e-Journal of Analysis and Applied Mathematics **2025** (2025), 17–32
DOI 10.62780/ejaam/2025-003 EJAAM 2025

# On the mean integral of the reciprocal and ratio of trigonometrically convex functions

Christophe Chesneau

**Abstract.** Trigonometrically convex functions have recently emerged as an important class of functions in the field of convex analysis. Despite the growing interest, several aspects of their theory remain unexplored. In this article, we first establish lower and upper bounds for the mean integral of the reciprocal of a function that is either trigonometrically convex or trigonometrically concave. Building on these results, we further investigate the mean integral of the ratio of two functions, each of which is assumed to be either trigonometrically convex or concave. These investigations provide new knowledge about the integral behavior and comparative structure of such functions.

**Keywords.** Trigonometrically convex function, trigonometrically concave function, Hermite-Hadamard integral inequality, hyperbolic cotangent function.

**2020 Mathematics Subject Classification.** 26D15, 33E20.

## 1 Introduction

### 1.1 Basis of convex analysis

Before presenting our contributions, it is necessary to recall some basics of convex analysis. Essentially, convexity and concavity of functions are characterized by inequalities that describe how a function behaves between two points. The formal definition of a convex function is given below. In this definition, and throughout the article, we will only consider non-negative functions for the sake of simplicity in presentation.

**Definition 1.1** (Convex function). Let $a, b \in \mathbb{R}$ with $a < b$. We say that a function $f : [a, b] \to [0, +\infty)$ is convex if and only if, for any $\lambda \in [0, 1]$ and $x, y \in [a, b]$, we have

$$f(\lambda x + (1 - \lambda)y) \le \lambda f(x) + (1 - \lambda) f(y).$$

So the graph of a convex function lies below the straight line connecting any two points on the graph. Conversely, the inequality is reversed for a concave function, as formalized below.

**Definition 1.2** (Concave function). Let $a, b \in \mathbb{R}$ with $a < b$. We say that a function $f : [a, b] \to [0, +\infty)$ is concave if and only if, for any $\lambda \in [0, 1]$ and $x, y \in [a, b]$, we have

$$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y).$$

In this case, the graph of the function lies above the straight line connecting any two points on the graph. For the details on these basic notions, see [1–7].

Convex and concave functions play a central role in the study of inequalities, especially integral inequalities. One of the most fundamental results in this context is the Hermite-Hadamard integral inequality. It gives bounds for the mean integral of a function over an interval in terms of its values at the endpoints, i.e., $a$ and $b$, and at the midpoint, i.e., $(a+b)/2$. A formal statement of this inequality in the case where the function is convex is given below.

**Theorem 1.3** (Hermite-Hadamard integral inequality). *Let $a, b \in \mathbb{R}$ with $a < b$ and let $f : [a, b] \to [0, +\infty)$ be a convex function. Then we have*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

This result thus shows that, for convex functions, the midpoint value gives a lower bound on the mean integral, while the average of the values at the endpoints gives an upper bound. The statement below considers the opposite case, when the function is concave.

**Theorem 1.4** (Hermite-Hadamard integral inequality). *Let $a, b \in \mathbb{R}$ with $a < b$ and let $f : [a, b] \to [0, +\infty)$ be a concave function. Then we have*

$$\frac{f(a)+f(b)}{2} \leq \frac{1}{b-a}\int_a^b f(x)dx \leq f\left(\frac{a+b}{2}\right).$$

This version emphasizes the dual behavior: for concave functions, the average of the values at the endpoints is a lower bound, while the value at the midpoint gives an upper bound for the mean integral. Improvements and extensions of the Hermite-Hadamard integral inequality are still an active area of research, see [8–17].

## 1.2 Trigonometrically convex function

Exploring modified notions of convexity can help to improve understanding of certain inequalities involving functions. One such notion is trigonometric convexity, which modifies the classical convexity inequality by using sine and cosine weights.

It was introduced in 2018 in [18] and has attracted some attention since then. A formal presentation of a trigonometrically convex function is given below.

