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A note on fractional powers of first-order differential and difference operators in relation to fractional derivatives

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Abstract. This paper concentrates on fractional derivatives and fractional powers of first-order differential and difference operators. It is illustrated how fractional derivatives are related to fractional powers of a first-order positive operator with boundary condition. Moreover, difference formulas for fractional derivatives are derived.

Keywords. Fractional derivatives, positive operators, first-order differential operators, difference formulas, fractional powers.

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1 Introduction

An overview of fractional calculus and its applications is provided in the following selected works with many other resources available. Fundamental concepts of fractional derivatives and integrals are introduced and key theoretical foundations are established in [1]. Expanding on these ideas, the book [2] examines the properties and applications of fractional operators in mathematical analysis. Additionally, the book [3] focuses on applying fractional calculus to solve differential equations. In summary, these works offer a unified study of fractional calculus theory and its applications in science and engineering.

The exploration of the positivity of operators has found a variety of applications in functional analysis, particularly in the context of differential equations and integral operators. The book [5] addresses the well-posedness of parabolic difference equations, emphasizing the conditions required for the existence, uniqueness, and stability of their solutions. The book [6] extends the classical theory of linear differential equations by examining them within the framework of Banach spaces, thus contributing to the broader theory of operators in infinite-dimensional spaces. The focus of the book [7] is on integral operators within spaces of summable functions, which advance the understanding of positive operators in functional spaces. Together, these studies form a fundamental basis for analyzing positivity across various operator theories in the context of both finite and infinite-dimensional equations.

The article [8] analyzes the positivity of a differential operator associated with a hyperbolic system of equations. Key conditions are identified to ensure the positivity of the operator. The work enhances the understanding of the spectral properties of operators in hyperbolic systems and their potential applications in both theoretical and applied mathematics.

The theory and applications of positive operators in Banach spaces to partial differential equations and numerical analysis were investigated by Sobolevskii and several of his students (see [9] and references therein).

The positivity of operators is closely related to fractional derivatives. This relationship is explored in several forms across the following papers. Each paper provides a unique perspective on how fractional derivatives and positive operators are connected in different contexts.

In paper [10], fractional derivatives are introduced as fractional powers of differential operators $-i\frac{d}{dt}$. It is using Taylor and Fourier series to construct fractional powers of self-adjoint differential operators. Besides, Fourier integrals and the Weyl quantization procedure are employed to establish the definition of the fractional derivative operator.

In paper [4], the fractional derivatives and fractional powers of the first-order positive operator $\frac{d}{dt}$ subject to initial condition is researched. It explores the connection between fractional derivatives and fractional powers of positive operators. Additionally, the expression for the fractional difference derivative is presented.

Research analogous to the one in [4] is carried out in the present study. It investigates fractional derivatives and fractional powers of the first-order positive operator $-\frac{d}{dt}$ with boundary condition. The link between fractional derivatives and fractional powers of operators is observed. Furthermore, it derives difference formulas for fractional derivatives.

Now, we introduce all the essential definitions necessary for this work. For fractional derivatives and integrals, refer to [1–3], and for fractional powers and positivity of operators, refer to [5–9]. For further study, one can refer to [10, 11].

Let $f(x) \in L_2[a,b]$ and $\alpha > 0$. Then the integrals

$$I_{a+}^{\alpha} f(x) := \Gamma^{-1}(\alpha) \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad x > a \tag{1}$$

and

$$I_{b-}^{\alpha} f(x) := \Gamma^{-1}(\alpha) \int_x^b \frac{f(t)}{(t-x)^{1-\alpha}} dt, \quad x < b \tag{2}$$

are said to be the left-sided and the right-sided Riemann-Liouville fractional integrals of order α (see [2]), respectively. Here, Γ^{-1} denotes $\frac{1}{\Gamma}$, where the gamma function (Γ) is given by the integral [3]

$$\Gamma(\alpha) = \int_0^\infty z^{\alpha-1} e^{-z} dz, \quad \alpha > 0. \tag{3}$$

