e-Journal of Analysis and Applied Mathematics **2025** (2025), 57–78
DOI 10.62780/ejaam/2025-006 EJAAM 2025

Results on common fixed points in S_b-metric spaces

Pramodkumar Sohanlal Sharma and Chintaman Tukaram Aage

Abstract. In this paper, we establish new common fixed point theorems for pairs of weakly compatible mappings within the framework of S_b-metric spaces. By introducing generalized contractive conditions, we demonstrate the existence and uniqueness of common fixed points for such self-mappings. Our results extend and generalize several well known fixed point theorems in the existing literature. An example is also provided to support and clarify the main result.

Keywords. S-metric space, b-metric space, S_b-metric space.

2020 Mathematics Subject Classification. 54H25, 47H10.

1 Introduction

Banach [5] proved the fundamental fixed point theorem, known as the Banach Contraction Principle. It has wide applications in analysis, differential and integral equations, optimization, and other areas of mathematics. Over the years, numerous researchers have extended and generalized this theorem in various directions. The concept of a b-metric space was proposed by Czerwik [6]. Many researchers have contributed to the study of fixed point theorems in b-metric spaces. Boriceanu et al. [7] extended the study of fractal operator theory for multivalued operators on complete b-metric spaces. Aydi et al. [4] established a common fixed point theorem for single-valued and multivalued mappings satisfying a weak φ-contraction in b-metric spaces. Shatanawi et al. [18] considered the setting of b-metric spaces to establish results on the common fixed points of two mappings, using a contraction condition defined by a comparison function. Abbas et al. [2] developed common fixed point results for generalized b-order contractive mappings and applied them to an integral equation. Zada et al. [20] established some fixed point results for rational type contractive mappings in b-metric spaces, generalizing and extending existing results. Iqbal et al. [8] introduced a generalized multivalued (α, L)-almost contraction in b-metric spaces and proved existence and uniqueness of fixed points, extending earlier results in the literature. Iqbal et al. [9] introduced a class of generalized (ψ, α, β)-weak contractions and proved several fixed point theorems in b-metric spaces. Latif et al. [11] proved several fixed point results for α-admissible mappings satisfying a Suzuki type contractive condition in the framework of b-metric spaces.

Sedghi et al. [15] introduced S-metric spaces, explored their properties, and established several common fixed point theorems for self-mappings on complete S-metric spaces. Several researchers have studied S-metric spaces and extended numerous results concerning the existence of fixed points. Singh and Hooda [16] obtained coupled fixed point results in S-metric spaces. Mlaiki [12] introduced the complex valued S-metric space and proved that two self mappings in this space possess a unique common fixed point. Mlaiki [13] introduced α–ψ-contractive mappings in S-metric spaces, and the existence of fixed points for such mappings is obtained under certain conditions. Prudhvi [14] established two fixed point theorems in S-metric spaces, and the results presented here extend and enhance known findings.

Recently, Souayah and Mlaiki [19] introduced the S_b-metric space as a generalization of both S-metric and b-metric spaces, and established several fixed point theorems for various contractive mappings in complete S_b-metric spaces. In this paper, we establish some common fixed point theorems for pairs of weakly compatible mappings within the setting of S_b-metric spaces. By employing generalized contractive conditions, we prove the existence and uniqueness of common fixed points for these self mappings.

2 Preliminaries

Czerwik [6] introduced the concept of a b-metric spacs and it is defined as follows:

Definition 2.1. [6] Let X be a non-empty set and $d : X \times X \to [0, \infty)$ be a mapping satisfying following properties:

(i) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) there exists a real number $s \geq 1$ such that

$$d(x, y) \leq s[d(x, z) + d(z, y)],$$

for all $x, y, z \in X$.

Then d is called a b-metric on X and the ordered pair (X, d) is called b-metric space with coefficient s.

Sedghi et al. [15] introduced the notion of an S-metric space, which is defined as follows:

Definition 2.2. [15] Let X be a non-empty set and $S : X \times X \times X \to [0, \infty)$ be a mapping satisfying following properties:

(i) $S(x, y, z) = 0$ if and only if $x = y = z$;

(ii) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a), \ \forall\, a, x, y, z \in X$ (rectangle inequality).

Then (X, S) is called a S-metric space.

Souayah et al. [19] integrated the ideas of b-metric spaces and S-metric spaces to introduce a new category of metric spaces, referred to as S_b-metric spaces, which is defined as follows:

Definition 2.3. [19] Let X be a non-empty set and $s \geq 1$ be a given real number. Then a mapping $S_b : X \times X \times X \to [0, \infty)$ is said to be S_b-metric on X, if following properties are satisfied:

(i) $S_b(x, y, z) = 0$ if and only if $x = y = z$;

(ii) $S_b(x, x, y) = S_b(y, y, x)$;

(iii) $S_b(x, y, z) \leq s\big[\, S_b(x, x, a) + S_b(y, y, a) + S_b(z, z, a)\big], \ \forall\, x, y, z, a \in X.$

Then (X, S_b) is called a S_b-metric space.

Example 2.4. [19] Let X be a set with card$(X) \geq 5$. Assume $X = X_1 \cup X_2$ is a partition of X such that card$(X_1) \geq 4$. Let $s \geq 1$. Then

$$S_b(x, y, z) = \begin{cases} 0 & \text{if } x = y = z = 0, \\ 3s & \text{if } (x, y, z) \in X_1^3, \\ 1 & \text{if } (x, y, z) \notin X_1^3, \end{cases}$$

for all $x, y, z \in X$, is a S_b-metric on X with coefficient $s \geq 1$.

Example 2.5. [17] Let X be the set of real numbers and define $S_b : X \times X \times X \to \mathbb{R}^+ \cup \{0\}$ by

$$S_b(x, y, z) = |x - y| + |y - z| + |z - x|.$$

Then S_b is an S_b-metric on X with coefficient $s \geq 1$.

