\left|\alpha_{22}\tau\right|\left|v_{k+1}^h\right|_h^2 \\
+\frac{1}{2}\left|\alpha_{22}\tau-2\right|\widetilde{M_8}\left(\left|v_{k-1}^h\right|_h^2+\left|v_{k+1}^h\right|_h^2\right) \\
+\widetilde{M_9}\left(\left|v_k^h\right|_h^2+\left|v_{k+1}^h\right|_h^2\right) \\
+\tau^2\left|\gamma_2\right|\left|c_4\right|\left|\delta_{21}\right|\widetilde{M_{10}}\left(\left|u_k^h\right|_h^2+\left|v_{k+1}^h\right|_h^2\right) \\
+\tau^2\left|\gamma_2\right|\left|c4\right|\left|\delta_{22}\right|\widetilde{M_{10}}\left(\left|v_k^h\right|_h^2+\left|v_{k+1}^h\right|_h^2\right).
\end{array}\right. \tag{51}$$

Collecting the like terms of equations (50), (51) respectively, we get

$$\left\{\begin{array}{l}
\left|u_{k+1}^h\right|_h^2-\left|u_k^h\right|_h^2+\left|u_{k+1}^h-u_k^h\right|_h^2 \\
+2\beta_1\mu_1\tau^2\left\|u_k^h\right\|^2\leq\frac{8c_1^2}{\beta_1\eta_1}\left|f_k\right|_\infty^2+\left(\frac{1}{2}\left|\alpha_{12}\tau\right|M_1\right. \\
+\frac{1}{2}\left|\alpha_{12}\tau\right|M_2+\left|\alpha_{11}\tau\right|+\frac{1}{2}\left|\alpha_{11}\tau-2\right|M_3+M_4 \\
\left.+\tau^2\left|\gamma_1\right|\left|c_2\right|\left|\delta_{11}\right|M_5+\tau^2\left|\gamma_1\right|\left|c_2\right|\left|\delta_{12}\right|M_5\right)\left|u_{k+1}^h\right|_h^2 \\
+\frac{1}{2}\left|\alpha_{11}\tau-2\right|M_3\left|u_{k-1}^h\right|_h^2 \\
+\left(M_4+\tau^2\left|\gamma_1\right|\left|c_2\right|\left|\delta_{11}\right|M_5\right)\left|u_k^h\right|_h^2 \\
+\frac{1}{2}\left|\alpha_{12}\tau\right|M_1\left|v_{k+1}^h\right|_h^2+\frac{1}{2}\left|\alpha_{12}\tau\right|M_2\left|v_{k-1}^h\right|_h^2 \\
+\tau^2\left|\gamma_1\right|\left|c_2\right|\left|\delta_{12}\right|M_5\left|v_k^h\right|_h^2
\end{array}\right. \tag{52}$$

and

$$\left\{\begin{array}{l} \left|v_{k+1}^h\right|_h^2-\left|v_k^h\right|_h^2+\left|v_{k+1}^h-v_k^h\right|_h^2 \\ +2\beta_2\mu_2\tau^2\left\|v_k^h\right\|^2 \leq \frac{8c_3^2}{\beta_2\eta_2}\left|g_k\right|_\infty^2+\Big(\frac{1}{2}\left|\alpha_{21}\tau\right|\widetilde{M_6} \\ +\frac{1}{2}\left|\alpha_{21}\tau\right|\widetilde{M_7}+\left|\alpha_{22}\tau\right|+\frac{1}{2}\left|\alpha_{22}\tau-2\right|\widetilde{M_8}+\widetilde{M_9} \\ +\tau^2\left|\gamma_2\right|\left|c_4\right|\left|\delta_{21}\right|\widetilde{M_{10}}+\tau^2\left|\gamma_2\right|\left|c4\right|\left|\delta_{22}\right|\widetilde{M_{10}}\Big)\left|v_{k+1}^h\right|_h^2 \\ +\frac{1}{2}\left|\alpha_{22}\tau-2\right|\widetilde{M_8}\left|v_{k-1}^h\right|_h^2 \\ +\left(\widetilde{M_9}+\tau^2\left|\gamma_2\right|\left|c_4\right|\left|\delta_{22}\right|\widetilde{M_{10}}\right)\left|v_k^h\right|_h^2 \\ +\frac{1}{2}\left|\alpha_{21}\tau\right|\widetilde{M_6}\left|u_{k+1}^h\right|_h^2+\frac{1}{2}\left|\alpha_{21}\tau\right|\widetilde{M_7}\left|u_{k-1}^h\right|_h^2 \\ +\tau^2\left|\gamma_2\right|\left|c_4\right|\left|\delta_{21}\right|\widetilde{M_{10}}\left|u_k^h\right|_h^2 . \end{array}\right. \tag{53}$$