**Definition 1.5** (Trigonometrically convex function). [18] Let $a, b \in \mathbb{R}$ with $a < b$. We say that a function $f : [a, b] \to [0, +\infty)$ is trigonometrically convex if and only if, for any $\lambda \in [0, 1]$ and $x, y \in [a, b]$, we have

$$f(\lambda x + (1-\lambda)y) \leq \sin\left(\lambda\frac{\pi}{2}\right) f(x) + \cos\left(\lambda\frac{\pi}{2}\right) f(y).$$

This inequality is reversed for trigonometrically concave functions. These functions have a similar structure but represent the dual behavior, as formally presented below.

**Definition 1.6** (Trigonometrically concave function). [18] Let $a, b \in \mathbb{R}$ with $a < b$. We say that a function $f : [a, b] \to [0, +\infty)$ is trigonometrically concave if and only if, for any $\lambda \in [0, 1]$ and $x, y \in [a, b]$, we have

$$f(\lambda x + (1-\lambda)y) \geq \sin\left(\lambda\frac{\pi}{2}\right) f(x) + \cos\left(\lambda\frac{\pi}{2}\right) f(y).$$

These two definitions provide a framework for studying inequalities involving trigonometric weights that offer flexibility beyond classical convexity. It is particularly relevant to the analysis of functionals and integrals where trigonometric structures play a central role. In a sense, it extends classical convex analysis to settings that are more naturally described in terms of trigonometric behavior.

As previously discussed, the shape of a function, particularly whether it is convex or concave, plays a critical role in the analysis of integral inequalities (see Theorems 1.3 and 1.4). For trigonometrically convex functions, it is possible to establish a two-sided bound for the mean integral value of the function over a given interval. A key result of this type, originally established in [18], is presented below.

**Theorem 1.7.** [18, Combination of Theorems 4 and 5] *Let $a, b \in \mathbb{R}$ with $a < b$ and let $f : [a, b] \to [0, +\infty)$ be a trigonometrically convex function. Then we have*

$$\frac{1}{\sqrt{2}} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{2}{\pi}\left[f(a) + f(b)\right].$$

This result thus provides lower and upper bounds for the mean integral with respect to the function at the midpoint and endpoints. In contrast, if the function is trigonometrically concave, the direction of the inequalities is reversed, reflecting the different behavior of such functions. This is what is formalized below.

**Theorem 1.8.** [18, Concave version of the combination of Theorems 4 and 5] *Let $a, b \in \mathbb{R}$ with $a < b$ and let $f : [a, b] \to [0, +\infty)$ be a trigonometrically concave function. Then we have*

$$\frac{2}{\pi}\left[f(a) + f(b)\right] \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{1}{\sqrt{2}}f\left(\frac{a+b}{2}\right).$$

Taken together, Theorems 1.7 and 1.8 illustrate how the curvature of a function determines the bounds of its mean integral within the interval. Recent developments on integral inequalities involving trigonometrically convex or concave functions can be found in [19–23].

### 1.3 Contributions

As discussed above, the study of integral inequalities has long been central to analysis. More recently, attention has shifted to integrals involving the reciprocal of a function, especially when the function is non-negative and has structural properties such as convexity, concavity, or generalized convex forms. See [13, 17]. This shift is driven by the emergence of applications in areas such as harmonic analysis and probability, as well as the potential to reveal deeper functional behavior through reciprocal inequalities.

In this article, we contribute to this subject by establishing lower and upper bounds for the mean integral of the reciprocal of a function that is either trigonometrically convex or concave. We thus give bounds for an integral of the form

$$\frac{1}{b-a}\int_a^b \frac{1}{f(x)}dx,$$

where $f$ is assumed to be either trigonometrically convex or concave. Such bounds are well established for convex or concave functions (see [13, 17]), but remain unexplored for trigonometrically convex or concave functions. We therefore fill this gap. A key ingredient in the definitions of our bounds is the inverse hyperbolic cotangent function and the function $f$ at the midpoint and endpoints.

We then extend our analysis to study the mean integral of the ratio of two such functions. This allows us to obtain different bounds for an integral of the following form:

$$\frac{1}{b-a}\int_a^b \frac{g(x)}{f(x)}dx,$$

where $f$ and $g$ are assumed to be either trigonometrically convex or concave. Some monotonicity assumptions are made on $f$ and $g$, allowing the application of the Chebyshev integral inequality, which will play a central role in our approach. These results enrich the understanding of integral behavior and comparative structure within the framework of trigonometrically modified convexity.