Example 2.6. [17] Let $X = \{a, b, c\}$ be a set. Define the mapping $S_b : X \times X \times X \to [0, \infty)$ as follows:

$$S_b(x, y, z) = \begin{cases} 0 & \text{if } x = y = z, \\ 1 & \text{if exactly two of } x, y, z \text{ are equal}, \\ \dfrac{1}{4} & \text{if all three elements } x, y, z \text{ are distinct}. \end{cases}$$

Then S_b is S_b-metric on X with coefficient $s \geq 1$.

Definition 2.7. [19] Let (X, S_b) be an S_b-metric space and $\{x_n\}$ be a sequence in X. Then

(i) A sequence $\{x_n\}$ is called convergent if and only if there exists $z \in X$ such that $S_b(x_n, x_n, z) \to 0$ as $n \to \infty$. In this case we write

$$\lim_{n\to\infty} x_n = z.$$

(ii) A sequence $\{x_n\}$ is called a Cauchy sequence if and only if $S_b(x_n, x_n, x_m) \to 0$ as $n, m \to \infty$.

(iii) (X, S_b) is said to be a complete S_b-metric space if every Cauchy sequence $\{x_n\}$ converges to a point $x \in X$ such that

$$\lim_{n,m\to\infty} S_b(x_n, x_n, x_m) = \lim_{n,m\to\infty} S_b(x_n, x_n, x) = S_b(x, x, x).$$

Definition 2.8. [1] Let f and g be self maps of a set X. If $w = fx = gx$, for some $x \in X$, then x is called coincidence point of f and g, and w is called a point of coincidence of f and g.

Definition 2.9. [10] Let f and g be self maps of a set X. Then f and g are said to be weakly compatible, if they commute at any coincidence point. That is $fgx = gfx$, for $x \in X$.

Proposition 2.10. [3] *Let f and g be weakly compatible self maps of a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is unique common fixed point of f and g.*

3 Main results

Theorem 3.1. *Let (X, S_b) be an S_b-metric space, and let $f, g : X \to X$ be two mappings satisfying*

$$\begin{aligned} S_b(fx, fy, fz) \leq \frac{1}{4}\left(S_b(gx, gx, fx) + S_b(gy, gy, fy) + S_b(gz, gz, fz)\right) \\ - \phi\left(S_b(gx, gx, fx), S_b(gy, gy, fy), S_b(gz, gz, fz)\right), \end{aligned}$$

for all $x, y, z \in X$, where $\phi : [0, \infty)^3 \to [0, \infty)$ is a continuous function such that $\phi(x, y, z) = 0$ if and only if $x = y = z$. Suppose further that s is a real number satisfying $1 \leq s < \frac{3}{2}$, and that the following conditions hold:

(i) $f(X) \subseteq g(X)$;

(ii) $g(X)$ *is complete.*

Then f and g have a unique coincidence point in X. If f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof. Choose an arbitrary element $x_0 \in X$. Because $f(X) \subseteq g(X)$, there exists an element $x_1 \in X$ Satisfying $f(x_0) = g(x_1)$. Repeating this construction, for each $x_n \in X$ one can select an element $x_{n+1} \in X$ such that $f(x_n) = g(x_{n+1})$. For sequence $\{gx_n\}$, consider

$$
\begin{aligned}
&S_b(gx_n, gx_n, gx_{n+1}) = S_b(fx_{n-1}, f_{n-1}, fx_n) \\
&\leq \frac{1}{4}\left(S_b(gx_{n-1}, gx_{n-1}, fx_{n-1}) + S_b(gx_{n-1}, gx_{n-1}, fx_{n-1}) + S_b(gx_n, gx_n, fx_n)\right) \\
&\quad - \phi\left(S_b(gx_{n-1}, gx_{n-1}, fx_{n-1}), S_b(gx_{n-1}, gx_{n-1}, fx_{n-1}), S_b(gx_n, gx_n, fx_n)\right) \\
&= \frac{1}{4}\left(2S_b(gx_{n-1}, gx_{n-1}, gx_n) + S_b(gx_n, gx_n, gx_{n+1})\right) \\
&\quad - \phi\left(S_b(gx_{n-1}, gx_{n-1}, gx_n), S_b(gx_{n-1}, gx_{n-1}, gx_n), S_b(gx_n, gx_n, gx_{n+1})\right).
\end{aligned}
$$

Since $\phi(t_1, t_2, t_3) \geq 0$, for all $t_1, t_2, t_3 \geq 0$, it follows that

$$S_b(gx_n, gx_n, gx_{n+1}) \leq \frac{1}{4}\left(2S_b(gx_{n-1}, gx_{n-1}, gx_n) + S_b(gx_n, gx_n, gx_{n+1})\right).$$

It implies that

$$S_b(gx_n, gx_n, gx_{n+1}) \leq \frac{2}{3} S_b(gx_{n-1}, gx_{n-1}, gx_n).$$

Letting $\alpha = \dfrac{2}{3} < 1$, we obtain

$$S_b(gx_n, gx_n, gx_{n+1}) \leq \alpha S_b(gx_{n-1}, gx_{n-1}, gx_n) \tag{1}$$

Similarly, we can show that

$$S_b(gx_{n-1}, gx_{n-1}, gx_n) \leq \alpha S_b(gx_{n-2}, gx_{n-2}, gx_{n-1}). \tag{2}$$

Using inequalities (1) and (2), we get

$$S_b(gx_n, gx_n, gx_{n+1}) \leq \alpha^2 S_b(gx_{n-2}, gx_{n-2}, gx_{n-1}).$$

Continuing this process, we establish

$$S_b(gx_n, gx_n, gx_{n+1}) \leq \alpha^n S_b(gx_0, gx_0, gx_1).$$

By setting $S_n = S_b(gx_n, gx_n, gx_{n+1})$, we obtain

$$S_n \leq \alpha^n S_0, \qquad n \in \mathbb{N}. \tag{3}$$

We now show that the sequence $\{gx_n\}$ is a Cauchy sequence in $g(X)$. Let $m > n > n_0$, for some $n_0 \in \mathbb{N}$. Then by repeated use of (iii) in the Definition 2.3, we obtain