By choosing $\alpha = m - k + 1$ and $z = t(1 + hs)$, it follows from (3) that

$$(1 + hs)^{k-m-1} = \frac{1}{\Gamma(m - k + 1)} \int_0^\infty t^{m-k} e^{-t(1+hs)} dt. \tag{4}$$

Assume that $f(x) \in L_2[a, b]$ and $\alpha \in (0, 1)$. Then the following expressions

$$D^\alpha_{a+} f(x) := \Gamma^{-1}(1 - \alpha) \frac{d}{dx} \int_a^x \frac{f(t)}{(x - t)^\alpha} dt \tag{5}$$

and

$$D^\alpha_{b-} f(x) := -\Gamma^{-1}(1 - \alpha) \frac{d}{dx} \int_x^b \frac{f(t)}{(t - x)^\alpha} dt \tag{6}$$

are referred to as the left-sided and the right-sided Riemann-Liouville fractional derivatives of order α (see [2]), respectively. Notice that if $f(a) = 0$, then the fractional derivative (5) (see [2]) becomes

$$D^\alpha_{a+} f(x) = \Gamma^{-1}(1 - \alpha) \int_a^x \frac{f'(t)}{(x - t)^\alpha} dt. \tag{7}$$

Moreover, if $f(b) = 0$, then the fractional derivative (6) (see [2]) becomes

$$D^\alpha_{b-} f(x) = -\Gamma^{-1}(1 - \alpha) \int_x^b \frac{f'(t)}{(t - x)^\alpha} dt. \tag{8}$$

The operator A is said to be positive if its spectrum $\sigma(A)$ is contained within the interior of a sector with an angle of φ, where $0 < 2\varphi < 2\pi$, symmetric with respect to the real axis. For the points on the edges of this sector, denoted by $S_1 = \{\lambda e^{i\varphi} : 0 \le \lambda < \infty\}$, $S_2 = \{\lambda e^{-i\varphi} : 0 \le \lambda < \infty\}$ and outside the sector, the resolvent $(\lambda I - A)^{-1}$ satisfies the following constraint

$$\|(\lambda I - A)^{-1}\|_{E \to E} \le \frac{M(\varphi)}{1 + |\lambda|}, \tag{9}$$

for some constant $M(\varphi)$ and complex number λ (see [5]). The infimum of all these angles φ is referred to as the spectral angle of the positive operator A, and is represented as $\varphi(A) = \varphi(A, E)$. For a positive operator A, negative fractional power $(-\alpha)$ can be defined [6] as follows

$$A^{-\alpha} = \frac{1}{2\pi i} \int_C \lambda^{-\alpha} R(\lambda)\, d\lambda, \tag{10}$$

where $\alpha > 0$, $R(\lambda) = (A - \lambda I)^{-1}$ and $C = S_1 \cup S_2$. Another formulation for (10) is given [6] as follows:

$$A^{-\alpha} = \Gamma^{-1}(\alpha)\Gamma^{-1}(1-\alpha)\int_0^\infty s^{-\alpha}R(-s)ds, \quad \alpha > 0. \tag{11}$$

In addition, one can define A^α for $\alpha > 0$, positive fractional power of A, as the inverse of operator to the negative power. The positive fractional power α of A is formulated [6] by

$$A^\alpha x = A^{\alpha-1}Ax = \Gamma^{-1}(\alpha)\Gamma^{-1}(1-\alpha)\int_0^\infty s^{\alpha-1}R(-s)Axds, \tag{12}$$

where $0 < \alpha < 1$ and $x \in \mathcal{D}(A)$.