### 1.4 Organization

The remainder of this article is organized as follows: Section 2 is dedicated to establishing bounds for the mean integral of the reciprocal of a trigonometrically convex function. Section 3 presents the analogous results for trigonometrically concave functions. In Section 4, we investigate bounds for the mean integral of the ratio of two functions, each assumed to be either trigonometrically convex or concave. Finally, Section 5 provides concluding remarks and outlines potential directions for future research.

## 2 On the mean integral of the reciprocal of a trigonometrically convex function

The main theorem of this section is presented below. A lower bound is given for the mean integral of interest.

**Theorem 2.1.** *Let $a, b \in \mathbb{R}$ with $a < b$ and let $f : [a, b] \to [0, +\infty)$ be a trigonometrically convex function. Then we have*

$$\frac{1}{b-a}\int_a^b \frac{1}{f(t)}dt$$

$$\geq \max\left\{\frac{4}{\pi\sqrt{f^2(a)+f^2(b)}}\operatorname{cotanh}^{-1}\left[\frac{f(a)+f(b)}{\sqrt{f^2(a)+f^2(b)}}\right], \frac{\pi}{2[f(a)+f(b)]}\right\},$$

*where*

$$\operatorname{cotanh}^{-1}(y) = \frac{1}{2}\log\left(\frac{y+1}{y-1}\right),$$

*with $y > 1$.*

*Proof.* Let us prove the first inequality, that is, the lower bound corresponding to the first term into the braces of the maximum term. Making the change of variables $t = \lambda a + (1-\lambda)b$ with respect to $\lambda \in [0, 1]$, we get

$$\frac{1}{b-a}\int_a^b \frac{1}{f(t)}dt = \int_0^1 \frac{1}{f(\lambda a + (1-\lambda)b)}d\lambda. \tag{1}$$

Since $f$ trigonometrically convex, for any $\lambda \in [0,1]$ and $x, y \in [a,b]$, we have

$$f(\lambda x + (1-\lambda)y) \leq \sin\left(\lambda\frac{\pi}{2}\right) f(x) + \cos\left(\lambda\frac{\pi}{2}\right) f(y),$$

and, in particular,

$$f(\lambda a + (1-\lambda)b) \leq \sin\left(\lambda\frac{\pi}{2}\right) f(a) + \cos\left(\lambda\frac{\pi}{2}\right) f(b).$$

This and the fact that the functions involved are non-negative imply that

$$\begin{aligned} &\int_0^1 \frac{1}{f(\lambda a + (1-\lambda)b)} d\lambda \geq \int_0^1 \frac{1}{\sin(\lambda\pi/2) f(a) + \cos(\lambda\pi/2) f(b)} d\lambda \\ &= \frac{1}{f(a)} \int_0^1 \frac{1}{\sin(\lambda\pi/2) + \cos(\lambda\pi/2)\gamma} d\lambda, \end{aligned} \tag{2}$$

where $\gamma = f(b)/f(a)$. Using standard primitives involving the inverse hyperbolic tangent function, i.e.,

$$\tanh^{-1}(z) = \frac{1}{2}\log\left(\frac{z+1}{1-z}\right),$$

with $z < 1$, we get

$$\begin{aligned} &\frac{1}{f(a)} \int_0^1 \frac{1}{\sin(\lambda\pi/2) + \cos(\lambda\pi/2)\gamma} d\lambda \\ &= \frac{1}{f(a)} \left[ -\frac{4}{\pi\sqrt{1+\gamma^2}} \tanh^{-1}\left[\frac{1-\gamma\tan(\pi\lambda/4)}{\sqrt{1+\gamma^2}}\right]\right]_{\lambda=0}^{\lambda=1} \\ &= \frac{1}{f(a)} \times \frac{4}{\pi\sqrt{1+\gamma^2}} \operatorname{cotanh}^{-1}\left[\frac{1+\gamma}{\sqrt{1+\gamma^2}}\right] \\ &= \frac{4}{f(a)\pi\sqrt{1+f^2(b)/f^2(a)}} \operatorname{cotanh}^{-1}\left[\frac{1+f(b)/f(a)}{\sqrt{1+f^2(b)/f^2(a)}}\right] \\ &= \frac{4}{\pi\sqrt{f^2(a)+f^2(b)}} \operatorname{cotanh}^{-1}\left[\frac{f(a)+f(b)}{\sqrt{f^2(a)+f^2(b)}}\right]. \end{aligned} \tag{3}$$