$$\begin{aligned}
&S_b(gx_n, gx_n, gx_m)\\
&\leq s\left(S_b(gx_n, gx_n, gx_{n+1}) + S_b(gx_n, gx_n, gx_{n+1}) + S_b(gx_m, gx_m, gx_{n+1})\right)\\
&= 2sS_b(gx_n, gx_n, gx_{n+1}) + sS_b(gx_{n+1}, gx_{n+1}, gx_m)\\
&\leq 2sS_b(gx_n, gx_n, gx_{n+1}) + s\left\{s\left(2S_b(gx_{n+1}, gx_{n+1}, gx_{n+2}) + S_b(gx_m, gx_m, gx_{n+2})\right)\right\}\\
&= 2sS_b(gx_n, gx_n, gx_{n+1}) + 2s^2 S_b(gx_{n+1}, gx_{n+1}, gx_{n+2}) + s^2 S_b(gx_m, gx_m, gx_{n+1})\\
&= 2sS_n + 2s^2 S_{n+1} + s^2 S_b(gx_{n+2}, gx_{n+2}, gx_m).
\end{aligned}$$

Proceeding inductively, we obtain

$$\begin{aligned}
S_b(gx_n, gx_n, gx_m) &\leq 2sS_n + 2s^2 S_{n+1} + 2s^3 S_{n+2} + \cdots + 2s^{m-n-1} S_{m-1}\\
&\leq 2sS_n + 2s^2 S_{n+1} + 2s^3 S_{n+2} + \cdots .
\end{aligned}$$

Inequality (3) implies that

$$\begin{aligned}
S_b(gx_n, gx_n, gx_m) &\leq 2s\alpha^n S_0 + 2s^2\alpha^{n+1} S_0 + 2s^3\alpha^{n+2} S_0 + \cdots\\
&= 2s\alpha^n \left(1 + (\alpha s) + (\alpha s)^2 + (\alpha s)^3 + \cdots\right) S_0\\
&= 2s\alpha^n \left(\frac{1}{1-\alpha s}\right) S_0.
\end{aligned}$$

Since $\alpha = \dfrac{2}{3} < 1$, it follows that $\alpha^n \to 0$ as $n \to \infty$. Hence, letting $n \to \infty$, we obtain

$$\lim_{n\to\infty} S_b(gx_n, gx_n, gx_m) = 0.$$

Consequently, the sequence $\{gx_n\}$ is a Cauchy sequence in $g(X)$. As $g(X)$ is complete, there exists an element $q \in g(X)$ for which $gx_n \to q$. That is

$$\lim_{n\to\infty} gx_n = q = \lim_{n\to\infty} fx_{n-1}.$$

Since $q \in g(X)$, there exists $p \in X$ such that $g(p) = q$. We now aim to prove that

$q = f(p)$. For this purpose, consider

$$\begin{aligned} S_b(gx_{n+1}, gx_{n+1}, fp) &= S_b(fx_n, fx_n, fp) \\ &\leq \frac{1}{4}\left(S_b(gx_n, gx_n, fx_n) + S_b(gx_n, gx_n, fx_n) + S_b(gp, gp, fp)\right) \\ &\quad - \phi\left(S_b(gx_n, gx_n, fx_n), S_b(gx_n, gx_n, fx_n), S_b(gp, gp, fp)\right) \\ &= \frac{1}{4}\left(2S_b(gx_n, gx_n, gx_{n+1}) + S_b(gp, gp, fp)\right) \\ &\quad - \phi\left(S_b(gx_n, gx_n, fx_n), S_b(gx_n, gx_n, fx_n), S_b(gp, gp, fp)\right). \end{aligned}$$

Since $\phi(t_1, t_2, t_3) \geq 0$, for all $t_1, t_2, t_3 \geq 0$, it follows that

$$S_b(gx_{n+1}, gx_{n+1}, fp) \leq \frac{1}{4}\left(2S_b(gx_n, gx_n, gx_{n+1}) + S_b(gp, gp, fp)\right).$$

Letting $n \to \infty$, we obtain

$$S_b(q, q, fp) \leq \frac{1}{4}\left(2S_b(q, q, q) + S_b(q, q, fp)\right).$$

It implies that

$$\frac{3}{4}S_b(q, q, fp) \leq 0.$$

We must have $S_b(q, q, fp) \leq 0$. However, we also have $S_b(q, q, fp) \geq 0$. Therefore $S_b(q, q, fp) = 0$, which implies that $fp = q = gp$. This shows that p is a coincidence point of f and g. Now, we claim that f and g have a unique coincidence point. Suppose, for the sake of contradiction, that there is an another coincidence point, say $r \neq p$, of f and g. Consider

$$\begin{aligned} S_b(gr, gr, gp) &= S_b(fr, fr, fp) \\ &\leq \frac{1}{4}\left(S_b(gr, gr, fr) + S_b(gr, gr, fr) + S_b(gp, gp, fp)\right) \\ &\quad - \phi\left(S_b(gr, gr, fr), S_b(gr, gr, fr), S_b(gp, gp, fp)\right) \\ &= \frac{1}{4}\left(S_b(gr, gr, gr) + S_b(gr, gr, gr) + S_b(gp, gp, gp)\right) \\ &\quad - \phi\left(S_b(gr, gr, gr), S_b(gr, gr, gr), S_b(gp, gp, gp)\right) \\ &= \frac{1}{4}(0) - \phi(0, 0, 0) \\ &= 0. \end{aligned}$$

Hence, $S_b(gr, gr, gp) \leq 0$. However, $S_b(gr, gr, gp) \geq 0$. Combining these inequalities, we obtain $S_b(gr, gr, gp) = 0$, which implies that $gr = gp$. Thus, the mappings f and g have a unique coincidence point. Furthermore, if f and g are weakly compatible, then Proposition 2.10 ensures that f and g have a unique common fixed point in X. □

