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Cards of fixed points of some Lotka-Volterra operators

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Abstract. The paper considers a special type of the Lotka-Volterra operator operating in a four-dimensional simplex. The tournament corresponding to this operator has four cyclic triples. All kinds of fixed point cards are built for it. It is proved which types of cards exist for Lotka-Volterra operators in the general position, and which cannot be. The paper also proposes a new approach to constructing an oriented card of fixed points.

Keywords. Lotka-Volterra operator, card of fixed points, skew-symmetric matrix, graph, Hamilton cycle, tournament.

2020 Mathematics Subject Classification. 37B25, 37C25, 37C27.

1 Introduction

Starting the work, we need to note that in the works of R.N. Ganikhodjaev [1]-[2] a special type of quadratic Lotka-Volterra operators was introduced, having the form

$$V : x'_k = x_k \left(1 + \sum_{i=1}^{m} a_{ki} x_i\right), \quad k = \overline{1, m}. \tag{1}$$

This operator preserves a finite-dimensional simplex

$$S^{m-1} = \left\{ x = (x_1, \ldots, x_m) : \sum_{i=1}^{m} x_i = 1, x_i \geq 0 \right\} \subset \mathbb{R}^m.$$

It should be noted that the operator is completely determined by the skew-symmetric matrix $A = (a_{ki})$. The elements of the matrix A satisfy the conditions $a_{ki} = -a_{ik}$, $|a_{ki}| \leq 1$, $k, i = \overline{1, m}$. We owe the connections of these mappings with elements of graph theory, such as tournaments, to the works [6, 7]. Partially oriented graphs and their relation to operators of this type are given in [3, 4]. These operators are relevant for research, since they can rightfully act as discrete models of airborne viral diseases, in the case when partially oriented graphs correspond to operators

[6–8]. In the case when they are in a general position and in the card of fixed points we have a strong triple (Hamiltonian cycle) and a transitive triple, then they describe an ecological model, that is, a model of the cycle of biogens, in particular carbon and phosphorus [11]. This paper is a continuation of the work [9], i.e. in it we give a solution to the problem posed in [10]. At the same time, in this article we propose a new method for constructing a fixed point card for the Lotka-Volterra operator, according to the signs of the principal minors of the fourth order.

2 Preliminaries

We start the article by characterizing the skew-symmetric matrix, since the operators which we consider here are completely based on and determined by matrices of this type.

Let A be a real matrix satisfying the relation

$$A = -A^T,$$

where A^T is the transposed matrix. Then we call this matrix skew-symmetric [5].

It follows from the definition that a skew-symmetric matrix can only be square, and its elements must satisfy the relation:

$$a_{ki} = -a_{ik} \text{ by } \{k, i\} \subset \{1, \dots, m\}.$$

It follows from this condition that all elements of the main diagonal of the skew-symmetric matrix must be equal to zero, and the matrix itself has the form:

$$\begin{pmatrix} 0 & a_{12} & a_{13} & \dots & a_{1m} \\ -a_{12} & 0 & a_{23} & \dots & a_{2m} \\ -a_{13} & -a_{23} & 0 & \dots & a_{3m} \\ \dots & \dots & \dots & \dots & \dots \\ -a_{1m} & -a_{2m} & -a_{3m} & \dots & 0 \end{pmatrix}. \tag{2}$$

The determinant of an odd-order skew-symmetric matrix is always zero.

Based on the definition of a skew-symmetric matrix, we have:

$$A = -A^T \quad \Longrightarrow \quad \det(A) = \det(-A^T) = (-1)^n \det(A).$$

If m is odd, the last equality implies that $\det(A) = 0$.

Now, let m be even. Then the determinant of a skew-symmetric matrix of even order m is the square of a homogeneous polynomial of degree $\frac{m}{2}$ with respect to its elements [11].

Definition 2.1. A skew-symmetric matrix $A = (a_{ki})$ is called a general position matrix if all principal minors of even order are nonzero.

