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Weakly compact embedding of non-commutative symmetric spaces

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Abstract. In this paper, we prove that the embeddings of certain well-known symmetric spaces are weakly compact. Our main results concern noncommutative Lorentz and Marcinkiewicz spaces on finite von Neumann algebras.

Keywords. Noncommutative function spaces, noncommutative Lorentz spaces, noncommutative Marcinkiewicz spaces.

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1 Introduction

Let E and F be Banach spaces, and let $j : E \to F$ be a continuous embedding of E into F. Let us briefly elaborate on what the term "weakly compact embedding of E into F" stands for. To this end, let us denote by F^* the Banach dual to space F, and by $\sigma(F, F^*)$ the weakest topology on F in which all continuous linear functionals on F are still continuous (for an extended discussion of this notion and its early history, see [1, Section 9.2]).

In this article, we are interested in the question *when every bounded set in $j(E)$ is weakly (that is $\sigma(F, F^*)-$) compact in F.*

In Banach space geometry, the discussion regarding the weakly compact embedding of E into F is frequently framed in the language of weakly compact operators. That is, the embedding $E \subset F$ is weakly compact if and only if j is a weakly compact operator, i.e., j transforms each norm-bounded set in E into a set that is relatively compact in F in the $\sigma(F, F^*)$-topology.

This question plays a substantial role in the Banach space geometry [2] and the interpolation theory [3]. For example, let $(E, F)_{\theta,q}$ be the space constructed by the method of real interpolation (see the detailed exposition of the real interpolation method in [2, 3]). It has been shown by B. Beauzamy (see also [2] for detailed exposition and discussion of this important result) that $(E, F)_{\theta,q}$ is a reflexive space for $0 < \theta < 1$ and $1 < q < \infty$ if and only if j is a weakly compact operator.

The following problem is partly motivated by the above observation and is of substantial interest for the interpolation theory of symmetric function (and operator) spaces.

Let $E \subset F$ be (function or operator) symmetric spaces. What is the general criterion guaranteeing that the embedding $j : E \subset F$ is weakly compact?

This question was earlier stated and considered by Kuzin-Aleksinskii [4, 5] in the setting of symmetric function spaces. In the present article, we resolve this question for symmetric operator spaces $E = E(\mathcal{M}, \tau)$ and $F = F(\mathcal{M}, \tau)$, where $(\mathcal{M}, \tau)$ is a finite von Neumann algebra equipped with a faithful normal finite trace. Our work extends the results in [4,5]. In fact, when we specialize our main result (Theorem 3.1 below) to the situation that von Neumann algebra $\mathcal{M}$ coincides with $L_\infty(0, 1)$, we recover results stated in [5]. In Section 4, we demonstrate corollaries of our main result in the special case of Lorentz and Marcinkiewics operator spaces. The results given in that section imply those announced in [5].

The presentation of results requires a few definitions and notions from theory of noncommutative integration, theory of symmetric function and operator spaces and some classical facts from standard course of functional analysis. A few selected definitions and results are cited in the next section. For all unexplained terminology from the theory of symmetric operator spaces, we refer to the recent monograph [6]. For classical facts from Banach function space theory (and interpolation in such spaces), we refer the reader to [2, 3]. For exposition of classical chapters of functional analysis concerning weak compactness, we refer to [1].

2 Preliminaries

Consider a Banach space $(X, \|\cdot\|_X)$ of real-valued Lebesgue measurable functions (with identification a.e.) on the interval $J = (0, 1)$. Let x^* denote the nonincreasing, right-continuous rearrangement of $|x|$ given by

$$x^*(t) = \inf\{s \geq 0 : \lambda(\{|x| > s\}) \leq t\}, \quad t > 0,$$

where λ denotes the Lebesgue measure. The space X is called a rearrangement invariant or a symmetric space [2, 3] if

(i) X is an ideal lattice, that is if $y \in X$, and x is any measurable function on J with $0 \leq |x| \leq |y|$, then $x \in X$ and $\|x\|_X \leq \|y\|_X$;

(ii) if $y \in X$ and if x is any measurable function on J with $x^* = y^*$, then $x \in X$ and $\|x\|_X = \|y\|_X$.