Theorem 3.2. *Let (X, S_b) be an S_b-metric space, and let $f, g : X \to X$ be two mappings. Suppose that there exists a real number α with $0 \leq \alpha < \frac{1}{s}$, $s \geq 1$ is a given real number such that for all $x, y, z \in X$,*

$$S_b(fx, fy, fz) \leq \alpha \max\{S_b(gx, gx, fx), S_b(gy, gy, fy), S_b(gz, gz, fz)\} - \phi\left(S_b(gx, gx, fx), S_b(gy, gy, fy), S_b(gz, gz, fz)\right),$$

where $\phi : [0, \infty)^3 \to [0, \infty)$ is a continuous function such that $\phi(x, y, z) = 0$ if and only if $x = y = z$. If

(i) $f(X) \subseteq g(X)$;

(ii) $g(X)$ *is complete.*

Then f and g have a unique coincidence point in X. If f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be chosen arbitrarily. Because $f(X) \subseteq g(X)$, there exists an element $x_1 \in X$ satisfying $f(x_0) = g(x_1)$. Repeating this process, for each $x_n \in X$, we can find corresponding $x_{n+1} \in X$ such that $f(x_n) = g(x_{n+1})$. For sequence $\{gx_n\}$, consider

$$\begin{aligned} &S_b(gx_n, gx_n, gx_{n+1}) = S_b(fx_{n-1}, f_{n-1}, fx_n) \\ &\leq \alpha \max\{S_b(gx_{n-1}, gx_{n-1}, fx_{n-1}), S_b(gx_{n-1}, gx_{n-1}, fx_{n-1}), S_b(gx_n, gx_n, fx_n)\} \\ &\quad - \phi\left(S_b(gx_{n-1}, gx_{n-1}, fx_{n-1}), S_b(gx_{n-1}, gx_{n-1}, fx_{n-1}), S_b(gx_n, gx_n, fx_n)\right) \\ &= \alpha \max\{S_b(gx_{n-1}, gx_{n-1}, gx_n), S_b(gx_n, gx_n, gx_{n+1})\} \\ &\quad - \phi\left(S_b(gx_{n-1}, gx_{n-1}, gx_n), S_b(gx_{n-1}, gx_{n-1}, gx_n), S_b(gx_n, gx_n, gx_{n+1})\right). \end{aligned}$$

Since $\phi(t_1, t_2, t_3) \geq 0$, for all $t_1, t_2, t_3 \geq 0$, it follows that

$$S_b(gx_n, gx_n, gx_{n+1}) \leq \alpha \max\{S_b(gx_{n-1}, gx_{n-1}, gx_n),\ S_b(gx_n, gx_n, gx_{n+1})\}. \quad (4)$$

If

$$\max\{S_b(gx_{n-1}, gx_{n-1}, gx_n),\ S_b(gx_n, gx_n, gx_{n+1})\} = S_b(gx_n, gx_n, gx_{n+1}),$$

then inequality (4) implies that

$$S_b(gx_n, gx_n, gx_{n+1}) \leq \alpha S_b(gx_n, gx_n, gx_{n+1}).$$

This is a contradiction since $\alpha < 1$. Therefore, we must have

$$\max\{S_b(gx_{n-1}, gx_{n-1}, gx_n),\ S_b(gx_n, gx_n, gx_{n+1})\} = S_b(gx_{n-1}, gx_{n-1}, gx_n).$$

Hence, inequality (4) implies that

$$S_b(gx_n, gx_n, gx_{n+1}) \le \alpha S_b(gx_{n-1}, gx_{n-1}, gx_n). \tag{5}$$

Similarly,

$$S_b(gx_{n-1}, gx_{n-1}, gx_n) \le \alpha S_b(gx_{n-2}, gx_{n-2}, gx_{n-1}). \tag{6}$$

Using inequalities (5) and (6), we obtain

$$S_b(gx_n, gx_n, gx_{n+1}) \le \alpha^2 S_b(gx_{n-2}, gx_{n-2}, gx_{n-1}).$$

By repeating this process, we get

$$S_b(gx_n, gx_n, gx_{n+1}) \le \alpha^n S_b(gx_0, gx_0, gx_1).$$

Setting $S_n = S_b(gx_n, gx_n, gx_{n+1})$, we obtain

$$S_n \le \alpha^n S_0, \qquad n \in \mathbb{N}. \tag{7}$$

We now show that the sequence $\{gx_n\}$ is a Cauchy sequence in $g(X)$. Let $m > n > n_0$, for some $n_0 \in \mathbb{N}$. Then by repeated use of (iii) in Definition 2.3, we obtain

$$\begin{aligned}
&S_b(gx_n, gx_n, gx_m)\\
&\le s\left(S_b(gx_n, gx_n, gx_{n+1}) + S_b(gx_n, gx_n, gx_{n+1}) + S_b(gx_m, gx_m, gx_{n+1})\right)\\
&= 2sS_b(gx_n, gx_n, gx_{n+1}) + sS_b(gx_{n+1}, gx_{n+1}, gx_m)\\
&\le 2sS_b(gx_n, gx_n, gx_{n+1}) + s\left\{s\left(2S_b(gx_{n+1}, gx_{n+1}, gx_{n+2}) + S_b(gx_m, gx_m, gx_{n+2})\right)\right\}\\
&= 2sS_b(gx_n, gx_n, gx_{n+1}) + 2s^2 S_b(gx_{n+1}, gx_{n+1}, gx_{n+2}) + s^2 S_b(gx_m, gx_m, gx_{n+1})\\
&= 2sS_n + 2s^2 S_{n+1} + s^2 S_b(gx_{n+2}, gx_{n+2}, gx_m).
\end{aligned}$$

Using the same reasoning, it follows that

$$\begin{aligned}
S_b(gx_n, gx_n, gx_m) &\le 2sS_n + 2s^2 S_{n+1} + 2s^3 S_{n+2} + \cdots + 2s^{m-n-1} S_{m-1}\\
&\le 2sS_n + 2s^2 S_{n+1} + 2s^3 S_{n+2} + \cdots .
\end{aligned}$$

Using inequality (7), we get

$$\begin{aligned}
S_b(gx_n, gx_n, gx_m) &\le 2s\alpha^n S_0 + 2s^2\alpha^{n+1} S_0 + 2s^3\alpha^{n+2} S_0 + \cdots\\
&= 2s\alpha^n \left(1 + (\alpha s) + (\alpha s)^2 + (\alpha s)^3 + \cdots\right) S_0\\
&= 2s\alpha^n \left(\frac{1}{1-\alpha s}\right) S_0.
\end{aligned}$$

