e-Journal of Analysis and Applied Mathematics **2025** (2025), 33–43
DOI 10.62780/ejaam/2025-004 EJAAM 2025

On the behavior of the solutions for certain neutral delay integro-differential equations

Ali Fuat Yeniçerioğlu

Abstract. Some results are given on the behavior of solutions of scalar linear and constant coefficient neutral delay integro-differential equations. These results are obtained using two different real roots of the relevant characteristic equation. Finally, an example of solutions to neutral delay integro-differential equations is given.

Keywords. Neutral delay integro-differential equation, double inequality, characteristic equation, characteristic root, characteristic solution.

2020 Mathematics Subject Classification. 34K06, 34K12, 34K40.

1 Introduction

The neutral delay integro-differential equations play an important role in the theory of functional differential equations. In recent years, the theory of this class of equations has become an independent field of study. Many results concerning the theory of neutral delay differential equations were given in the excellent books by Hale and Verduyn Lunel [1], Kolmanovski and Myshkis [2], Lakshmikantham, Wen and Zhang [3], and Bellman and Cooke [4]. Besides its theoretical interest, the study of integro-differential equations has some importance in applications. For the basic theory of integral equations, we choose to refer to the books by Burton [5], Corduneanu [6], and Agarwal and O'Regan [7]. Since the first systematic study was performed by Volterra [8], such equations have been investigated in various fields such as mathematical biology and control theory (see, for example, [9]). This system can be found in a wide variety of scientific and engineering fields such as biology, physics, ecology, medicine, etc. (see [10, 11]). In particular, the delay integro-differential system has been observed to play an important role in modeling many different phenomena in circuit analysis and chemical process simulation; a comprehensive list of these can be found in [12].

In general, the theory of neutral delay integro-differential equations presents some additional complications, which are not presented in the theory of the corresponding delay differential equations. So, it is not easy to extend results concerning delay differential equations to neutral delay integro-differential equations.

This paper deals with the neutral delay integro-differential equation with constant coefficients

$$y'(t) = ay(t) + by(t-\tau) + cy'(t-\tau) + d\int_{t-\tau}^{t} y(s)ds, \quad t \geq 0, \tag{1}$$

$$y(t) = \phi(t) \quad \text{ for } -\tau \leq t \leq 0, \tag{2}$$

where a, b, c, d are real numbers, τ is a positive real number, and ϕ is a continuous initial function given on the interval $[-\tau, 0]$.

By a *solution* y to the neutral delay integro-differential equation (1), we mean a continuous real-valued function defined on $[-\tau, \infty)$, which is continuously differentiable on $[0, \infty)$ and satisfies (1).

Along with the neutral delay integro-differential equation (1), we associate the following equation

$$\lambda = a + be^{-\lambda\tau} + c\lambda e^{-\lambda\tau} + d\int_{0}^{\tau} e^{-\lambda s}ds, \tag{3}$$

which will be called the *characteristic equation* of (1). Equation (3) is obtained from (1) by looking for solutions of the form $y(t) = e^{\lambda t}$ for $t \geq -\tau$.

Philos and Purnaras [13-15] studied some results on the asymptotic properties and stability of solutions of linear autonomous delay and neutral delay differential equations (see also a similar study reference [16]).

Yeniçerioğlu and other authors [17] investigated the asymptotic behavior and stability of solutions of neutral-type functional differential equations. Yeniçerioğlu [18] obtained some results on the behavior of solutions of linear impulsive neutral-delay differential equations with constant coefficients. Yeniçerioğlu and Yalçınbaş [19] investigated the asymptotic behavior and stability of solutions of delay integro-differential equations. Later, Yeniçerioğlu [20] extended the asymptotic behavior and stability of solutions of neutral-delay integro-differential equations. The authors in [21] obtained some results of first-order delay integro-differential equations. In [22], Wu and Gan obtained the numerical and analytical stability of solutions of neutral delay integro-differential equations of the form (1). Our work in this paper is mainly inspired by the results in [22].