For example, for $m = 2$ we find

$$\begin{vmatrix} 0 & a_{12} \\ -a_{12} & 0 \end{vmatrix} = a_{12}^2 \neq 0,$$

and for $m = 4$

$$\triangle = \begin{vmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{vmatrix},$$

so

$$\triangle = (a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})^2 = (a_{12}a_{34} + a_{13}a_{24} + a_{14}a_{23})^2 \neq 0.$$

Since $a_{ki} = -a_{ik}$ and $a_{ki} \neq 0$ for $k \neq i$, we connect skew-symmetric matrices with elements of graph theory. We can match tournaments to a skew-symmetric matrix, the elements of which satisfy such conditions. As we recalled above, such matrices are in a general position. To do this, we give definitions from classical graph theory [12, 13].

Definition 2.2 ([12]). A graph G is a finite nonempty set W containing p vertices and a given set E consisting of q unordered pairs of different vertices from W.

Each pair $x = \{u, v\}$ of vertices in E is said to be an edge of the graph G, and this notation means that x connects u and v. We write $x = uv$, and this notation means that u and v are adjacent vertices of the edge x. The vertex u and the edge x, as well as v and x are incident.

If two different edges x and y are incident to the same vertex, then they are called adjacent.

A graph with p vertices and q edges is called the (p, q)-graph.

It is clear from the definitions that a graph cannot have loops, that is, edges connecting vertices with themselves.

Definition 2.3 ([8]). An oriented graph or digraph D is a finite nonempty set containing vertices and a given set E of ordered pairs of different vertices.

The elements of E of an oriented graph are called oriented edges or arcs.

Definition 2.4. Pairs of vertices that are connected by more than one edge are called multiple pairs.

There are no loops or multiple arcs in a digraph.

Definition 2.5. A directed graph is a digraph in which no pair of vertices is connected by a symmetric pair of arcs.

The definition implies that every orientation of a graph generates a directed graph.

In the classical definition of the tournament [6]-[7] a round-robin tournament is considered, in which a set of players leading the game is given.

The rules of the game are that any two players meet each other only once and as a result of the game, the outcome of the $-$ draw is prohibited. This means that the tournament is a digraph in which each pair of vertices is connected by only one arc.

We will associate the concepts we have given from classical graph theory with the considered skew-symmetric matrix (2).

Let $A = (a_{ki})$ be a skew-symmetric matrix of general position (2). Suppose that $a_{ki} \neq 0$ for $k \neq i$. Taking m points on the plane, we number them with the numbers $1, 2, \ldots, m$, and then connect the point k to the point i with an arrow directed from k to i if $a_{ki} < 0$, and back if $a_{ki} > 0$. Let's call the graph constructed in this way a tournament corresponding to the skew-symmetric matrix $A = (a_{ki})$ and denote it by T_m, i.e. a digraph is called a tournament, if for any two distinct vertices i and k, one and only one of the ordered pairs (i, k) or (k, i) is an edge of the digraph.

In the case when the principal minors of even order are zero, we get a degenerate skew-symmetric matrix. This case is possible only if some coefficients of the skew-symmetric matrix are zero, i.e. $a_{ki} = 0$.

For degenerate matrices, we introduce the concept of a partially-oriented graph [14]. By the name, it can be understood that a partially-oriented graph is a graph that contains both oriented and undirected edges.

An undirected graph is a graph that has no oriented edges, and, generally speaking, it can be included in the composition of partially-oriented graphs [15].

3 Main results

Let us consider a graph shown in Figure 1. We represent the graphs based on John W. Moon's monograph [6].

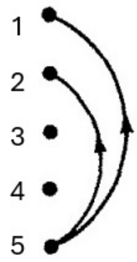

Figure 1. The tournament T_5 corresponding to the operator V and the matrix A (a representation of this type is given in the monograph [6]).