Since $\alpha = \dfrac{1}{s} < 1$, it follows that $\alpha^n \to 0$ as $n \to \infty$. Hence, letting $n \to \infty$, we obtain

$$\lim_{n\to\infty} S_b(gx_n, gx_n, gx_m) = 0.$$

Hence, the sequence $\{gx_n\}$ is Cauchy in $g(X)$. Because $g(X)$ is complete, there exists an element $q \in g(X)$ such that $gx_n \to q$. That is

$$\lim_{n\to\infty} gx_n = q = \lim_{n\to\infty} fx_{n-1}.$$

As $q \in g(X)$, there exists an element $p \in X$ with $g(p) = q$. We now aim to prove that $q = f(p)$. For this purpose, consider

$$\begin{aligned}
&S_b(gx_{n+1}, gx_{n+1}, fp) = S_b(fx_n, fx_n, fp)\\
&\le \alpha \max\{S_b(gx_n, gx_n, fx_n), S_b(gx_n, gx_n, fx_n), S_b(gp, gp, fp)\}\\
&\quad - \phi\left(S_b(gx_n, gx_n, fx_n), S_b(gx_n, gx_n, fx_n), S_b(gp, gp, fp)\right)\\
&= \alpha \max\{S_b(gx_n, gx_n, gx_{n+1}), S_b(gp, gp, fp)\}\\
&\quad - \phi\left(S_b(gx_n, gx_n, gx_{n+1}), S_b(gx_n, gx_n, gx_{n+1}), S_b(gp, gp, fp)\right).
\end{aligned}$$

Since $\phi(t_1, t_2, t_3) \ge 0$, for all $t_1, t_2, t_3 \ge 0$, it follows that

$$S_b(gx_{n+1}, gx_{n+1}, fp) \le \alpha \max\{S_b(gx_n, gx_n, gx_{n+1}), S_b(gp, gp, fp)\}.$$

Letting $n \to \infty$, we obtain

$$S_b(q, q, fp) \le \alpha \max\{S_b(q, q, q), S_b(q, q, fp)\}.$$

It implies that

$$S_b(q, q, fp) \le \alpha S_b(q, q, fp).$$

Therefore

$$(1-\alpha) S_b(q, q, fp) \le 0.$$

Since $\alpha < 1$, we have $(1-\alpha) > 0$. Hence, it follows that $S_b(q, q, fp) \le 0$. However, by definition of S_b, we know that $S_b(q, q, fp) \ge 0$.Thus, $S_b(q, q, fp) = 0$. This implies that $fp = q = gp$. This shows that p is a coincidence point of f and g. Now, we claim that f and g have a unique coincidence point. Suppose, for the sake of contradiction, that there is an another coincidence point, say $r \neq p$, of

f and g. Consider

$$\begin{aligned}
S_b(gr, gr, gp) &= S_b(fr, fr, fp) \\
&\leq \alpha \max\{S_b(gr, gr, fr), S_b(gr, gr, fr), S_b(gp, gp, fp)\} \\
&\quad - \phi\left(S_b(gr, gr, fr), S_b(gr, gr, fr), S_b(gp, gp, fp)\right) \\
&= \alpha \max\{S_b(gr, gr, gr), S_b(gr, gr, gr), S_b(gp, gp, gp)\} \\
&\quad - \phi\left(S_b(gr, gr, gr), S_b(gr, gr, gr), S_b(gp, gp, gp)\right) \\
&= \alpha \max\{0, 0, 0\} - \phi(0, 0, 0) \\
&= 0.
\end{aligned}$$

Therefore, $S_b(gr, gr, gp) \leq 0$. However, by definition of S_b, we know that $S_b(gr, gr, gp) \geq 0$. Combining these inequalities, we obtain $S_b(gr, gr, gp) = 0$, which implies that $gr = gp$. Hence, f and g have a unique coincidence point. Furthermore, if f and g are weakly compatible, then Proposition 2.10 ensures that f and g possess a unique common fixed point in X. □

Theorem 3.3. *Let (X, S_b) be an S_b-metric space, and let $f, g : X \to X$ be two mappings satisfying*

$$\begin{aligned}
S_b(fx, fy, fz) &\leq \frac{1}{4}\left(S_b(gx, gx, fx) + S_b(gy, gy, fy) + S_b(gz, gz, fz)\right) \\
&\quad - \phi\left(S_b(gx, gx, fx), S_b(gy, gy, fy)\right) - \psi\left(S_b(gy, gy, fy), S_b(gz, gz, fz)\right),
\end{aligned}$$

for all $x, y, z \in X$, where $\phi, \psi : [0, \infty)^2 \to [0, \infty)$ are continuous functions such that $\phi(x, y) = 0$ if and only if $x = y$ and $\psi(x, y) = 0$ if and only if $x = y$. Suppose further that s is a real number satisfying $1 \leq s < \frac{3}{2}$, and that the following conditions hold:

(i) *$f(X) \subseteq g(X)$;*

(ii) *$g(X)$ is complete.*

Then f and g have a unique coincidence point in X. If f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof. Since $f(X) \subseteq g(X)$, for any arbitrary $x_0 \in X$, we can find $x_1 \in X$ such that $f(x_0) = g(x_1)$. Proceeding similarly, for each $x_n \in X$, we can find