Denoting the coefficients

$$K_1=\frac{1}{2}\left|\alpha_{12}\tau\right|M_1+\frac{1}{2}\left|\alpha_{12}\tau\right|M_2+\left|\alpha_{11}\tau\right|+\frac{1}{2}\left|\alpha_{11}\tau-2\right|M_3+M_4$$

$$+\tau^2\left|\gamma_1\right|\left|c_2\right|\left|\delta_{11}\right|M_5+\tau^2\left|\gamma_1\right|\left|c_2\right|\left|\delta_{12}\right|M_5$$

$$K_2=\frac{1}{2}\left|\alpha_{11}\tau-2\right|M_3, K_3=M_4+\tau^2\left|\gamma_1\right|\left|c_2\right|\left|\delta_{11}\right|M_5$$

$$K_4=\frac{1}{2}\left|\alpha_{12}\tau\right|M_1, K_5=\frac{1}{2}\left|\alpha_{12}\tau\right|M_2$$

$$K_6=\tau^2\left|\gamma_1\right|\left|c_2\right|\left|\delta_{12}\right|M_5$$

for equation (52), and

$$\widetilde{K}_1=\frac{1}{2}\left|\alpha_{21}\tau\right|\widetilde{M_6}+\frac{1}{2}\left|\alpha_{21}\tau\right|\widetilde{M_7}+\left|\alpha_{22}\tau\right|+\frac{1}{2}\left|\alpha_{22}\tau-2\right|\widetilde{M_8}$$

$$+\widetilde{M_9}+\tau^2\left|\gamma_2\right|\left|c_4\right|\left|\delta_{21}\right|\widetilde{M_{10}}+\tau^2\left|\gamma_2\right|\left|c4\right|\left|\delta_{22}\right|\widetilde{M_{10}}$$

$$\widetilde{K}_2=\frac{1}{2}\left|\alpha_{22}\tau-2\right|\widetilde{M_8}, \widetilde{K}_3=\widetilde{M_9}+\tau^2\left|\gamma_2\right|\left|c_4\right|\left|\delta_{22}\right|\widetilde{M_{10}}$$

$$\widetilde{K}_4=\frac{1}{2}\left|\alpha_{21}\tau\right|\widetilde{M_6}, \widetilde{K}_5=\frac{1}{2}\left|\alpha_{21}\tau\right|\widetilde{M_7}$$

$$\widetilde{K}_6=\tau^2\left|\gamma_2\right|\left|c_4\right|\left|\delta_{21}\right|\widetilde{M_{10}}$$

for equation (53), we get

$$\left\{\begin{array}{l} \left|u_{k+1}^h\right|_h^2-\left|u_k^h\right|_h^2+\left|u_{k+1}^h-u_k^h\right|_h^2+2\beta_1\mu_1\tau^2\left\|u_k^h\right\|^2 \\ \leq \frac{8c_1^2}{\beta_1\eta_1}\left|f_k\right|_\infty^2+K_1\left|u_{k+1}^h\right|_h^2+K_2\left|u_{k-1}^h\right|_h^2+K_3\left|u_k^h\right|_h^2 \\ +K_4\left|v_{k+1}^h\right|_h^2+K_5\left|v_{k-1}^h\right|_h^2+K_6\left|v_k^h\right|_h^2 \end{array}\right. \tag{54}$$

and

$$\begin{cases} \left|v_{k+1}^h\right|_h^2 - \left|v_k^h\right|_h^2 + \left|v_{k+1}^h - v_k^h\right|_h^2 + 2\beta_2\mu_2\tau^2 \left\|v_k^h\right\|^2 \\ \leq \frac{8c_3^2}{\beta_2\eta_2} \left|g_k\right|_\infty^2 + \widetilde{K}_1 \left|v_{k+1}^h\right|_h^2 + \widetilde{K}_2 \left|v_{k-1}^h\right|_h^2 + \widetilde{K}_3 \left|v_k^h\right|_h^2 \\ +\widetilde{K}_4 \left|u_{k+1}^h\right|_h^2 + \widetilde{K}_5 \left|u_{k-1}^h\right|_h^2 + \widetilde{K}_6 \left|u_k^h\right|_h^2 . \end{cases} \tag{55}$$