Joining relations (1), (2) and (3), we obtain

$$\frac{1}{b-a}\int_a^b \frac{1}{f(t)} dt \geq \frac{4}{\pi\sqrt{f^2(a)+f^2(b)}} \operatorname{cotanh}^{-1}\left[\frac{f(a)+f(b)}{\sqrt{f^2(a)+f^2(b)}}\right], \tag{4}$$

which is the desired first lower bound. Now let us find the other lower bound. Using the Cauchy-Schwarz integral inequality, we obtain

$$b-a=\int_a^b dt=\int_a^b \frac{\sqrt{f(t)}}{\sqrt{f(t)}}dt \leq \sqrt{\int_a^b f(t)dt}\sqrt{\int_a^b \frac{1}{f(t)}dt}. \tag{5}$$

Since $f$ is trigonometrically convex, using the right-hand side inequality of Theorem 1.7, we get

$$\int_a^b f(t)dt=(b-a)\left[\frac{1}{b-a}\int_a^b f(t)dt\right] \leq (b-a)\frac{2}{\pi}[f(a)+f(b)]. \tag{6}$$

Joining inequalities (5) and (6), we find that

$$b-a \leq \sqrt{(b-a)\frac{2}{\pi}[f(a)+f(b)]}\sqrt{\int_a^b \frac{1}{f(t)}dt},$$

so that

$$\frac{1}{b-a}\int_a^b \frac{1}{f(t)}dt \geq \frac{\pi}{2[f(a)+f(b)]}. \tag{7}$$

It follows from inequalities (4) and (7) that

$$\begin{aligned}&\frac{1}{b-a}\int_a^b \frac{1}{f(t)}dt\\ &\geq \max\left\{\frac{4}{\pi\sqrt{f^2(a)+f^2(b)}}\operatorname{cotanh}^{-1}\left[\frac{f(a)+f(b)}{\sqrt{f^2(a)+f^2(b)}}\right], \frac{\pi}{2[f(a)+f(b)]}\right\}.\end{aligned}$$

This ends the proof of Theorem 2.1. □

We can note that the term $\operatorname{cotanh}^{-1}$ is valid because

$$\sqrt{f^2(a)+f^2(b)}<\sqrt{f^2(a)}+\sqrt{f^2(b)}=f(a)+f(b),$$

implying that

$$\frac{f(a)+f(b)}{\sqrt{f^2(a)+f^2(b)}}>1.$$

We also mention that

$$\operatorname{cotanh}^{-1}\left[\frac{f(a)+f(b)}{\sqrt{f^2(a)+f^2(b)}}\right]=\tanh^{-1}\left[\frac{\sqrt{f^2(a)+f^2(b)}}{f(a)+f(b)}\right]$$

and that

$$\begin{aligned}&\operatorname{cotanh}^{-1}\left[\frac{f(a)+f(b)}{\sqrt{f^2(a)+f^2(b)}}\right]\\&=\frac{1}{2}\log\left[1+\frac{\sqrt{f^2(a)+f^2(b)}}{f(a)+f(b)}\right]-\frac{1}{2}\log\left[1-\frac{\sqrt{f^2(a)+f^2(b)}}{f(a)+f(b)}\right].\end{aligned}$$

These expressions show the complexity of the "first" lower bound in Theorem 2.1, depending on a sophisticated transformation of the values of $f$ at the endpoints. The other lower bound is much more tractable, i.e.,

$$\frac{1}{b-a}\int_a^b \frac{1}{f(t)}dt \geq \frac{\pi}{2[f(a)+f(b)]},$$

but there is no evidence that it is better than the first. These bounds must be seen as complementary, justifying the role of the maximum term.