The operator corresponding to this tournament T_5 has the form

$$V: \begin{cases} x_1' = x_1(1 - a_{12}x_2 - a_{13}x_3 - a_{14}x_4 + a_{15}x_5) \\ x_2' = x_2(1 + a_{12}x_1 - a_{23}x_3 - a_{24}x_4 + a_{25}x_5) \\ x_3' = x_3(1 + a_{13}x_1 + a_{23}x_2 - a_{34}x_4 - a_{35}x_5) \\ x_4' = x_4(1 + a_{14}x_1 + a_{24}x_2 + a_{34}x_3 - a_{45}x_5) \\ x_5' = x_5(1 - a_{15}x_1 - a_{25}x_2 + a_{35}x_3 + a_{45}x_4), \end{cases} \tag{3}$$

and the skew-symmetric matrix looks like this:

$$A = \begin{pmatrix} 0 & -a_{12} & -a_{13} & -a_{14} & a_{15} \\ a_{12} & 0 & -a_{23} & -a_{24} & a_{25} \\ a_{13} & a_{23} & 0 & -a_{34} & -a_{35} \\ a_{14} & a_{24} & a_{34} & 0 & -a_{45} \\ -a_{15} & -a_{25} & a_{35} & a_{45} & 0 \end{pmatrix}.$$

For the V operator of the general position, it is necessary that the principal minors of the skew-symmetric matrix A be nonzero

$$A_1^{11} = \begin{pmatrix} 0 & -a_{23} & -a_{24} & a_{25} \\ a_{23} & 0 & -a_{34} & -a_{35} \\ a_{24} & a_{34} & 0 & -a_{45} \\ -a_{25} & a_{35} & a_{45} & 0 \end{pmatrix},$$

the determinant of the skew-symmetric matrix A_1^{11} is equal to

$$\begin{vmatrix} 0 & -a_{23} & -a_{24} & a_{25} \\ a_{23} & 0 & -a_{34} & -a_{35} \\ a_{24} & a_{34} & 0 & -a_{45} \\ -a_{25} & a_{35} & a_{45} & 0 \end{vmatrix} = (a_{24}a_{35} - a_{23}a_{45} + a_{25}a_{34})^2 \neq 0.$$

Similarly, we calculate the remaining determinants. The expressions in parentheses are denoted as follows:

$$\begin{gathered} \Delta_1^{11} = a_{24}a_{35} - a_{23}a_{45} + a_{25}a_{34}, \;\; \Delta_2^{22} = a_{14}a_{35} - a_{13}a_{45} + a_{15}a_{34}, \\ \Delta_3^{33} = a_{14}a_{25} - a_{15}a_{24} + a_{12}a_{45}, \;\; \Delta_4^{44} = a_{12}a_{35} - a_{15}a_{23} + a_{13}a_{25}, \\ \Delta_5^{55} = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}. \end{gathered} \tag{4}$$

Since A is in general position, $\Delta_i^{ii} \neq 0$, $i = \overline{1,5}$.

In system (3) there are four cyclic triples, i.e. the simplex has four strong sub tournaments. The concept of strong sub tournaments was given in the works [9]-[11]. The two-dimensional faces of S^4 corresponding to strong sub tournaments are $\overline{135}$, $\overline{145}$, $\overline{235}$ and $\overline{245}$. Since they are strong, i.e. they make up a cyclic triple, then according to [11], they have one interior fixed point each:

$$M_{135}\left(\frac{a_{35}}{a_{13}+a_{15}+a_{35}},\ 0,\ \frac{a_{15}}{a_{13}+a_{15}+a_{35}},\ 0,\ \frac{a_{13}}{a_{13}+a_{15}+a_{35}}\right),$$

$$M_{145}\left(\frac{a_{45}}{a_{14}+a_{15}+a_{45}},\ 0,\ 0,\ \frac{a_{15}}{a_{14}+a_{15}+a_{45}},\ \frac{a_{14}}{a_{14}+a_{15}+a_{45}}\right),$$