$x_{n+1} \in X$ such that $f(x_n) = g(x_{n+1})$. For sequence $\{gx_n\}$, consider

$$\begin{aligned}
&S_b(gx_n, gx_n, gx_{n+1}) = S_b(fx_{n-1}, f_{n-1}, fx_n) \\
&\leq \frac{1}{4}\left(S_b(gx_{n-1}, gx_{n-1}, fx_{n-1}) + S_b(gx_{n-1}, gx_{n-1}, fx_{n-1}) + S_b(gx_n, gx_n, fx_n)\right) \\
&\quad - \phi\left(S_b(gx_{n-1}, gx_{n-1}, fx_{n-1}), S_b(gx_{n-1}, gx_{n-1}, fx_{n-1})\right) \\
&\quad - \psi\left(S_b(gx_{n-1}, gx_{n-1}, fx_{n-1}), S_b(gx_n, gx_n, fx_n)\right) \\
&= \frac{1}{4}\left(2S_b(gx_{n-1}, gx_{n-1}, gx_n) + S_b(gx_n, gx_n, gx_{n+1})\right) \\
&\quad - \phi\left(S_b(gx_{n-1}, gx_{n-1}, gx_n), S_b(gx_{n-1}, gx_{n-1}, gx_n)\right) \\
&\quad - \psi\left(S_b(gx_{n-1}, gx_{n-1}, gx_n), S_b(gx_n, gx_n, gx_{n+1})\right).
\end{aligned}$$

Since $\phi(t_1, t_2) \geq 0$ and $\psi(t_1, t_2) \geq 0$ for all $t_1, t_2 \geq 0$, we deduce that

$$S_b(gx_n, gx_n, gx_{n+1}) \leq \frac{1}{4}\left(2S_b(gx_{n-1}, gx_{n-1}, gx_n) + S_b(gx_n, gx_n, gx_{n+1})\right).$$

Hence, we have

$$S_b(gx_n, gx_n, gx_{n+1}) \leq \frac{2}{3} S_b(gx_{n-1}, gx_{n-1}, gx_n).$$

Letting $\alpha = \frac{2}{3} < 1$, we obtain

$$S_b(gx_n, gx_n, gx_{n+1}) \leq \alpha S_b(gx_{n-1}, gx_{n-1}, gx_n). \tag{8}$$

Similarly, we can show that

$$S_b(gx_{n-1}, gx_{n-1}, gx_n) \leq \alpha S_b(gx_{n-2}, gx_{n-2}, gx_{n-1}). \tag{9}$$

Using inequalities (8) and (9), we get

$$S_b(gx_n, gx_n, gx_{n+1}) \leq \alpha^2 S_b(gx_{n-2}, gx_{n-2}, gx_{n-1}).$$

Continuing this process, it follows that

$$S_b(gx_n, gx_n, gx_{n+1}) \leq \alpha^n S_b(gx_0, gx_0, gx_1).$$

By setting $S_n = S_b(gx_n, gx_n, gx_{n+1})$, we obtain

$$S_n \leq \alpha^n S_0, \qquad n \in \mathbb{N}. \tag{10}$$

We now show that the sequence $\{gx_n\}$ is a Cauchy sequence in $g(X)$. Let $m > n > n_0$, for some $n_0 \in \mathbb{N}$. Then by repeated use of (iii) in Definition 2.3, we obtain

$$\begin{aligned}
&S_b(gx_n, gx_n, gx_m)\\
&\leq s\left(S_b(gx_n, gx_n, gx_{n+1}) + S_b(gx_n, gx_n, gx_{n+1}) + S_b(gx_m, gx_m, gx_{n+1})\right)\\
&= 2sS_b(gx_n, gx_n, gx_{n+1}) + sS_b(gx_{n+1}, gx_{n+1}, gx_m)\\
&\leq 2sS_b(gx_n, gx_n, gx_{n+1}) + s\left\{s\left(2S_b(gx_{n+1}, gx_{n+1}, gx_{n+2}) + S_b(gx_m, gx_m, gx_{n+2})\right)\right\}\\
&= 2sS_b(gx_n, gx_n, gx_{n+1}) + 2s^2S_b(gx_{n+1}, gx_{n+1}, gx_{n+2}) + s^2S_b(gx_m, gx_m, gx_{n+1})\\
&= 2sS_n + 2s^2S_{n+1} + s^2S_b(gx_{n+2}, gx_{n+2}, gx_m).
\end{aligned}$$

Continuing in the same manner, we obtain

$$\begin{aligned}
S_b(gx_n, gx_n, gx_m) &\leq 2sS_n + 2s^2S_{n+1} + 2s^3S_{n+2} + \cdots + 2s^{m-n-1}S_{m-1}\\
&\leq 2sS_n + 2s^2S_{n+1} + 2s^3S_{n+2} + \cdots .
\end{aligned}$$

Using inequality (10), we get

$$\begin{aligned}
S_b(gx_n, gx_n, gx_m) &\leq 2s\alpha^nS_0 + 2s^2\alpha^{n+1}S_0 + 2s^3\alpha^{n+2}S_0 + \cdots\\
&= 2s\alpha^n\left(1 + (\alpha s) + (\alpha s)^2 + (\alpha s)^3 + \cdots\right)S_0\\
&= 2s\alpha^n\left(\frac{1}{1-\alpha s}\right)S_0.
\end{aligned}$$

Since $\alpha = \dfrac{2}{3} < 1$, it follows that $\alpha^n \to 0$ as $n \to \infty$. Hence, letting $n \to \infty$, we obtain

$$\lim_{n\to\infty} S_b(gx_n, gx_n, gx_m) = 0.$$

Hence, the sequence $\{gx_n\}$ is Cauchy in $g(X)$. As $g(X)$ is complete, there exists an element $q \in g(X)$ such that $gx_n \to q$. That is

$$\lim_{n\to\infty} gx_n = q = \lim_{n\to\infty} fx_{n-1}.$$

Since $q \in g(X)$, there exists an element $p \in X$ such that $g(p) = q$. We now aim to prove that $q = f(p)$. For this purpose, consider

$$\begin{aligned}
&S_b(gx_{n+1}, gx_{n+1}, fp) = S_b(fx_n, fx_n, fp) \\
&\le \frac{1}{4}\left(S_b(gx_n, gx_n, fx_n) + S_b(gx_n, gx_n, fx_n) + S_b(gp, gp, fp)\right) \\
&\quad - \phi\left(S_b(gx_n, gx_n, fx_n), S_b(gx_n, gx_n, fx_n)\right) \\
&\quad - \psi\left(S_b(gx_n, gx_n, fx_n), S_b(gp, gp, fp)\right) \\
&= \frac{1}{4}\left(2S_b(gx_n, gx_n, gx_{n+1}) + S_b(gp, gp, fp)\right) \\
&\quad - \phi\left(S_b(gx_n, gx_n, gx_{n+1}), S_b(gx_n, gx_n, gx_{n+1})\right) \\
&\quad - \psi\left(S_b(gx_n, gx_n, gx_{n+1}), S_b(gp, gp, fp)\right).
\end{aligned}$$