In this paper, our aim is to extend the results obtained in [21] to the neutral delay integro-differential equation. This article concerns the behavior of solutions to scalar first-order linear neutral delay integro-differential equations (1). Namely that, a fundamental criterion for the lower and upper bounds of the solutions of equation (1) is established. The results obtained in this article are obtained using two appropriately different real roots of the relevant characteristic equation (3). The techniques used to obtain the results are a combination of the methods used in [13-21].

2 Statement of results

Lemma 2.1. *Suppose that*

$$c \geq 0, \quad b < 0, \quad and \quad d < 0. \tag{4}$$

Let λ_0 *be a nonpositive real root of the characteristic equation* (3) *and let*

$$\beta(\lambda_0) = ((b + c\lambda_0)\tau - c)\, e^{-\lambda_0 \tau} + d \int_0^\tau s e^{-\lambda_0 s} ds.$$

Then

$$1 + \beta(\lambda_0) > 0$$

if (3) *has another real root less than* λ_0*, and*

$$1 + \beta(\lambda_0) < 0$$

if (3) *has another real root greater than* λ_0*.*

Proof. Let $F(\lambda)$ denote the characteristic function of (3), i.e.,

$$F(\lambda) = \lambda - a - be^{-\lambda\tau} - c\lambda e^{-\lambda\tau} - d \int_0^\tau e^{-\lambda s} ds, \quad \text{for } \lambda \in \mathbb{R}. \tag{5}$$

We obtain immediately

$$F'(\lambda) = 1 + b\tau e^{-\lambda\tau} + ce^{-\lambda\tau}(\tau\lambda - 1) + d \int_0^\tau s e^{-\lambda s} ds, \quad \text{for } \lambda \in \mathbb{R}. \tag{6}$$

Furthermore,

$$F''(\lambda) = -b\tau^2 e^{-\lambda\tau} + c\tau e^{-\lambda\tau}(2 - \tau\lambda) - d \int_0^\tau s^2 e^{-\lambda s} ds, \quad \text{for } \lambda \in \mathbb{R}.$$

So, considering (4), we conclude that

$$F''(\lambda) > 0, \quad \text{for all } \lambda \in (-\infty, 0]. \tag{7}$$

Now, assume that (3) has another real root λ_1 with $\lambda_1 < \lambda_0$ (respectively, $\lambda_1 > \lambda_0$). From the definition of the function F by (5) it follows that $F(\lambda_1) = F(\lambda_0) = 0$, and consequently Rolle's Theorem guarantees the existence of a point α with $\lambda_1 < \alpha < \lambda_0$ (resp., $\lambda_1 > \alpha > \lambda_0$) such that $F'(\alpha) = 0$. But, (7) implies that F' is positive on (α, ∞) (resp., F' is negative on $(-\infty, \alpha)$). Thus we must have $F'(\lambda_0) > 0$ (resp., $F'(\lambda_0) < 0$). The proof of Lemma 2.1 can be completed, by observing that

$$F'(\lambda_0) = 1 + \beta(\lambda_0).$$

□

Theorem 2.2. *Suppose that*

$$c \geq 0, \quad b < 0, \quad and \quad d < 0.$$

Let λ_0 *be a non-positive real root of the characteristic equation* (3) *with* $1 + \beta(\lambda_0) \neq 0$ *where* $\beta(\lambda_0)$ *is defined as in Lemma 2.1, and let*

$$\begin{aligned} L(\lambda_0;\phi) =& \phi(0) - c\phi(-\tau) + (b + c\lambda_0)\, e^{-\lambda_0\tau} \int_{-\tau}^{0} e^{-\lambda_0 s}\phi(s)ds \\ & + d \int_0^{\tau} e^{-\lambda_0 s} \left\{ \int_{-s}^{0} e^{-\lambda_0 u}\phi(u)du \right\} ds. \end{aligned}$$