$$M_{235}\left(0,\ \frac{a_{35}}{a_{23}+a_{25}+a_{35}},\ \frac{a_{25}}{a_{23}+a_{25}+a_{35}},\ 0,\ \frac{a_{23}}{a_{23}+a_{25}+a_{35}}\right),$$

$$M_{245}\left(0,\ \frac{a_{45}}{a_{24}+a_{25}+a_{45}},\ 0,\ \frac{a_{25}}{a_{24}+a_{25}+a_{45}},\ \frac{a_{24}}{a_{24}+a_{25}+a_{45}}\right),$$

where all coefficients are positive. Recall the following works [9]-[11].

Let A be a skew-symmetric matrix. Then the sets

$$P = \{x \in S^{m-1} : Ax \geq 0\} \quad \text{and} \quad Q = \{x \in S^{m-1} : Ax \leq 0\} \tag{5}$$

are nonempty convex polyhedra [2].

Recall the definition of a card of fixed points for mappings in a general position.

Let $I = \{1, \dots, m\}$, $\alpha \subset I$, and $X = \{x(\alpha) : \alpha \subset I\}$ be the set of fixed points. We say that the fixed points $x(\alpha)$ and $x(\beta)$ form a pair (P, Q) if there exists a face Γ_γ of the simplex, such that $\gamma = \alpha \cup \beta$ and the following inequalities hold

$$A_\gamma x(\alpha) \geq 0, \quad A_\gamma x(\beta) \leq 0.$$

Here A_γ is the narrowing of the skew-symmetric matrix A at the face Γ_γ of the simplex S^{m-1}.

Now the fixed points of $x(\alpha)$ and $x(\beta)$ we draw on the plane: if they form a (P, Q) pair, then we connect them with an arc from $x(\alpha)$ to $x(\beta)$. The graph constructed in this way is called a card of fixed points. The card of fixed points for the operator V is denoted by G_V.

In the case when all principal minors of the second and fourth order are nonzero, we obtain a skew-symmetric matrix $A = (a_{ki})$, $k \neq i$ in the general position, then the fixed points card G_V have the form shown in Figure 2.

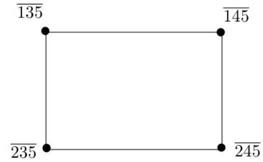

Figure 2. General view of the card of fixed points of the operator V.

Figure 2 shows a card with non-directional edges. The directions on the edges of the card (graph) are set according to the signs of the principal minors of the fourth order $\Delta_i^{ii} \neq 0$, $i = \overline{1, 5}$, given in (4). From the graph theory [7] we know that the number oriented graphs is $2^4 = 16$. Of these, the number of non-isomorphic digraphs is equal to 4. These oriented graphs are shown in the Figure 3.

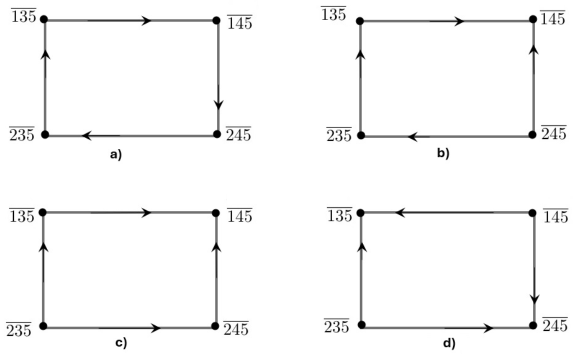

Figure 3. All kinds of fixed point cards for the operator V.

Theorem 3.1. *If the skew-symmetric matrix is in general position, then the fixed point card for the operator V has following the form*

(i) *it has the form a), if* $\Delta_1^{11}, \Delta_4^{44} < 0$

(ii) *it has the form b), if* $\Delta_1^{11}, \Delta_3^{33}, \Delta_4^{44} < 0$

(iii) *it has the form c), if* $\Delta_3^{33}, \Delta_4^{44} < 0$.