With every symmetric space $X = X(0, 1)$ and every finite von Neumann algebra $(\mathcal{M}, \tau)$, $\tau(\mathbf{1}) = 1$ (here, $\mathbf{1}$ is the unit of $\mathcal{M}$) we associate a noncommutative symmetric space $X(\mathcal{M},\tau)$ defined by (see [6, 7]) the formula

$$X(\mathcal{M},\tau) = \{x \in S(\mathcal{M},\tau) : \mu(x) \in X\}, \ \|x\|_{X(\mathcal{M},\tau)} = \|\mu(x)\|_X,$$

where $S(\mathcal{M},\tau)$ is the algebra of all τ-measurable operators affiliated with $(\mathcal{M},\tau)$ (see [6, Section 2]) and $\mu(x)$ is the singular value function of the operator x (see [6, Sections 2 and 3]). The fundamental fact that $\|\cdot\|_{X(\mathcal{M},\tau)}$ is indeed a norm on $X(\mathcal{M},\tau)$ was established in full generality in [7]. The detailed proof of this fact under small additional conditions imposed on the space $X = X(0,1)$ (namely, when X is a strongly symmetric space) may be found in [6].

Our main object of investigation in this article concerns the weakly compact embedding of symmetric operator space $E := E(\mathcal{M},\tau)$ into symmetric operator space $F := F(\mathcal{M},\tau)$.

3 Main part

In the statement of our main result and in its proof, we need a notion of *symmetric operator spaces with order continuous norm.* In the setting of Banach function spaces, such spaces are well known (e.g., all Lebesgue L_p-spaces, $1 \le p < \infty$) have order continuous norm [2, 3]). We recall a definition of symmetric operator spaces with order continuous norm in Remark 3.2 below. Numerous equivalent descriptions of this important class of operator spaces may be found in [6, Section 3]. The importance of such spaces in the theory may be seen from the following fundamental fact: if E is a symmetric operator space with order continuous norm, then its Banach dual E^* coincides with its Köthe dual $E^\times$ (see [6, Sections 3 and 4]). Of course, this fact has its predecessor in the theory of Banach function spaces [2, 3]. In the proof of Theorem 3.1 below, we will use this fact.

Finally, we shall need to recall one more technical fact which is connected with just mentioned coincidence between Banach and Köthe duals already in the setting of general symmetric operator spaces (not necessarily having order continuous norm). Let $X = X(\mathcal{M},\tau)$ be an arbitrary symmetric operator space. Consider the norm closure of the linear subspace $\mathcal{M}$ in X. This subspace is the largest symmetric operator subspace in X with order continuous norm. For brevity, everywhere below we denote this closure by the symbol X°. We will further elaborate this point immediately after stating our main result below in Remark 3.2.

Our main result is the following theorem.

Theorem 3.1. *Let $E = E(\mathcal{M},\tau)$ and $F = F(\mathcal{M},\tau)$ be symmetric operator spaces with order continuous norm such that $E \subset F$. The natural embedding $j : E \hookrightarrow F$ is weakly compact if and only if we have*

$$(a) \ \ F^* \subseteq (E^*)^\circ;$$

$$(b)\ \ ((E^*)^\circ)^* \subseteq F.$$

Remark 3.2. In the theorem above, using the assumption that E and F have order continuous norm (that is $E = E^{oc} = \{x \in E : |x| \geq x_\alpha \downarrow 0 \Rightarrow \|x_\alpha\|_E \to 0\}$), we identify the Banach dual E^* with the space of all order continuous functionals on E (the Köthe dual $E^\times$), see Definition 4.3.1, Lemma 4.3.5, Lemma 4.3.6 and Proposition 5.4.6 in [6]. The symbol $(E^*)^\circ$ denotes the subspace of $E^* = E^*(\mathcal{M};\tau)$ of all elements in the Köthe dual $E^\times(\mathcal{M},\tau)$ with order continuous norm, which coincides with the closure of the subspace M in E with respect to the norm $\|\cdot\|_E$.

Proof of Theorem 3.1. Our proof is using a classical fact from [1] which gives an equivalent description of a weakly compact operator T acting from Banach space E into Banach space F. Namely, a linear bounded operator $T : E \to F$ is weakly compact if and only if $T^* : F^* \to E^*$ is weakly compact from F^* into E^* if and only if there exists a reflexive Banach space R and operators $U : E \to R$, $V : R \to F$ such that $T = UV$ [1]. We refer a reader to [1] for definition and discussion of the notion of a weakly compact operators. In our setting, we will always assume that $T = j$, the embedding operator $E \subset F$. Note that the linear operator j is weakly compact if and only if the embedding $j : E \subset F$ is weakly compact.

Sufficiency. Suppose, firstly, that the assumptions (a) and (b) hold. Let us show that $j^{**}(E^{**}) \subseteq F$. Appealing to the assumption (a), we have

$$j^* : F^* \to (E^*)^\circ \subset E^*.$$

Hence, $j^{**} : E^{**} \to ((E^*)^\circ)^* \to F^{**}$, and employing the assumption (b), we conclude that

$$j^{**}(E^{**}) \subseteq F,$$

which implies that j is a weakly compact embedding by the classical result from [1, Corollory 9.3.2].