Taking the sum of (54) and (55), and using the inequalities presented so far, we get

$$\begin{cases} \left|u_{k+1}^h\right|_h^2 - \left|u_k^h\right|_h^2 + \left|u_{k+1}^h - u_k^h\right|_h^2 + \left|v_{k+1}^h\right|_h^2 - \left|v_k^h\right|_h^2 + \left|v_{k+1}^h - v_k^h\right|_h^2 \\ +2\beta_1\mu_1\tau^2 \left\|u_k^h\right\|^2 + 2\beta_2\mu_2\tau^2 \left\|v_k^h\right\|^2 \\ \leq \frac{8c_1^2}{\beta_1\eta_1} \left|f_k\right|_\infty^2 + \frac{8c_3^2}{\beta_2\eta_2} \left|g_k\right|_\infty^2 + \left(K_1 + \widetilde{K}_4\right) \left|u_{k+1}^h\right|_h^2 \\ + \left(K_4 + \widetilde{K}_1\right) \left|v_{k+1}^h\right|_h^2 + \left(K_2 + \widetilde{K}_5\right) \left|u_{k-1}^h\right|_h^2 + \left(K_5 + \widetilde{K}_2\right) \left|v_{k-1}^h\right|_h^2 \\ + \left(K_3 + \widetilde{K}_6\right) \left|u_k^h\right|_h^2 + \left(K_6 + \widetilde{K}_3\right) \left|v_k^h\right|_h^2 . \end{cases} \tag{56}$$

By dropping some positive terms we rewrite the system as follows

$$\begin{cases} \left|u_{k+1}^h\right|_h^2 - \left|u_k^h\right|_h^2 + \left|v_{k+1}^h\right|_h^2 - \left|v_k^h\right|_h^2 \leq \frac{8c_1^2}{\beta_1\eta_1} \left|f_k\right|_\infty^2 \\ +\frac{8c_3^2}{\beta_2\eta_2} \left|g_k\right|_\infty^2 + \left(K_1 + \widetilde{K}_4\right) \left|u_{k+1}^h\right|_h^2 + \left(K_3 + \widetilde{K}_6 - 2\beta_1\mu_1\tau^2\right) \left|u_k^h\right|_h^2 \\ + \left(K_2 + \widetilde{K}_5\right) \left|u_{k-1}^h\right|_h^2 + \left(K_4 + \widetilde{K}_1\right) \left|v_{k+1}^h\right|_h^2 \\ + \left(K_6 + \widetilde{K}_3 - 2\beta_2\mu_2\tau^2\right) \left|v_k^h\right|_h^2 + \left(K_5 + \widetilde{K}_2\right) \left|v_{k-1}^h\right|_h^2 . \end{cases} \tag{57}$$

We apply Gronwall lemma for (57) by letting $\left|u_{k+1}^h\right|_h^2 = \xi_n$ $\left(\text{and } \xi_n = \left|v_{k+1}^h\right|_h^2\right)$. The assumptions of Lemma 2.2 are satisfied since there exist upper bounds such that

$$\left(K_3 + \widetilde{K}_6 - 2\beta_1\mu_1\tau^2\right) \leq a_1\left(n_1, n_*\right), \left(K_2 + \widetilde{K}_5\right) \leq a_2\left(n_1, n_*\right),$$

$$\left(K_1 + \widetilde{K}_4\right) \leq a_3\left(n_1, n_*\right), \left(K_6 + \widetilde{K}_3 - 2\beta_2\mu_2\tau^2\right) \leq \widetilde{a}_1\left(n_1, n_*\right),$$

$$\left(K_5 + \widetilde{K}_2\right) \leq \widetilde{a}_2\left(n_1, n_*\right), \left(K_4 + \widetilde{K}_1\right) \leq \widetilde{a}_3\left(n_1, n_*\right).$$