## 3 On the mean integral of the reciprocal of a trigonometrically concave function

The main theorem of this section is presented below, considering a trigonometrically concave function. A lower bound and an upper bound are given for the mean integral of interest.

**Theorem 3.1.** *Let $a, b \in \mathbb{R}$ with $a < b$ and let $f : [a, b] \to [0, +\infty)$ be a trigonometrically concave function. Then we have*

$$\frac{\sqrt{2}}{f[(a+b)/2]} \leq \frac{1}{b-a}\int_a^b \frac{1}{f(t)}dt \leq \frac{4}{\pi\sqrt{f^2(a)+f^2(b)}}\operatorname{cotanh}^{-1}\left[\frac{f(a)+f(b)}{\sqrt{f^2(a)+f^2(b)}}\right].$$

*Proof.* Let us prove the upper bound. Making the change of variables $t = \lambda a + (1-\lambda)b$ with respect to $\lambda \in [0, 1]$, we get

$$\frac{1}{b-a}\int_a^b \frac{1}{f(t)}dt = \int_0^1 \frac{1}{f(\lambda a+(1-\lambda)b)}d\lambda. \tag{8}$$

Since $f$ trigonometrically concave, for any $\lambda \in [0, 1]$, we have

$$f(\lambda a+(1-\lambda)b) \geq \sin\left(\lambda\frac{\pi}{2}\right)f(a)+\cos\left(\lambda\frac{\pi}{2}\right)f(b).$$

This and the fact that the functions involved are non-negative imply that

$$\int_0^1 \frac{1}{f(\lambda a+(1-\lambda)b)}d\lambda \le \int_0^1 \frac{1}{\sin(\lambda\pi/2)\, f(a)+\cos(\lambda\pi/2)\, f(b)}d\lambda$$
$$= \frac{1}{f(a)}\int_0^1 \frac{1}{\sin(\lambda\pi/2)+\cos(\lambda\pi/2)\,\gamma}d\lambda, \tag{9}$$

where $\gamma = f(b)/f(a)$. Reusing equality (3) for the integral calculus, we get

$$\frac{1}{f(a)}\int_0^1 \frac{1}{\sin(\lambda\pi/2)+\cos(\lambda\pi/2)\,\gamma}d\lambda$$
$$= \frac{4}{\pi\sqrt{f^2(a)+f^2(b)}}\,\mathrm{cotanh}^{-1}\left[\frac{f(a)+f(b)}{\sqrt{f^2(a)+f^2(b)}}\right]. \tag{10}$$

Joining relations (8), (9) and (10), we obtain

$$\frac{1}{b-a}\int_a^b \frac{1}{f(t)}dt \le \frac{4}{\pi\sqrt{f^2(a)+f^2(b)}}\,\mathrm{cotanh}^{-1}\left[\frac{f(a)+f(b)}{\sqrt{f^2(a)+f^2(b)}}\right], \tag{11}$$

which is the desired upper bound.

Let us now investigate the lower bound. Using the Cauchy-Schwarz integral inequality, we obtain

$$b-a = \int_a^c dt = \int_a^b \frac{\sqrt{f(t)}}{\sqrt{f(t)}}dt \le \sqrt{\int_a^b f(t)dt}\sqrt{\int_a^b \frac{1}{f(t)}dt}. \tag{12}$$

Since $f$ is trigonometrically concave, using the right-hand side inequality of Theorem 1.8, we get

$$\int_a^b f(t)dt = (b-a)\left[\frac{1}{b-a}\int_a^b f(t)dt\right] \le (b-a)\frac{1}{\sqrt{2}}f\left(\frac{a+b}{2}\right). \tag{13}$$

Joining inequalities (12) and (13), we have

$$b-a \le \sqrt{(b-a)\frac{1}{\sqrt{2}}f\left(\frac{a+b}{2}\right)}\sqrt{\int_a^b \frac{1}{f(t)}dt},$$

so that

$$\frac{1}{b-a}\int_a^b \frac{1}{f(t)}dt \ge \frac{\sqrt{2}}{f[(a+b)/2]}. \tag{14}$$

It follows from inequalities (11) and (14) that

$$\frac{\sqrt{2}}{f[(a+b)/2]} \leq \frac{1}{b-a}\int_a^b \frac{1}{f(t)}dt \leq \frac{4}{\pi\sqrt{f^2(a)+f^2(b)}}\,\text{cotanh}^{-1}\left[\frac{f(a)+f(b)}{\sqrt{f^2(a)+f^2(b)}}\right].$$

This ends the proof of Theorem 3.1. □

In contrast to Theorem 2.1, Theorem 3.1 provides an upper bound. It offers tighter control over the mean integral of interest. In this precise context, it can be argued that working with trigonometrically concave functions is more convenient than dealing with trigonometrically convex functions.