Let also λ_1 *be a real root of* (3) *with* $\lambda_0 \neq \lambda_1$. *Then, the solution* y *of* (1)-(2) *satisfies*

$$D_1(\lambda_0,\lambda_1;\phi) \leq e^{-\lambda_1 t}\left[y(t) - \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}e^{\lambda_0 t}\right] \leq D_2(\lambda_0,\lambda_1;\phi) \tag{8}$$

for all $t \geq 0$*, where*

$$D_1(\lambda_0,\lambda_1;\phi) = \min_{-\tau \leq t \leq 0}\left\{e^{-\lambda_1 t}\left[\phi(t) - \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}e^{\lambda_0 t}\right]\right\}$$

and

$$D_2(\lambda_0,\lambda_1;\phi) = \max_{-\tau \leq t \leq 0}\left\{e^{-\lambda_1 t}\left[\phi(t) - \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}e^{\lambda_0 t}\right]\right\}.$$

Note: Since $\lambda_0 \neq \lambda_1$*, according to the Lemma 2.1,* $1 + \beta(\lambda_0) \neq 0$.

Proof. Let λ_0 be a real root of the characteristic equation (3) and let y be the solution of (1)-(2). Define

$$x(t) = e^{-\lambda_0 t}y(t) \quad \text{for} \;\; t \geq -\tau. \tag{9}$$

Then, for every $t \geq 0$, we have

$$\begin{aligned} & \left\{y'(t) - ay(t) - by(t-\tau) - cy'(t-\tau) - d\int_{t-\tau}^{t} y(s)ds\right\}e^{-\lambda_0 t} \\ &= x'(t) + \lambda_0 x(t) - ax(t) - bx(t-\tau)e^{-\lambda_0\tau} \\ &\quad - ce^{-\lambda_0\tau}\left(x'(t-\tau) + \lambda_0 x(t-\tau)\right) - d\int_0^{\tau} e^{-\lambda_0 s}x(t-s)ds = 0. \end{aligned}$$

Thus, the fact that y satisfies (1) for all $t \geq 0$ is equivalet to

$$\begin{aligned} x'(t) =&(a-\lambda_0)x(t) + (b + c\lambda_0)\, e^{-\lambda_0 \tau} x(t-\tau) \\ &+ ce^{-\lambda_0\tau} x'(t-\tau) + d\int_0^\tau e^{-\lambda_0 s} x(t-s)ds. \end{aligned} \tag{10}$$

Moreover, the initial condition (2) can be equivalently written

$$x(t) = e^{-\lambda_0 t}\phi(t) \quad \text{for} \quad -\tau \leq t \leq 0. \tag{11}$$

Furthermore, by using the fact that λ_0 is a root of (3) and taking into account $L(\lambda_0;\phi)$ and (11), we can verify that (10) is equivalent to

$$\begin{aligned} x(t) &= x(0) + (a-\lambda_0)\int_0^t x(s)ds + (b+c\lambda_0)\, e^{-\lambda_0\tau}\int_0^t x(s-\tau)ds \\ &\quad + ce^{-\lambda_0\tau}\left(x(t-\tau) - x(-\tau)\right) + d\int_0^\tau e^{-\lambda_0 s}\left\{\int_0^t x(u-s)du\right\}ds \\ &= \phi(0) + (a-\lambda_0)\int_0^t x(s)ds + (b+c\lambda_0)\, e^{-\lambda_0\tau}\int_{-\tau}^{t-\tau} x(s)ds \\ &\quad + ce^{-\lambda_0\tau}\left(x(t-\tau) - e^{\lambda_0\tau}\phi(-\tau)\right) + d\int_0^\tau e^{-\lambda_0 s}\left\{\int_{-s}^{t-s} x(u)du\right\}ds \\ &= \phi(0) + (a-\lambda_0)\int_0^t x(s)ds + (b+c\lambda_0)\, e^{-\lambda_0\tau}\int_{-\tau}^{0} e^{-\lambda_0 s}\phi(s)ds \\ &\quad + (b+c\lambda_0)\, e^{-\lambda_0\tau}\int_0^{t-\tau} x(s)ds + ce^{-\lambda_0\tau}x(t-\tau) - c\phi(-\tau) \\ &\quad + d\int_0^\tau e^{-\lambda_0 s}\left\{\int_{-s}^{0} e^{-\lambda_0 u}\phi(u)du + \int_0^{t-s} x(u)du\right\}ds \\ &= L(\lambda_0;\phi) + (a-\lambda_0)\int_0^t x(s)ds + (b+c\lambda_0)\, e^{-\lambda_0\tau}\int_0^{t-\tau} x(s)ds \\ &\quad + ce^{-\lambda_0\tau}x(t-\tau) + d\int_0^\tau e^{-\lambda_0 s}\left\{\int_0^{t-s} x(u)du\right\}ds \\ &= L(\lambda_0;\phi) - \left(be^{-\lambda_0\tau} + c\lambda_0 e^{-\lambda_0\tau} + d\int_0^\tau e^{-\lambda_0 s}ds\right)\int_0^t x(s)ds \\ &\quad + (b+c\lambda_0)\, e^{-\lambda_0\tau}\int_0^{t-\tau} x(s)ds + ce^{-\lambda_0\tau}x(t-\tau) \\ &\quad + d\int_0^\tau e^{-\lambda_0 s}\left\{\int_0^{t-s} x(u)du\right\}ds \end{aligned}$$