Proof. Solving $Ax = 0$, we get the following expressions:

$$\begin{aligned} -\Delta_4^{44}x_2 - \Delta_2^{22}x_4, \\ -\Delta_3^{33}x_2 + \Delta_2^{22}x_3, \\ \Delta_4^{44}x_1 - \Delta_1^{11}x_4, \\ \Delta_1^{11}x_3 + \Delta_3^{33}x_1. \end{aligned} \tag{6}$$

If $\Delta_1^{11}, \Delta_4^{44} < 0$, then (6) has the form:

$$\begin{aligned} \Delta_4^{44}x_2 - \Delta_2^{22}x_4, \\ -\Delta_3^{33}x_2 + \Delta_2^{22}x_3, \\ -\Delta_4^{44}x_1 + \Delta_1^{11}x_4, \\ -\Delta_1^{11}x_3 + \Delta_3^{33}x_1. \end{aligned} \tag{7}$$

Using these expressions, we determine the directions on the edges of the fixed point cards:
– let us set $\gamma = \{1, 2, 3, 5\}$ and $X = \{x(\alpha), x(\beta) : \alpha, \beta \subset \gamma\}$ be the set of fixed points, where $\alpha = 135$, $\beta = 235$, i.e. $\gamma = \alpha \cup \beta$ and $x(\alpha) = \overline{135}$ and $x(\beta) = \overline{235}$ fixed points belonging to strong triples $\overline{135}$, $\overline{235}$. The edge connecting these fixed points is denoted by Γ_γ. The direction on the edge Γ_γ defines the sign before the expression Δ_4^{44}.

Analyzing expression (7), we obtain the following:
– in the first expression, there is a positive sign before Δ_4^{44}, and in the third expression, the negative sign before Δ_4^{44}. This means that we set the direction from the second vertex to the first, i.e. $2 \to 1$. More accurately, we set the direction from a fixed point belonging to face $\overline{235}$ to a fixed point $\overline{135}$.

Now, as the set γ, let's take the set $\gamma = \{1, 3, 4, 5\}$. Here $x(\alpha) = \overline{135}$ and $x(\beta) = \overline{145}$. The direction of the edge Γ_γ here defines the sign before Δ_2^{22}. Consider the first expression in (7), where there is a negative sign before Δ_2^{22}, and in the second expression (6) before Δ_2^{22} we see a positive sign. This means that we set the direction from $\overline{135}$ to $\overline{145}$. We define the remaining directions on the edges in the same way and as a result we get the cards of fixed points shown in Figure 3, cases $a), b), c)$. □

For the case $d)$, shown in Figure 3, we have obtained the following theorem.

Theorem 3.2. *If the skew-symmetric matrix is in general position, then the card of fixed points of the operator V cannot be represented as $d)$.*

Proof. The proof of this theorem follows from Theorem 3.1. If the skew-symmetric matrix is in general position, then according to Theorem 3.1, the sets P and Q consist of a single fixed point. But as you can see from Figure 3, the case of $d)$ each of the sets P and Q consists of two points, that is, $P = \{\overline{145}, \overline{235}\}$ and $Q = \{\overline{135}, \overline{245}\}$. This contradicts the condition of Theorem 3.1. □

4 Conclusion

As we indicated above, earlier in [3, 10, 11] Lotka-Volterra mappings operating in a four-dimensional simplex with homogeneous tournaments were investigated. Mappings of this kind will be in a general position, since the skew-symmetric matrices corresponding to them will also be in a general position. Fixed points were found for them and their characters were investigated, on the basis of which fixed point cards were constructed.

The paper is devoted to solving the problem posed in the works [10, 11]. The Lotka-Volterra mappings, which are not in the general position, are investigated here. We give a new definition of the construction of a map of fixed points in contrast to the works [2, 3, 10, 11]. The considered mapping in this paper proves that fixed point cards are partially oriented graphs.

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