2 Main Results

Theorem 2.1. *Let an operator A acting on $E = L_2[a,b]$ be defined as*

$$Au(x) = -u'(x),$$

with the domain $\mathcal{D}(A) = \{u(x) : u'(x) \in E, u(b) = 0\}$. Then the operator A is positive in E and it is valid for every function $f(x) \in \mathcal{D}(A)$ that

$$A^\alpha f(x) = D^\alpha_{b-} f(x).$$

Proof. Note that the inverse of operator $\lambda I + A$ is bounded for $\lambda \geq 0$ and is given by the formula

$$\left[(\lambda I + A)^{-1} f\right](x) = \int_x^b e^{-\lambda(t-x)}f(t)dt, \tag{13}$$

which implies that the operator A is positive in E. Then formulas (12) and (13) are applied as follows

$$A^\alpha f(x) = -\Gamma^{-1}(\alpha)\Gamma^{-1}(1-\alpha)\int_x^b \left[\int_0^\infty s^{\alpha-1}e^{-s(t-x)}ds\right] f'(t)dt.$$

Let us substitute $z = s(t-x)$ in the formula (3) to evaluate the following integral

$$\int_0^\infty s^{\alpha-1}e^{-s(t-x)}ds = \frac{\Gamma(\alpha)}{(t-x)^\alpha}.$$

We conclude by substituting this value above and using the formula (8). Thus, we prove that

$$A^\alpha f(x) = -\Gamma^{-1}(1-\alpha)\int_x^b \frac{f'(t)}{(t-x)^\alpha}dt = D^\alpha_{b-}f(x).$$

□

Theorem 2.2. *Assume that an operator A, acting on the space $E = L_2[a,b]$, is defined by*

$$Au(x) = -u'(x),$$

with the domain $\mathcal{D}(A) = \{u(x) : u'(x) \in E, u(b) = 0\}$. Then the operator A is positive in E and it holds for all $f(x) \in \mathcal{D}(A)$ that

$$A^{-\alpha} f(x) = I^{\alpha}_{b-} f(x).$$

Proof. Using formulas (11) and (13), we achieve that

$$A^{-\alpha} f(x) = \Gamma^{-1}(\alpha)\Gamma^{-1}(1-\alpha) \int_x^b \left[\int_0^\infty s^{-\alpha} e^{-s(t-x)} ds \right] f(t)dt.$$

Using $z = s(t-x)$ in the formula (3), we compute the value of the following integral

$$\int_0^\infty s^{-\alpha} e^{-s(t-x)} ds = \frac{\Gamma(1-\alpha)}{(t-x)^{1-\alpha}}.$$

To conclude, we substitute this value above and use the formula (2). Hence, we show that

$$A^{-\alpha} f(x) = \Gamma^{-1}(\alpha) \int_x^b \frac{f(t)}{(t-x)^{1-\alpha}} dt = I^{\alpha}_{b-} f(x).$$

□

We denote $E_h = L_2[a,b]_h$ with the mesh points

$$[a,b]_h = \left\{ x_k : x_k = kh + a, k = 0, 1, \ldots, N, h = \frac{b-a}{N} \right\}.$$

Theorem 2.3. *Suppose A_h is an operator in $E_h = L_2[a,b]_h$ that is given by*

$$A_h u^h(x) = \left\{ -\frac{u_k - u_{k-1}}{h} \right\}_1^N,$$

with $u_N = 0$. Then the operator A_h is positive in E_h and the following holds

$$A_h^{\alpha} f^h(x) = \left\{ -\Gamma^{-1}(1-\alpha) \sum_{m=k}^{N} \frac{\Gamma(m-k-\alpha+1)}{\Gamma(m-k+1)} \frac{f_m - f_{m-1}}{h^{\alpha}} \right\}_1^N.$$

Proof. Notice that the inverse of operator $\lambda I + A_h$ is bounded for $\lambda \geq 0$ and is formulated by

$$(\lambda I + A_h)^{-1} f^h(x) = \left\{ \sum_{m=k}^{N} (1 + hs)^{k-m-1} f_{m-1} h \right\}_{k=1}^N. \tag{14}$$