$$= L(\lambda_0;\phi) + ce^{-\lambda_0\tau}x(t-\tau) - (b+c\lambda_0)\,e^{-\lambda_0\tau}\int_{t-\tau}^{t} x(s)ds$$
$$- d\int_0^{\tau} e^{-\lambda_0 s}\left\{\int_{t-s}^{t} x(u)du\right\}ds.$$

Next, we define

$$z(t) = x(t) - \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)} \quad \text{for} \quad -\tau \le t.$$

Thus we can see that the last equation reduces to the following equivalent equation:

$$\begin{aligned} z(t) &= ce^{-\lambda_0\tau}z(t-\tau) - (b+c\lambda_0)\,e^{-\lambda_0\tau}\int_{t-\tau}^{t} z(s)ds \\ &\quad - d\int_0^{\tau} e^{-\lambda_0 s}\left\{\int_{t-s}^{t} z(u)du\right\}ds \quad \text{for} \;\; t \ge 0. \end{aligned} \tag{12}$$

Also, due to the x and z transformations, the following initial condition is obtained using the initial condition (2):

$$z(t) = e^{-\lambda_0 t}\phi(t) - \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)} \quad \text{for} \quad -\tau \le t \le 0. \tag{13}$$

Now, we define

$$h(t) = e^{(\lambda_0-\lambda_1)t}z(t) \quad \text{for} \quad -\tau \le t. \tag{14}$$

Because of the x and z transformations, the following expression is obtained for the function h:

$$h(t) = e^{-\lambda_1 t}\left[y(t) - e^{\lambda_0 t}\frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}\right] \quad \text{for} \quad -\tau \le t. \tag{15}$$

Moreover, by using the h function, (12) can be written as equivalent

$$\begin{aligned} h(t) &= ce^{-\lambda_1\tau}h(t-\tau) - (b+c\lambda_0)\,e^{-\lambda_0\tau}\int_{-\tau}^{0} e^{(\lambda_1-\lambda_0)s}h(s+t)ds \\ &\quad - d\int_0^{\tau} e^{-\lambda_0 s}\left\{\int_{-s}^{0} e^{(\lambda_1-\lambda_0)u}h(u+t)du\right\}ds \quad \text{for} \;\; t \ge 0. \end{aligned} \tag{16}$$

Also, (13) becomes

$$h(t) = e^{-\lambda_1 t}\left[\phi(t) - e^{\lambda_0 t}\frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}\right] \quad \text{for} \quad -\tau \le t \le 0. \tag{17}$$

As solution y satisfies the initial condition (2), we can use the function h as well as the definitions of $D_1(\lambda_0, \lambda_1; \phi)$ and $D_2(\lambda_0, \lambda_1; \phi)$, to see that

$$D_1(\lambda_0, \lambda_1; \phi) = \min_{-\tau \le t \le 0} h(t) \text{ and } D_2(\lambda_0, \lambda_1; \phi) = \max_{-\tau \le t \le 0} h(t). \tag{18}$$