The rest of the article investigates bounds for the mean integral of the ratio of two trigonometrically convex or concave functions.

## 4 On the mean integral of the ratio of two trigonometrically convex or concave functions

The theorem below is our first result on bounds for the mean integral of the ratio of a trigonometrically concave function and a trigonometrically convex function. Additional monotonicity assumptions are made on these functions, mainly to apply the Chebyshev integral inequality. Essential tools for the proof are also Theorems 1.7 and 3.1.

**Theorem 4.1.** *Let $a, b \in \mathbb{R}$ with $a < b$ and let $f, g : [a, b] \to [0, +\infty)$ be two monotonic functions with the same monotonicity with $f$ trigonometrically concave and $g$ trigonometrically convex. Then we have*

$$\frac{1}{b-a}\int_a^b \frac{g(x)}{f(x)}dx \leq \frac{8[g(a)+g(b)]}{\pi^2\sqrt{f^2(a)+f^2(b)}}\,\text{cotanh}^{-1}\left[\frac{f(a)+f(b)}{\sqrt{f^2(a)+f^2(b)}}\right].$$

*Proof.* Since $f$ and $g$ are monotonic with the same monotonicity, $1/f$ and $g$ are with opposite monotonicity. It follows from the Chebyshev integral inequality that

$$\frac{1}{b-a}\int_a^b \frac{g(x)}{f(x)}dx \leq \left[\frac{1}{b-a}\int_a^b \frac{1}{f(x)}dx\right]\left[\frac{1}{b-a}\int_a^b g(x)dx\right]. \tag{15}$$

Since $f$ is trigonometrically concave and $g$ is trigonometrically convex, applying the right-hand side inequality of Theorem 3.1 to $f$ and the right-hand side inequality of Theorem 1.7 to $g$, we obtain

$$\begin{aligned}
&\left[\frac{1}{b-a}\int_a^b \frac{1}{f(x)}dx\right]\left[\frac{1}{b-a}\int_a^b g(x)dx\right]\\
&\leq \frac{4}{\pi\sqrt{f^2(a)+f^2(b)}}\operatorname{cotanh}^{-1}\left[\frac{f(a)+f(b)}{\sqrt{f^2(a)+f^2(b)}}\right]\times\frac{2}{\pi}[g(a)+g(b)]\\
&=\frac{8[g(a)+g(b)]}{\pi^2\sqrt{f^2(a)+f^2(b)}}\operatorname{cotanh}^{-1}\left[\frac{f(a)+f(b)}{\sqrt{f^2(a)+f^2(b)}}\right]. \qquad (16)
\end{aligned}$$

It follows from inequalities (15) and (16) that

$$\frac{1}{b-a}\int_a^b \frac{g(x)}{f(x)}dx \leq \frac{8[g(a)+g(b)]}{\pi^2\sqrt{f^2(a)+f^2(b)}}\operatorname{cotanh}^{-1}\left[\frac{f(a)+f(b)}{\sqrt{f^2(a)+f^2(b)}}\right].$$

This concludes the proof of Theorem 4.1. □

The upper bound obtained can be used as a benchmark for further purposes of this type of mean integrals.

The theorem below is the analogue of Theorem 4.1, but with an opposite monotonicity assumption on the main functions. The proof uses the same ingredients.