Thus by the Gronwall lemma the following inequality

$$\begin{aligned} \left|u_{k+1}^h\right|_h^2 + \left|v_{k+1}^h\right|_h^2 \leq & \left(\frac{a_3\left(n_1, n_*\right)}{\tau n_2} + a_2\left(n_1, n_*\right)\right) \exp\left(a_1\left(n_1, n_*\right)\right) \\ & + \left(\frac{\widetilde{a}_3\left(n_1, n_*\right)}{\tau n_2} + \widetilde{a}_2\left(n_1, n_*\right)\right) \exp\left(\widetilde{a}_1\left(n_1, n_*\right)\right) \end{aligned} \tag{58}$$

holds. Next, adding inequality (56), from $k = 0, \ldots, N-1$, we get

$$\begin{cases} \left|u_N^h\right|_h^2 + \left|v_N^h\right|_h^2 + \sum\limits_{k=1}^{N-1} \left|u_{k+1}^h - u_k^h\right|_h^2 + \sum\limits_{k=1}^{N-1} \left|v_{k+1}^h - v_k^h\right|_h^2 \\ +2\beta_1\mu_1\tau^2 \sum\limits_{k=1}^{N-1} \left\|u_k^h\right\|_h^2 + 2\beta_2\mu_2\tau^2 \sum\limits_{k=1}^{N-1} \left\|v_k^h\right\|^2 \\ \leq \left|u_0^h\right|_h^2 + \left|v_0^h\right|_h^2 + \frac{8c_1^2}{\beta_1\eta_1} \sum\limits_{k=1}^{N-1} \left|f_k\right|_\infty^2 + \frac{8c_3^2}{\beta_2\eta_2} \sum\limits_{k=1}^{N-1} \left|g_k\right|_\infty^2 \\ + \left(K_1 + \widetilde{K}_4\right) \sum\limits_{k=1}^{N-1} \left|u_{k+1}^h\right|_h^2 + \left(K_4 + \widetilde{K}_1\right) \sum\limits_{k=1}^{N-1} \left|v_{k+1}^h\right|_h^2 \\ + \left(K_2 + \widetilde{K}_5\right) \sum\limits_{k=1}^{N-1} \left|u_{k-1}^h\right|_h^2 + \left(K_5 + \widetilde{K}_2\right) \sum\limits_{k=1}^{N-1} \left|v_{k-1}^h\right|_h^2 \\ + \left(K_3 + \widetilde{K}_6\right) \sum\limits_{k=1}^{N-1} \left|u_k^h\right|_h^2 + \left(K_6 + \widetilde{K}_3\right) \sum\limits_{k=1}^{N-1} \left|v_k^h\right|_h^2 . \end{cases} \tag{59}$$

Using estimate (44) the following inequality

$$\begin{cases} \left|u_N^h\right|_h^2 + \left|v_N^h\right|_h^2 + \sum\limits_{k=1}^{N-1} \left|u_{k+1}^h - u_k^h\right|_h^2 + \sum\limits_{k=1}^{N-1} \left|v_{k+1}^h - v_k^h\right|_h^2 \\ +d_1 \sum\limits_{k=1}^{N-1} \left\|u_k^h\right\|_h^2 + d_2 \sum\limits_{k=1}^{N-1} \left\|v_k^h\right\|^2 + d_3 \left|u_{k+1}^h\right|_h^2 + d_4 \left|v_{k+1}^h\right|_h^2 \\ \leq \left|u_0^h\right|_h^2 + \left|v_0^h\right|_h^2 + d_5 \left|u_{k-1}^h\right|_h^2 + d_6 \left|v_{k-1}^h\right|_h^2 \\ +\tau^2 d_7 \sum\limits_{k=1}^{N-1} \left|f_k\right|_\infty^2 + \tau^2 d_8 \sum\limits_{k=1}^{N-1} \left|g_k\right|_\infty^2 . \end{cases} \tag{60}$$

is obtained. Here,

$$d_1 = \left(2\beta_1\tau^2\mu_1 - c_5\left(K_3 + \widetilde{K}_6\right)\right), d_2 = \left(2\beta_2\tau^2\mu_2 - c_6\left(K_6 + \widetilde{K}_3\right)\right),$$

$$d_3 = -\left(K_1 + \widetilde{K}_4\right), d_4 = -\left(K_4 + \widetilde{K}_1\right) d_5 = \left(K_2 + \widetilde{K}_5\right),$$

$$d_6 = \left(K_5 + \widetilde{K}_2\right), d_7 = \frac{8c_1^2}{\beta_1\eta_1}, d_8 = \frac{8c_3^2}{\beta_2\eta_2}.$$