Necessity. Assume that j is weakly compact. By Theorem 2.g.21 and Proposition 2.g.22 in [2], the symmetric operator space (see [6, Section 7]) $R = [E,F]_{\theta,p}$ with $0 < \theta < 1, 1 < p < \infty$ is reflexive and hence with order continuous norm [6], and so, the embeddings $j_0 : E \to R$, $j_1 : R \to F$ are dense. Hence,

$$j^* : F^* \xrightarrow{j_1^*} R^* \xrightarrow{j_0^*} E^*.$$

Since the space R has order continuous norm, and since its Banach (or Köthe) dual also has order continuous norm (due to its reflexivity), guaranteeing

$$j^* : F^* \xrightarrow{j_1^*} R^* \xrightarrow{j_2} (E^*)^\circ \xrightarrow{j_3} E^*,$$

where j_2 and j_3 are embeddings such that $j_0^* = j_3 \circ j_2$. The preceding display shows that the condition (a) holds.

Next, observing that j_1^* and j_2 are dense embeddings, we conclude that $F^* \subseteq (E^*)^\circ$. Therefore,

$$j^{**} : E^{**} \xrightarrow{j_3^*} ((E^*)^\circ)^* \xrightarrow{j_2^*} R^{**} \xrightarrow{j_1^{**}} F^{**}.$$

Since the space R is reflexive, we have that

$$R^{**}|_{j_1^{**}} = R|_{j_1^{**}}.$$

Recalling that for the second adjoint operator $j^{**}{}_1$, we always have that

$$R|_{j_1^{**}} = R|_{j_1},$$

and recalling that j_1 is the embedding of R into F, we infer that

$$((E^*)^\circ)^* \xrightarrow{j_2^*} R \xrightarrow{j_1} F.$$

The preceding display shows that the condition (b) holds and the proof is completed.

□

4 Application to Lorentz and Marcinkiewicz operator spaces

In this section we present corollaries of Theorem 3.1 from the preceding section to the study of weakly compact embeddings of Lorentz and Marcinkiewicz operator spaces consisting of τ-measurable operators affiliated with a finite von Neumann algebra $\mathcal{M}$. For detailed study of such spaces in the classical setting, we refer to [3] and to [6] (in the noncommutative settings). We recall some definitions.

Let Ψ denote the set of all concave increasing functions $\psi : [0, 1) \mapsto [0, \infty)$ with $\psi(0) = 0$ and

$$\lim_{t \to 0^+} \frac{t}{\psi(t)} = 0.$$

Let Ω denote the set of all $\psi \in \Psi$ such that $\lim_{t \to 0^+} \psi(t) = 0$. The important function from Ω is the function $t \mapsto t^\alpha$, $(log(1 + t)^\alpha)$ for $0 < \alpha < 1$.

Let $\psi \in \Psi$. Define the weighted mean function

$$\alpha(x, t) = \frac{1}{\psi(t)} \int_0^t x^*(s) ds, \quad 0 < t < 1.$$

and denote by $M_\psi = M_\psi(0,1))$ the Matcinkiewicz space of measurable functions x on $(0,1)$ such that

$$\|x\|_{M_\psi} := \sup_{0<t\leq 1} \alpha(x,t) < \infty.$$

The largest separable symmetric subspace of M_ψ is denoted by M_ψ°.

For a given $\psi \in \Omega$, the Lorentz space $\Lambda_\psi = \Lambda_\psi(0,1)$ consists of all Lebesque measurable functions f on $[0,1]$ such that

$$\|f\|_{\Lambda_\psi} = \int_0^1 f^*(s)d\psi(s) < \infty$$

Here, f^* is decreasing rearrangement of the function $|f|$ defined in Preliminaries. We recall the definitions of Lorentz spaces $\Lambda_\psi(\mathcal{M},\tau)$ and Marcinkiewicz spaces $M_\psi(\mathcal{M},\tau)$, where $(\mathcal{M},\tau)$ is a finite von Neumann algebra equipped with a faithful normal finite trace τ (we always assume that $\tau(\mathbf{1}) = 1$) where 1 is the unit element in $\mathcal{M}$). The extensive discussion of these spaces and their properties can be found in Sections 6.3 and 6.4 of [6], respectively. We have