Considering (15) and (18), the double inequality (8) can be written equivalent to

$$\min_{-\tau \le t \le 0} h(t) \le h(t) \le \max_{-\tau \le t \le 0} h(t) \quad \text{for all } t \ge 0. \tag{19}$$

We need to prove that the inequality (19) holds. We will use the fact that h satisfies (16) for all $t \ge 0$ to show that (19) is valid. We just need to prove the following inequality

$$\min_{-\tau \le t \le 0} h(t) \le h(t) \quad \text{for all } t \ge 0. \tag{20}$$

The proof of the inequality

$$h(t) \le \max_{-\tau \le t \le 0} h(t) \quad \text{for all } t \ge 0$$

can be obtained in a similar manner and is therefore omitted. We will obtain (20) for the rest of the proof. To do this, we are considering an arbitrary real number A with $A < \min\limits_{-\tau \le t \le 0} h(t)$, i.e., with

$$A < h(t) \quad \text{for } -\tau \le t \le 0. \tag{21}$$

We will show that

$$A < h(t) \quad \text{for all } t \ge 0. \tag{22}$$

For this purpose, suppose that (22) does not hold. Then, due to (21), there is a point $t_0 > 0$ such that

$$A < h(t) \quad \text{for } -\tau \le t < t_0 \quad \text{and } h(t_0) = A.$$

Thus, by using (3), from (16) we obtain

$$
\begin{aligned}
A =& h(t_0) = ce^{-\lambda_1\tau}h(t_0-\tau) - (b+c\lambda_0)\,e^{-\lambda_0\tau}\int_{-\tau}^{0} e^{(\lambda_1-\lambda_0)s}h(s+t_0)ds \\
& - d\int_0^{\tau} e^{-\lambda_0 s}\left\{\int_{-s}^{0} e^{(\lambda_1-\lambda_0)u}h(u+t_0)du\right\}ds \\
& > A\left(ce^{-\lambda_1\tau} - (b+c\lambda_0)\,e^{-\lambda_0\tau}\int_{-\tau}^{0} e^{(\lambda_1-\lambda_0)s}ds\right. \\
& \qquad \left. -d\int_0^{\tau} e^{-\lambda_0 s}\left\{\int_{-s}^{0} e^{(\lambda_1-\lambda_0)u}du\right\}ds\right) \\
& = A\left(ce^{-\lambda_1\tau} - (b+c\lambda_0)\,e^{-\lambda_0\tau}\left(\frac{1}{\lambda_1-\lambda_0}\right)\left[1-e^{-(\lambda_1-\lambda_0)\tau}\right]\right. \\
& \qquad \left. -\left(\frac{d}{\lambda_1-\lambda_0}\right)\int_0^{\tau} e^{-\lambda_0 s}\left[1-e^{-(\lambda_1-\lambda_0)s}\right]ds\right) \\
& = \frac{A}{\lambda_1-\lambda_0}\left((\lambda_1-\lambda_0)ce^{-\lambda_1\tau} - (b+c\lambda_0)\left[e^{-\lambda_0\tau}-e^{-\lambda_1\tau}\right]\right. \\
& \qquad \left. -d\int_0^{\tau}\left[e^{-\lambda_0 s}-e^{-\lambda_1 s}\right]ds\right) \\
& = \frac{A}{\lambda_1-\lambda_0}\left(\lambda_1 ce^{-\lambda_1\tau} + be^{-\lambda_1\tau} + d\int_0^{\tau} e^{-\lambda_1 s}ds\right. \\
& \qquad \left. -\lambda_0 ce^{-\lambda_0\tau} - be^{-\lambda_0\tau} - d\int_0^{\tau} e^{-\lambda_0 s}ds\right) \\
& = \frac{A}{\lambda_1-\lambda_0}\left((\lambda_1-a)-(\lambda_0-a)\right) = A.
\end{aligned}
$$

Thus, we arrive at a contradiction and therefore (22) is correct. Since (22) is satisfied for any real number A with $A < \min\limits_{-\tau\le t\le 0} h(t)$, it follows that (20) is always valid. The proof of the Theorem 2.2 is complete.

□

We can use Theorem 2.2 to derive the following corollary.