We refer to [1] for the useful inequality

$$\sum_{k=1}^{N-1} \left|f_k\right|_\infty^2 \leq \int_0^T \left|f\left(s\right)\right|^2 ds. \tag{61}$$

The initial u_0^h is the orthogonal projection of u_0^h onto V^h, in $L^2\left(\Omega\right)$. By this we have (see, [1])

$$\left|u_0^h\right| \leq \left|u_0\right|, \forall h. \tag{62}$$

Similar is true for v_0^h. Making use of estimates (61) and (62) it follows that the right hand side of (9) is bounded by

$$C = |u_0|^2 + |v_0|^2 + d_5 \left| u_{k-1}^h \right|_h^2 + d_6 \left| v_{k-1}^h \right|_h^2 + d_7 \int_0^T |f(s)|^2 \, ds + d_8 \int_0^T |g(s)|^2 \, ds.$$

This proves (34) and (35). Next adding the inequalities (56) for $k = 0, \dots r-1$, dropping some positive terms, we get

$$\left| u_r^h \right|_h^2 + \left| v_r^h \right|_h^2 \leq \left| u_0^h \right|^2 + \left| v_0^h \right|^2 + \tau^2 d_7 \sum_{k=1}^{r-1} |f_k|_\infty^2 + \tau^2 d_8 \sum_{k=1}^{r-1} |g_k|_\infty^2 \leq C.$$

Using this result and estimate (58) we prove (33). Hence, Theorem 3.2 is proved.

Next theorem states that the set of mesh functions $\left\{ u_k^h \right\}$ and $\left\{ v_k^h \right\}$ are compact in $L_{2h}(\Omega)$ topology.

Theorem 3.3. *Let the hypotheses of Theorem 3.2 are satisfied. Then there exist subsequences*

$$\left\{ u_{k_m}^h \right\} \subset \left\{ u_k^h \right\} \textit{ and } \left\{ v_{k_m}^h \right\} \subset \left\{ v_k^h \right\}$$

which converge in V_h to bounded measurable functions u^h and v^h, respectively. Moreover, the limit functions u^h and v^h are unique weak solutions satisfying (9).

Proof. Estimates (33), (34),(35), and Discrete Gronwall Lemma (see, [21–23]) imply that

$$\{u_k^h\} \text{ and } \{v_k^h\} \text{ are bounded in } L^\infty(0,T;V).$$

Then, by the Rellich Theorem [4] there exists a subsequence $\mathbf{w}_{k_m} = [u_{k_m}, v_{k_m}]^T$ of $\mathbf{w}_k = [u_k, v_k]^T$ and $\tilde{\mathbf{w}}_k \in L^\infty(0,T;\mathcal{V})$ such that

$$\tilde{\mathbf{w}}_k \in L^\infty(0,T;\mathcal{V}) \subset L^2(0,T;\mathcal{V}$$

and

$$\mathbf{w}_{k_m} \to \tilde{\mathbf{w}}_k \text{ weak star in } L^\infty(0,T;\mathcal{V} \text{ and weakly in } L^2(0,T;\mathcal{V}). \tag{63}$$

By the Aubin Compactness theorem [16], the above convergence results imply

$$\mathbf{w}_{k_m} \to \tilde{\mathbf{w}}_k \text{ strongly in } L^2(0,T;\mathcal{H}) \tag{64}$$

and by (64),

$$\sin \delta \mathbf{w}_{k_m} \to \sin \delta \tilde{\mathbf{w}}_k \text{ strongly in } L^2(0,T;\mathcal{H}).$$

which proves the existence of $\tilde{\mathbf{w}}_k$ a.e. in $\mathcal{H}$ and $\tilde{\mathbf{w}}_0 = \mathbf{w}_0$. Uniqueness follows from the results of Theorem 3.2 and convergence of difference scheme (9). Hence, Theorem 3.3 is proved.