Since $\phi(t_1, t_2) \ge 0$ and $\psi(t_1, t_2) \ge 0$ for all $t_1, t_2 \ge 0$, it follows that

$$S_b(gx_{n+1}, gx_{n+1}, fp) \le \frac{1}{4}\left(2S_b(gx_n, gx_n, gx_{n+1}) + S_b(gp, gp, fp)\right).$$

Letting $n \to \infty$, we obtain

$$S_b(q, q, fp) \le \frac{1}{4}\left(2S_b(q, q, q) + S_b(q, q, fp)\right).$$

It implies that

$$\frac{3}{4} S_b(q, q, fp) \le 0.$$

It follows that $S_b(q, q, fp) \le 0$. However, we also have $S_b(q, q, fp) \ge 0$. Therefore $S_b(q, q, fp) = 0$, which implies that $fp = q = gp$. This shows that p is a coincidence point of f and g. Now, we claim that f and g have a unique coincidence point. Suppose, for the sake of contradiction, that there is an another coincidence point, say $r \neq p$, of f and g. Consider

$$\begin{aligned}
&S_b(gr, gr, gp) = S_b(fr, fr, fp) \\
&\le \frac{1}{4}\left(S_b(gr, gr, fr) + S_b(gr, gr, fr) + S_b(gp, gp, fp)\right) \\
&\quad - \phi\left(S_b(gr, gr, fr), S_b(gr, gr, fr)\right) - \psi\left(S_b(gr, gr, fr), S_b(gp, gp, fp)\right) \\
&= \frac{1}{4}\left(S_b(gr, gr, gr) + S_b(gr, gr, gr) + S_b(gp, gp, gp)\right) \\
&\quad - \phi\left(S_b(gr, gr, gr), S_b(gr, gr, gr)\right) - \psi\left(S_b(gr, gr, gr), S_b(gp, gp, gp)\right) \\
&= \frac{1}{4}(0) - \phi(0, 0) - \psi(0, 0) \\
&= 0.
\end{aligned}$$

Therefore, $S_b(gr, gr, gp) \leq 0$. However, we know that $S_b(gr, gr, gp) \geq 0$. Combining these inequalities, we obtain $S_b(gr, gr, gp) = 0$, which implies that $gr = gp$. Hence, f and g have a unique coincidence point. Furthermore, if f and g are weakly compatible, then it follows from Proposition 2.10 that f and g possess a unique common fixed point in X. □

Theorem 3.4. *Let (X, S_b) be an S_b-metric space, and let $f, g : X \to X$ be two mappings. Suppose that there exists a real number α with $0 \leq \alpha < \frac{1}{s}$, $s \geq 1$ is a given real number such that for all $x, y, z \in X$,*

$$S_b(fx, fy, fz) \leq \alpha \max \{S_b(gx, gx, fx), S_b(gy, gy, fy), S_b(gz, gz, fz)\}$$
$$- \phi (S_b(gx, gx, fx), S_b(gy, gy, fy)) - \psi (S_b(gy, gy, fy), S_b(gz, gz, fz)),$$

where $\phi, \psi : [0, \infty)^2 \to [0, \infty)$ are continuous functions such that $\phi(x, y) = 0$ if and only if $x = y$ and $\psi(x, y) = 0$ if and only if $x = y$. If

(i) *$f(X) \subseteq g(X)$;*

(ii) *$g(X)$ is complete.*

Then f and g have a unique coincidence point in X. If f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof. Since $f(X) \subseteq g(X)$, for any arbitrary $x_0 \in X$, there exists $x_1 \in X$ such that $f(x_0) = g(x_1)$. Continuing in this way, for any $x_n \in X$, we can find $x_{n+1} \in X$ so that $f(x_n) = g(x_{n+1})$. For sequence $\{gx_n\}$, consider

$$S_b(gx_n, gx_n, gx_{n+1}) = S_b(fx_{n-1}, f_{n-1}, fx_n)$$
$$\leq \alpha \max \{S_b(gx_{n-1}, gx_{n-1}, fx_{n-1}), S_b(gx_{n-1}, gx_{n-1}, fx_{n-1}), S_b(gx_n, gx_n, fx_n)\}$$
$$- \phi (S_b(gx_{n-1}, gx_{n-1}, fx_{n-1}), S_b(gx_{n-1}, gx_{n-1}, fx_{n-1}))$$
$$- \psi (S_b(gx_{n-1}, gx_{n-1}, fx_{n-1}), S_b(gx_n, gx_n, fx_n))$$
$$= \alpha \max \{S_b(gx_{n-1}, gx_{n-1}, gx_n), S_b(gx_n, gx_n, gx_{n+1})\}$$
$$- \phi (S_b(gx_{n-1}, gx_{n-1}, gx_n), S_b(gx_{n-1}, gx_{n-1}, gx_n))$$
$$- \psi (S_b(gx_{n-1}, gx_{n-1}, gx_n), S_b(gx_n, gx_n, gx_{n+1})).$$

Since $\phi(t_1, t_2) \geq 0$ and $\psi(t_1, t_2) \geq 0$ for all $t_1, t_2 \geq 0$, it follows that

$$S_b(gx_n, gx_n, gx_{n+1}) \leq \alpha \max \{S_b(gx_{n-1}, gx_{n-1}, gx_n),\ S_b(gx_n, gx_n, gx_{n+1})\}. \quad (11)$$

If

$$\max \{S_b(gx_{n-1}, gx_{n-1}, gx_n),\ S_b(gx_n, gx_n, gx_{n+1})\} = S_b(gx_n, gx_n, gx_{n+1}),$$

then inequality (11) implies that

$$S_b(gx_n, gx_n, gx_{n+1}) \leq \alpha S_b(gx_n, gx_n, gx_{n+1}).$$

This is a contradiction since $\alpha < 1$. Therefore, we must have

$$\max\{S_b(gx_{n-1}, gx_{n-1}, gx_n),\ S_b(gx_n, gx_n, gx_{n+1})\} = S_b(gx_{n-1}, gx_{n-1}, gx_n).$$