Corollary 2.3. Assume that the conditions in Theorem 2.2 are provided. Then, for the solutions of (1)-(2), double inequality of the Theorem 2.2 can be written as follows:

$$
D_1\left(\lambda_0,\lambda_1;\phi\right)e^{(\lambda_1-\lambda_0)t} \le y(t)e^{-\lambda_0 t} - \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)} \le D_2\left(\lambda_0,\lambda_1;\phi\right)e^{(\lambda_1-\lambda_0)t}
$$

and so

$$\lim_{t\to\infty}\left(y(t)e^{-\lambda_0 t}\right)=\frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)},$$

provided that $\lambda_1 < \lambda_0$. Moreover, we immediately observe that this double inequality can equivalently be written in the form

$$\begin{aligned} D_1\left(\lambda_0,\lambda_1;\phi\right)e^{\lambda_1 t}+\frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}e^{\lambda_0 t} &\le y(t) \\ &\le D_2\left(\lambda_0,\lambda_1;\phi\right)e^{\lambda_1 t}+\frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}e^{\lambda_0 t} \end{aligned}$$

for all $t \geq 0$.

Example 2.4. Consider the differential equation (1) with

$$y'(t)=\frac{1}{2}y(t)-\frac{1}{4}y(t-1)+\frac{1}{e}y'(t-1)-\frac{1}{4}\int_{t-1}^{t}y(s)ds,\quad t\geq 0 \tag{23}$$

$$y(t)=\phi(t)\quad \text{for } -1\leq t\leq 0, \tag{24}$$

where ϕ is an arbitrary continuous function on the interval $[-1,0]$. In this example, we apply the characteristic equation (3). That is, the characteristic equation (3) is

$$\lambda=\frac{1}{2}-\frac{1}{4}e^{-\lambda}+\frac{1}{e}\lambda e^{-\lambda}-\frac{1}{4}\lambda^{-1}\left(1-e^{-\lambda}\right). \tag{25}$$

We see that $\lambda = 0$ and $\lambda \approx -0.4$ are real roots of (25). Let's choose $\lambda_0 = 0$ and $\lambda_1 = -0.4$. In this case, by applying Theorem 2.2, we obtain the following result: For any continuous real-valued function ϕ on $[-1, 0]$, the solution y of (1)–(2) satisfies

$$D_1\left(\lambda_0,\lambda_1;\phi\right)\leq e^{0.4t}\left[y(t)-\frac{L(0;\phi)}{1+\beta(0)}\right]\leq D_2\left(\lambda_0,\lambda_1;\phi\right)$$

for all $t \geq 0$, where

$$D_1\left(\lambda_0,\lambda_1;\phi\right)=\min_{-1\leq t\leq 0}\left\{e^{0.4t}\left[\phi(t)-\frac{L(0;\phi)}{1+\beta(0)}\right]\right\},$$

$$D_2\left(\lambda_0,\lambda_1;\phi\right)=\max_{-1\leq t\leq 0}\left\{e^{0.4t}\left[\phi(t)-\frac{L(0;\phi)}{1+\beta(0)}\right]\right\},$$

$$L(0;\phi)=\phi(0)-\frac{1}{e}\phi(-1)-\frac{1}{4}\int_{-1}^{0}\phi(s)ds-\frac{1}{4}\int_{0}^{1}\left\{\int_{-s}^{0}\phi(u)du\right\}ds$$

and

$$\beta(0) = -\left(\frac{1}{4} + \frac{1}{e}\right) - \frac{1}{4}\int_0^1 s ds \approx -0.743.$$

Furthermore, by applying Corollary 2.3, we have

$$D_1(\lambda_0, \lambda_1; \phi) e^{-0.4t} + \frac{L(0;\phi)}{0.257} \leq y(t) \leq D_2(\lambda_0, \lambda_1; \phi) e^{-0.4t} + \frac{L(0;\phi)}{0.257}$$

for all $t \geq 0$, and also, since $\lambda_1 < \lambda_0$, we get

$$\lim_{t\to\infty} y(t) = \frac{L(0;\phi)}{0.257}.$$

Bibliography

[1] J.K. Hale and S.M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Springer, Berlin, Heidelberg, New York, 1993.