4 Numerical Analysis

In this section, we verify the theoretical statements that are proved above by numerical experiments. A unified numerical method based on the fixed point iteration and finite difference schemes will be used. We aid the nonlinearity using the fixed point method. Fixed-point iterations are utilized to overcome the difficulties arising from the nonlinear source term. We introduce a composite numerical method to obtain accurate results for the solution of a system of PDEs with initial and boundary conditions for the coupled sine-Gordon equations. We consider the function

$$w(t,x)=\{u(t,x),v(t,x)\},$$

$$u(t,x)=\cos t\sin\pi x, v(t,x)=\cos 2t\sin\pi x$$

being the solution and using this function we formulate a mixed boundary value problem that leads to this solution. We consider the following problem for the system of sine-Gordon equations

$$\begin{cases} u_{tt}+u_t-u_{xx}+u=\left(\pi^2\cos t-\sin t\right)\sin\pi x \\ -\sin(\cos t\sin\pi x-\cos 2t\sin\pi x)+\sin(u-v), \\ 0<t<1,\ 0<x<1, \\ v_{tt}+v_t-v_{xx}+v=\left(\left(\pi^2-3\right)\cos 2t-2\sin 2t\right)\sin\pi x \\ -\sin\left(\cos t\sin\pi x-\cos 2t\sin\pi x\right)+\sin(u-v), \\ 0<t<1,\ 0<x<1, \\ u(0,x)=\sin\pi x, u_t(0,x)=0,\ 0\le x\le 1, \\ v(0,x)=\sin\pi x, v_t(0,x)=0,\ 0\le x\le 1, \\ u(t,0)=u(t,1)=0, 0\le t\le 1, \\ v(t,0)=v(t,1)=0,\ 0\le t\le 1. \end{cases} \tag{65}$$

The modified Gauss elimination method is used to solve the difference scheme (66) corresponding to the approximate solution of (65). The family of grid points is

$$\Omega_h=[0,1]_\tau\times[0,1]_h=\{(t_k,x_n):t_k=k\tau,\ \ 0\le k\le N,$$

$$N\tau=1, x_n=nh, 0\le n\le M, Mh=1\}$$

is used for the difference scheme

$$\begin{cases} \frac{{}_pu_n^{k+1}-2{}_pu_n^k+{}_pu_n^{k-1}}{\tau^2}+\frac{{}_pu_n^{k+1}-{}_pu_n^{k-1}}{2\tau}-\frac{{}_pu_{n+1}^k-2{}_pu_n^k+{}_pu_{n-1}^k}{h^2}+{}_p\,u_n^k \\ +\frac{\tau^2}{4}\frac{{}_pu_{n+2}^{k+1}-4{}_pu_{n+1}^{k+1}+6{}_pu_n^{k+1}-4{}_pu_{n-1}^{k+1}+{}_pu_{n-2}^{k+1}}{h^4}=\sin({}_pu_n^k-{}_p\,v_n^k) \\ +\left(\pi^2\cos t_k-\sin t_k\right)\sin\pi x \\ -\sin\left(\cos t_k\sin\pi x_n-\cos 2t_k\sin\pi x_n\right), \\ \frac{{}_pv_n^{k+1}-2{}_pv_n^k+{}_pv_n^{k-1}}{\tau^2}+\frac{{}_pv_n^{k+1}-{}_pv_n^{k-1}}{2\tau}-\frac{{}_pv_{n+1}^k-2{}_pu_n^k+{}_pu_{n-1}^k}{h^2}+{}_p\,v_n^k \\ +\frac{\tau^2}{4}\frac{{}_pv_{n+2}^{k+1}-4{}_pv_{n+1}^{k+1}+6{}_pv_n^{k+1}-4{}_pv_{n-1}^{k+1}+{}_pv_{n-2}^{k+1}}{h^4}=\sin({}_pu_n^k-{}_p\,v_n^k) \\ +\left(\left(\pi^2-3\right)\cos 2t_k-2\sin 2t_k\right)\sin\left(\pi x_n\right) \\ -\sin\left(\cos t_k\sin\pi x_n-\cos 2t_k\sin\pi x_n\right), \\ t_k=k\tau, 1\le k\le N-1,\ N\tau=1, x_n=nh, \\ 1\le n\le M-1, Mh=1, \\ {}_pu_n^0=\sin(\pi x_n), 1\le n\le M-1, \\ (2\tau)^{-1}\left(-3{}_pu_n^0\left(x_n\right)+4{}_pu_n^1\left(x_n\right)-{}_p\,u_n^2\left(x_n\right)\right)=0, \\ 1\le n\le M-1, {}_p\,v_n^0=\sin(\pi x_n), 1\le n\le M-1, \\ (2\tau)^{-1}\left(-3{}_pv_n^0\left(x_n\right)+4{}_pv_n^1\left(x_n\right)-{}_p\,v_n^2\left(x_n\right)\right)=0, \\ 1\le n\le M-1, \\ {}_pu_0^k={}_p\,u_M^k=0, {}_p\,v_0^k={}_p\,v_M^k=0, 0\le k\le N. \end{cases} \tag{66}$$