**Theorem 4.2.** *Let $a, b \in \mathbb{R}$ with $a < b$ and let $f, g : [a, b] \to [0, +\infty)$ be two monotonic functions with opposite monotonicity with $f$ trigonometrically concave and $g$ trigonometrically convex. Then we have*

$$\frac{1}{b-a}\int_a^b \frac{g(x)}{f(x)}dx \geq \frac{g[(a+b)/2]}{f[(a+b)/2]}.$$

*Proof.* Since $f$ and $g$ are monotonic with opposite monotonicity, $1/f$ and $g$ are with the same monotonicity. It follows from the Chebyshev integral inequality that

$$\frac{1}{b-a}\int_a^b \frac{g(x)}{f(x)}dx \geq \left[\frac{1}{b-a}\int_a^b \frac{1}{f(x)}dx\right]\left[\frac{1}{b-a}\int_a^b g(x)dx\right]. \qquad (17)$$

Since $f$ is trigonometrically concave and $g$ is trigonometrically convex, applying the left-hand side inequality of Theorem 3.1 to $f$ and the left-hand side inequality of Theorem 1.7 to $g$, we obtain

$$\begin{aligned}
&\left[\frac{1}{b-a}\int_a^b \frac{1}{f(x)}dx\right]\left[\frac{1}{b-a}\int_a^b g(x)dx\right] \geq \frac{\sqrt{2}}{f[(a+b)/2]}\times\frac{1}{\sqrt{2}}g\left(\frac{a+b}{2}\right)\\
&=\frac{g[(a+b)/2]}{f[(a+b)/2]}. \qquad (18)
\end{aligned}$$

It follows from inequalities (17) and (18) that

$$\frac{1}{b-a}\int_a^b \frac{g(x)}{f(x)}dx \geq \frac{g[(a+b)/2]}{f[(a+b)/2]}.$$

This concludes the proof of Theorem 4.2. □

In this context, we can remark that the function $h = g/f$ satisfies

$$\frac{1}{b-a}\int_a^b h(x)dx \geq h\left(\frac{a+b}{2}\right),$$

which corresponds to the left-hand side of the Hermite-Hadamard integral inequality in Theorem 1.3, but without the assumption that $h$ is convex; $h$ is defined by the ratio of a trigonometrically concave function and a trigonometrically convex function. In a sense, we have identified a new class of functions that satisfy the left-hand side of the Hermite-Hadamard integral inequality beyond the standard convexity assumption. This adds a new dimension to the study of ratio-type functions from the point of view of convex analysis.

The theorem below examines a lower bound for the mean integral of the ratio of two trigonometrically convex functions.

**Theorem 4.3.** *Let $a, b \in \mathbb{R}$ with $a < b$ and let $f, g : [a, b] \to [0, +\infty)$ be two monotonic functions trigonometrically convex with opposite monotonicity. Then we have*

$$\frac{1}{b-a}\int_a^b \frac{g(x)}{f(x)}dx \geq$$
$$\max\left\{\frac{4}{\pi\sqrt{f^2(a)+f^2(b)}}\operatorname{cotanh}^{-1}\left[\frac{f(a)+f(b)}{\sqrt{f^2(a)+f^2(b)}}\right], \frac{\pi}{2[f(a)+f(b)]}\right\}$$
$$\times \frac{1}{\sqrt{2}}g\left(\frac{a+b}{2}\right).$$

*Proof.* Since $f$ and $g$ are monotonic with opposite monotonicity, $1/f$ and $g$ are with the same monotonicity. It follows from the Chebyshev integral inequality that

$$\frac{1}{b-a}\int_a^b \frac{g(x)}{f(x)}dx \geq \left[\frac{1}{b-a}\int_a^b \frac{1}{f(x)}dx\right]\left[\frac{1}{b-a}\int_a^b g(x)dx\right]. \tag{19}$$

Since $f$ and $g$ are trigonometrically convex, applying Theorem 2.1 to $f$ and the left-hand side inequality of Theorem 1.7 to $g$, we obtain

$$\left[\frac{1}{b-a}\int_a^b \frac{1}{f(x)}dx\right]\left[\frac{1}{b-a}\int_a^b g(x)dx\right]$$
$$\geq \max\left\{\frac{4}{\pi\sqrt{f^2(a)+f^2(b)}}\,\mathrm{cotanh}^{-1}\left[\frac{f(a)+f(b)}{\sqrt{f^2(a)+f^2(b)}}\right], \frac{\pi}{2[f(a)+f(b)]}\right\}$$
$$\times \frac{1}{\sqrt{2}}g\left(\frac{a+b}{2}\right). \tag{20}$$