[2] V. Kolmanovski and A. Myshkis, *Applied Theory of Functional Differential Equations*, Kluver Academic, Dordrecht, 1992.

[3] V. Lakshmikantham, L. Wen and B. Zhang, *Theory of Differential Equations with Unbounded Delay*, Kluwer Academic Publishers, London, 1994.

[4] R. Bellman and K.L. Cooke, *Differential-Difference Equations*, Academic Press, New York, 1963.

[5] T.A. Burton, *Volterra Integral and Differential Equations*, Academic Press, New York, 1983.

[6] C. Corduneanu, *Integral Equations and Applications*, Cambridge University Press, New York, 1991.

[7] R.P. Agarwal and D.O'Regan, *Integral and Integro-differential Equations: Theory, Methods and Applications*, Gordon and Breach Science Publishers, 2000.

[8] V. Volterra, Sur la théorie mathématique des phénoménes héréditaires, *J. Math. Pures Appl.* **7:9** (1928), 249–298.

[9] Y. Kuang, *Delay Differential Equations with Applications in Population Dynamics*, Academic Press, San Diego, 1993.

[10] G.A. Bocharov and F.A. Rihan, Numerical modelling in biosciencess with delay differential equations, *J. Comput. and Appl. Math.* **125** (2000), 183–199.

[11] H. Brunner, The numerical solution of neutral Volterra integro-differential equations with delay arguments, *Ann. Numer. Math.* **1** (1994), 309–322.

[12] A. Jerri, *Introduction to Integral Equations with Applications*, Wiley, New York, 1999.

[13] Ch.G. Philos and I.K. Purnaras, Asymptotic properties, nonoscillation, and stability for scalar first order linear autonomous neutral delay differential equations, *Elect. J. Diff. Equ.* **2004:03** (2004), 1–17.

[14] Ch.G. Philos and I.K. Purnaras, On the behavior of the solutions for certain first order linear autonomous functional differential equations, *Rocky Mountain J. Math.* **36** (2006), 1999–2019.

[15] Ch.G. Philos and I. K. Purnaras, A result on the behavior of the solutions for scalar first order linear autonomous neutral delay differential equations, *Math. Proc. the Cambridge Phil. Soc.* **140** (2006), 349–358.

[16] I.G.E. Kordonis, N.T. Niyianni and Ch.G. Philos, On the behavior of the solutions of scalar first order linear autonomous neutral delay differential equations, *Arch. Math.* **71** (1998), 454–464.

[17] A.F. Yeniçerioğlu, C. Yazıcı and V. Yazıcı, Stability behaviour in functional differential equations of the neutral type, *Univer. J. Math. and Appl.* **4:1** (2021), 33–40.

[18] A.F. Yeniçerioğlu, Results on the behavior of the solutions for linear impulsive neutral delay differential equations with constant coefficients, *Commun. in Advan. Mathematical Sci.* **IV:4** (2021), 180–189.

[19] A.F. Yeniçerioğlu and S. Yalçınbaş, On the stability of delay integro-differential equations, *Math. and Comput. Appl.* **12:1** (2007), 51–58.

[20] A.F. Yeniçerioğlu, Stability properties of neutral delay integro-differential equation, *Selçuk J. Appl. Math.* **10:2** (2009), 15–26.

[21] A.F. Yeniçerioğlu and C. Yazıcı, Stability in first order delay integro-differential equations, *J. BAUN Inst. Sci. Technol.* **22:2** (2020), 660–668.

[22] S. Wu and S. Gan, Analytical and numerical stability of neutral delay integro-differential equations and neutral delay partial differential equations, *Comput. and Math. with Appl.* **55** (2008), 2426–2443.

Received October 17, 2025; revised December 15, 2025; accepted December 19, 2025.

Author information

Ali Fuat Yeniçerioğlu, Department of Mathematics, The Faculty of Education, Kocaeli University, 41001, Kocaeli, Türkiye.
E-mail: fuatyenicerioglu@kocaeli.edu.tr