$$\Lambda_\psi(\mathcal{M},\tau) = \{x \in S(\mu,\tau) : \mu(x) \in \Lambda_\psi\}),$$

$$M_\psi(\mathcal{M},\tau) = \{x \in S(\mu,\tau) : \mu(x) \in M_\psi\}).$$

Corollary 4.1. *Let $(\mathcal{M},\tau)$ be an atomless finite von Neumann algebra. Let $\varphi(t)$ and $\psi(t)$ be increasing concave functions on $[0,1]$ from Ω such that $\varphi(t) \leq \psi(t)$, $t \in [0,1]$. The embedding $M_\varphi^0(\mathcal{M},\tau) \subseteq M_\psi^0(\mathcal{M},\tau)$ is weakly compact if and only if*

$$\lim_{t\to 0} \frac{\varphi(t)}{\psi(t)} = 0.$$

Proof. We shall use our main result, Theorem 3.1. For brevity, we write simply M_ψ and M_ϕ omitting $(\mathcal{M},\tau)$ from the notation.

Necessity. Assume that the embedding $M_\varphi^0(\mathcal{M},\tau) \subseteq M_\psi^0(\mathcal{M},\tau)$ is weakly compact. It follows from classical duality results [3] that

$$(M_\varphi^0(0,1))^{**} = M_\varphi(0,1).$$

Combining this fact with results from [6, Chapter 6] ,we obtain the noncommutative analogue of this equality, that is

$$(M_\varphi^0)^{**} = M_\varphi, \quad (M_\psi^0)^{**} = M_\psi.$$

Hence, we have

$$j^{**}(M_\varphi) \subseteq M_\psi^0,$$

where j is the natural embedding M^0_φ into M^0_ψ. Since the algebra $(\mathcal{M}, \tau)$ is assumed to be atomless, we know (see [9, Th.2.3.1]) that M_φ contains an element x such that

$$\mu(t, x) = \varphi'(t), \;\; t \in [0, 1].$$

Hence, $j^{**}(x) \in M^0_\psi$, or simply $x \in M^0_\psi$. Again appealing to [9–11], we conclude that $\varphi' \in M^0_\psi[0, 1]$, which yields

$$\lim_{t\to 0} \frac{\varphi(t)}{\psi(t)} = \lim_{t\to 0} \frac{1}{\psi} \int_0^t \varphi'(s)ds = 0.$$

Sufficiency. Suppose that $x \in M_\varphi$ and the assumption $\lim_{t\to 0} \frac{\varphi(t)}{\psi(t)} = 0$ holds. We have

$$\lim_{t\to 0} \frac{1}{\psi(t)} \int_0^t \mu(s; x)ds = \lim_{t\to 0} \frac{\varphi(t)}{\psi(t)} \frac{1}{\varphi(t)} \int_0^t \mu(s; x)ds \le \|x\|_{M_\varphi} \lim_{t\to 0} \frac{\varphi(t)}{\psi(t)} = 0.$$

That is $x \in M^0_\psi$ and we arrive at the embedding

$$M_\phi \subseteq M^0_\psi.$$

This verifies condition (b) of Theorem 3.1. Indeed, we have (see [3, 6]) that

$$(M^0_\psi)^* = \Lambda_{\frac{t}{\psi(t)}} \text{ and that } (\Lambda_{\frac{t}{\psi(t)}})^* = M_\psi.$$

Thus, we have $(M^0_\psi)^* = M_\psi \subseteq M^0_\psi$.

The condition (a) in our present setting is equivalent to the verification of the embedding

$$(M^0_\psi)^* \subseteq ((M^0_\phi)^*)^0.$$

Observe (again appealing to [6]) that $(M^0_\psi)^* = \Lambda_{\frac{t}{\phi(t)}}$ and that $(M^0_\phi)^* = \Lambda_{\frac{t}{\psi(t)}}$. Since $\lim_{t\to 0} \frac{\phi(t)}{\psi(t)} = 0$, it follows that $\frac{t}{\psi(t)} \le c\frac{t}{\psi(t)}$, $t > 0$ for some $c > 0$. Hence, (see [3]) we obtain that

$$\Lambda_{\frac{t}{\psi(t)}} \subseteq \Lambda_{\frac{t}{\phi(t)}}.$$

The proof if completed. □

Arguing in a completely similar manner, we have the following result.

Corollary 4.2. *The embedding* $\Lambda_\phi(\mathcal{M}, \tau) \subseteq \Lambda_\psi(\mathcal{M}, \tau)$ *is weakly compact if and only if* $\lim_{t\to 0} \dfrac{\psi(t)}{\phi(t)} = 0.$

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