It follows from inequalities (19) and (20) that

$$\frac{1}{b-a}\int_a^b \frac{g(x)}{f(x)}dx \geq$$
$$\max\left\{\frac{4}{\pi\sqrt{f^2(a)+f^2(b)}}\,\mathrm{cotanh}^{-1}\left[\frac{f(a)+f(b)}{\sqrt{f^2(a)+f^2(b)}}\right], \frac{\pi}{2[f(a)+f(b)]}\right\}$$
$$\times \frac{1}{\sqrt{2}}g\left(\frac{a+b}{2}\right).$$

This concludes the proof of Theorem 4.3. □

Again, the lower bound obtained is new in this context. It can be used as a benchmark for similar purposes.

The theorem below completes the article by giving a lower bound for the mean integral of the ratio of two trigonometrically concave functions.

**Theorem 4.4.** *Let $a, b \in \mathbb{R}$ with $a < b$ and let $f, g : [a, b] \to [0, +\infty)$ be two monotonic functions trigonometrically concave with opposite monotonicity. Then we have*

$$\frac{1}{b-a}\int_a^b \frac{g(x)}{f(x)}dx \geq \frac{2\sqrt{2}\,[g(a)+g(b)]}{\pi f[(a+b)/2]}.$$

*Proof.* Since $f$ and $g$ are monotonic with opposite monotonicity, $1/f$ and $g$ are with the same monotonicity. It follows from the Chebyshev integral inequality that

$$\frac{1}{b-a}\int_a^b \frac{g(x)}{f(x)}dx \geq \left[\frac{1}{b-a}\int_a^b \frac{1}{f(x)}dx\right]\left[\frac{1}{b-a}\int_a^b g(x)dx\right]. \tag{21}$$

Since $f$ and $g$ are trigonometrically concave, applying the left-hand side inequality of Theorem 3.1 to $f$ and the left-hand side inequality of Theorem 1.8 to $g$, we obtain

$$\left[\frac{1}{b-a}\int_a^b \frac{1}{f(x)}dx\right]\left[\frac{1}{b-a}\int_a^b g(x)dx\right] \geq \frac{\sqrt{2}}{f[(a+b)/2]} \times \frac{2}{\pi}[g(a)+g(b)]$$
$$= \frac{2\sqrt{2}\,[g(a)+g(b)]}{\pi f[(a+b)/2]}. \tag{22}$$

It follows from inequalities (21) and (22) that

$$\frac{1}{b-a}\int_a^b \frac{g(x)}{f(x)}dx \geq \frac{2\sqrt{2}\,[g(a)+g(b)]}{\pi f[(a+b)/2]}.$$

This concludes the proof of Theorem 4.4. □

A new lower bound is thus obtained in this case.

## 5 Conclusion

In this article, we have advanced the study of trigonometrically convex and concave functions by establishing new bounds for the mean integral of their reciprocals and ratios. These results contribute to a deeper understanding of the integral behavior and comparative structure within this emerging class of functions.

From a theoretical point of view, our results open the door to further exploration of functional inequalities in higher dimensional settings. Following the spirit of [13] for convex functions and a particular two-dimensional case, we can envisage investigating bounds for multi-dimensional integrals of the following forms:

$$\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b \frac{1}{\sum_{i=1}^n f(x_i)}dx_1 \dots dx_n$$

and

$$\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b \frac{\sum_{i=1}^n g(x_i)}{\sum_{i=1}^n f(x_i)}dx_1 \dots dx_n,$$

where $n \in \mathbb{N}\backslash\{0\}$, and $f_1, \dots, f_n, g_1, \dots, g_n$ denote functions that are either trigonometrically convex or concave.

From an applied perspective, the lower and upper bounds derived here have potential implications in fields where periodic or angular structures are inherent, such as signal processing, harmonic analysis, or mathematical physics.

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Received August 15, 2025; revised October 1, 2025; accepted October 8, 2025.

## Author information

Christophe Chesneau, Department of Mathematics, LMNO, University of Caen-Normandie, 14032 Caen, France.
E-mail: `christophe.chesneau@gmail.com`