For different N, M values we present the errors, iteration numbers, and related CPU times in the following tables. MATLAB R2021b software package, by a PC System of 64 bit, Core i5 CPU, 1.80 GHz, 8 GB of RAM is used for numerical experiments. We use the following formula

$$\max_{u,v}\left[\max_{\substack{1\le k\le N-1\\1\le n\le M-1}}\left|u\left(t_k,x_n\right)-u_n^k\right|,\max_{\substack{1\le k\le N-1\\1\le n\le M-1}}\left|v\left(t_k,x_n\right)-v_n^k\right|\right]$$

to compute errors. The computation is carried our for $m=1,2,...,p$, which p depends on a stopping criterion also called error tolerance ε such that

$$\left|{}_pu_n-{}_{p-1}\,u_n\right|<\varepsilon \text{ and } \left|{}_pv_n-{}_{p-1}\,v_n\right|<\varepsilon.$$

Here m index represents the number of fixed point iterations. We denote the exact solution of problem (65) by $w\left(t_k,x_n\right)=\left[u\left(t_k,x_n\right),v\left(t_k,x_n\right)\right]^T$ and the approximate solution by $w_n^k=\left[u_n^k,v_n^k\right]^T$. The results of numerical implementations are presented in tables below.

Table 1. Results for problem (66)			
$N = M$	Error of w	m	CPU times
20	0.0041	9	0.443
40	0.0011	10	1.202
80	0.00028	10	5.382
160	0.000071	11	69.039

Table 1 gives the errors for the solution of (66), with a terminating criteria $\varepsilon = 10^{-15}$. The initials

$$ {}_0u_n^k = \left[r_{i,j}\right]_{i,j=1}^{N-1}, \quad r_{i,j} \sim \text{Uniform}(0,1), \tag{67} $$

$$ {}_0v_n^k = I_{N+1}, \quad \text{where } I_{N+1} \text{is the identity matrix} \tag{68} $$

where (67) is random matrix and (68) is a identity matrix are used.

Table 2. Results for problem (66)			
$N = M$	Error of w	m	CPU times
20	0.0041	10	0.504
40	0.0011	12	1.777
80	0.00028	12	6.425
160	0.000071	13	59.190

Table 2 presents the errors for the solution of (66), with terminating criteria $\varepsilon = 10^{-20}$. Here, the initials are

$$ {}_0u_n^k = \left[r_{i,j}\right]_{i,j=1}^{N-1}, \quad r_{i,j} \sim \text{Uniform}(0,1), $$

$$ {}_0v_n^k = \mathbf{0}_{N+1}, \tag{69} $$

where (69) is the zero matrix.

Table 3. Results for problem (66)			
$N = M$	Error of w	m	CPU times
20	0.0041	9	0.485
40	0.0011	10	1.223
80	0.00028	11	5.899
160	0.000071	12	49.307

Table 3 gives the errors for the solution of (66), with $\varepsilon = 10^{-20}$. In the iteration, the initials are taken as the identity matrices of the form

$$ {}_0u_n^k = {}_0 v_n^k = I_{N+1}. \quad (70)$$

Difference scheme (66) is used together with fixed point iteration to obtain numerical solutions. Difference scheme (66) converges for different $N = M$ values, initial vectors ${}_0v_n^k$, ${}_0w_n^k$, termination criteria ε in different iteration numbers m. When reaching the maximum difference value at specific grid points of two successive results yields less than ε, the iterative process stops. Note that when initials ${}_0v_n^k$, ${}_0w_n^k$ in (67), (68), (69), (70), and ε are varied, iteration numbers, and CPU times increase until the error becomes very small for some $N = M$ values.

5 Conclusion

In this work existence and uniqueness of weak solutions for the coupled system of finite difference schemes corresponding to the coupled sine-Gordon equations are studied. The existence and uniqueness of the solutions are proved using the variational methods. A numerical method that uses the second order of accuracy unconditionally stable difference scheme (9) with the fixed point iteration is presented. The theoretical statements are verified by numerical experiments with Matlab implementations.

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