It follows from this formula that the operator A_h is positive in the space E_h. Then applying formulas (12) and (14) we obtain that

$$A_h^{\alpha} f^h(x)$$
$$= \left\{ -\Gamma^{-1}(\alpha)\Gamma^{-1}(1-\alpha) \sum_{m=k}^{N} \left[\int_0^{\infty} s^{\alpha-1}(1+hs)^{k-m-1}\, ds \right] \frac{f_m - f_{m-1}}{h} h \right\}_1^N .$$

We use the formula (4) to evaluate the next integral:

$$\int_0^{\infty} s^{\alpha-1}(1+hs)^{k-m-1}\, ds$$
$$= \frac{1}{\Gamma(m-k+1)} \int_0^{\infty} t^{m-k} e^{-t} \left[\int_0^{\infty} s^{\alpha-1} e^{-ths} ds \right] dt.$$

By substitution $z = ths$ in the formula (3), we achieve

$$\int_0^{\infty} s^{\alpha-1}(1+hs)^{k-m-1}\, ds = \frac{\Gamma(\alpha)\Gamma(m-k-\alpha+1)}{\Gamma(m-k+1)h^{\alpha}}.$$

Finally, we substitute this value above. We demonstrate that

$$A_h^{\alpha} f^h(x) = \left\{ -\Gamma^{-1}(1-\alpha) \sum_{m=k}^{N} \frac{\Gamma(m-k-\alpha+1)}{\Gamma(m-k+1)} \frac{f_m - f_{m-1}}{h^{\alpha}} \right\}_1^N .$$

□

Theorem 2.4. *Let A_h be an operator in $E_h = L_2[a,b]_h$ formulated by*

$$A_h u^h(x) = \left\{ -\frac{u_k - u_{k-1}}{h} \right\}_1^N ,$$

with $u_N = 0$. Then A_h is a positive operator in E_h and it satisfies that

$$A_h^{-\alpha} f^h(x) = \left\{ \Gamma^{-1}(\alpha) \sum_{m=k}^{N} \frac{\Gamma(m-k+\alpha)}{\Gamma(m-k+1)} f_{m-1} h^{\alpha} \right\}_1^N .$$

Proof. Use formulas (11) and (14) to get that

$$A_h^{-\alpha} f^h(x) = \left\{ \Gamma^{-1}(\alpha)\Gamma^{-1}(1-\alpha) \sum_{m=k}^{N} \left[\int_0^{\infty} s^{-\alpha}(1+hs)^{k-m-1} ds \right] f_{m-1} h \right\}_1^N .$$

Applying the formula (4), we compute the integral as follows

$$\int_0^\infty s^{-\alpha}(1+hs)^{k-m-1}ds = \frac{1}{\Gamma(m-k+1)}\int_0^\infty t^{m-k}e^{-t}\left[\int_0^\infty s^{-\alpha}e^{-ths}ds\right]dt.$$

Putting $z = ths$ in the formula (3), we obtain

$$\int_0^\infty s^{-\alpha}(1+hs)^{k-m-1}ds = \frac{\Gamma(1-\alpha)\Gamma(m-k+\alpha)}{\Gamma(m-k+1)}h^{\alpha-1}.$$

To conclude, we substitute this value above. We prove that

$$A_h^{-\alpha}f^h(x) = \left\{\Gamma^{-1}(\alpha)\sum_{m=k}^{N}\frac{\Gamma(m-k+\alpha)}{\Gamma(m-k+1)}f_{m-1}h^\alpha\right\}_1^N.$$

□

3 Conclusion

In this paper, we have examined the relationship between fractional derivatives and fractional powers of positive operators. We have also deduced difference formulas for fractional derivatives. These findings contribute to a better understanding of the connections between these concepts and provide useful formulas for future applications in fractional calculus. In addition, one may consider fractional powers of various positive operators with nonlocal boundary conditions in future research.

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