Hence, inequality (11) implies that

$$S_b(gx_n, gx_n, gx_{n+1}) \leq \alpha S_b(gx_{n-1}, gx_{n-1}, gx_n). \tag{12}$$

Similarly,

$$S_b(gx_{n-1}, gx_{n-1}, gx_n) \leq \alpha S_b(gx_{n-2}, gx_{n-2}, gx_{n-1}). \tag{13}$$

Using inequalities (12) and (13), we obtain

$$S_b(gx_n, gx_n, gx_{n+1}) \leq \alpha^2 S_b(gx_{n-2}, gx_{n-2}, gx_{n-1}).$$

Proceeding by induction, we get

$$S_b(gx_n, gx_n, gx_{n+1}) \leq \alpha^n S_b(gx_0, gx_0, gx_1).$$

By setting $S_n = S_b(gx_n, gx_n, gx_{n+1})$, we obtain

$$S_n \leq \alpha^n S_0, \qquad n \in \mathbb{N}. \tag{14}$$

We now show that the sequence $\{gx_n\}$ is a Cauchy sequence in $g(X)$. Let $m > n > n_0$, for some $n_0 \in \mathbb{N}$. Then by repeated use of (iii) in Definition 2.3, we obtain

$$\begin{aligned}
&S_b(gx_n, gx_n, gx_m)\\
&\leq s\left(S_b(gx_n, gx_n, gx_{n+1}) + S_b(gx_n, gx_n, gx_{n+1}) + S_b(gx_m, gx_m, gx_{n+1})\right)\\
&= 2sS_b(gx_n, gx_n, gx_{n+1}) + sS_b(gx_{n+1}, gx_{n+1}, gx_m)\\
&\leq 2sS_b(gx_n, gx_n, gx_{n+1}) + s\left\{s\left(2S_b(gx_{n+1}, gx_{n+1}, gx_{n+2}) + S_b(gx_m, gx_m, gx_{n+2})\right)\right\}\\
&= 2sS_b(gx_n, gx_n, gx_{n+1}) + 2s^2 S_b(gx_{n+1}, gx_{n+1}, gx_{n+2}) + s^2 S_b(gx_m, gx_m, gx_{n+1})\\
&= 2sS_n + 2s^2 S_{n+1} + s^2 S_b(gx_{n+2}, gx_{n+2}, gx_m).
\end{aligned}$$

Continuing in the same manner, we obtain

$$\begin{aligned}
S_b(gx_n, gx_n, gx_m) &\leq 2sS_n + 2s^2 S_{n+1} + 2s^3 S_{n+2} + \cdots + 2s^{m-n-1} S_{m-1}\\
&\leq 2sS_n + 2s^2 S_{n+1} + 2s^3 S_{n+2} + \cdots .
\end{aligned}$$

Using inequality (14), we get

$$\begin{aligned} S_b(gx_n, gx_n, gx_m) &\leq 2s\alpha^n S_0 + 2s^2\alpha^{n+1} S_0 + 2s^3\alpha^{n+2} S_0 + \cdots \\ &= 2s\alpha^n \left(1 + (\alpha s) + (\alpha s)^2 + (\alpha s)^3 + \cdots\right) S_0 \\ &= 2s\alpha^n \left(\frac{1}{1-\alpha s}\right) S_0. \end{aligned}$$

Since $\alpha = \dfrac{1}{s} < 1$, it follows that $\alpha^n \to 0$ as $n \to \infty$. Hence, letting $n \to \infty$, we obtain

$$\lim_{n\to\infty} S_b(gx_n, gx_n, gx_m) = 0.$$

Therefore, $\{gx_n\}$ is a Cauchy sequence in $g(X)$. Since $g(X)$ is complete, there exists $q \in g(X)$ such that $gx_n \to q$. That is

$$\lim_{n\to\infty} gx_n = q = \lim_{n\to\infty} fx_{n-1}.$$

As $q \in g(X)$, there exists $p \in X$ such that $g(p) = q$. We now aim to prove that $q = f(p)$. For this purpose, consider

$$\begin{aligned} &S_b(gx_{n+1}, gx_{n+1}, fp) = S_b(fx_n, fx_n, fp) \\ &\leq \alpha \max\{S_b(gx_n, gx_n, fx_n), S_b(gx_n, gx_n, fx_n), S_b(gp, gp, fp)\} \\ &\quad - \phi\left(S_b(gx_n, gx_n, fx_n), S_b(gx_n, gx_n, fx_n)\right) \\ &\quad - \psi\left(S_b(gx_n, gx_n, fx_n), S_b(gp, gp, fp)\right) \\ &= \alpha \max\{S_b(gx_n, gx_n, gx_{n+1}), S_b(gp, gp, fp)\} \\ &\quad - \phi\left(S_b(gx_n, gx_n, gx_{n+1}), S_b(gx_n, gx_n, gx_{n+1})\right) \\ &\quad - \psi\left(S_b(gx_n, gx_n, gx_{n+1}), S_b(gp, gp, fp)\right). \end{aligned}$$

Since $\phi(t_1, t_2) \geq 0$ and $\psi(t_1, t_2) \geq 0$ for all $t_1, t_2, t_3 \geq 0$, it follows that

$$S_b(gx_{n+1}, gx_{n+1}, fp) \leq \alpha \max\{S_b(gx_n, gx_n, gx_{n+1}), S_b(gp, gp, fp)\}.$$

Letting $n \to \infty$, we obtain

$$S_b(q, q, fp) \leq \alpha \max\{S_b(q, q, q), S_b(q, q, fp)).$$

It implies that

$$S_b(q, q, fp) \leq \alpha S_b(q, q, fp).$$

Therefore

$$(1-\alpha) S_b(q, q, fp) \leq 0.$$

Since $\alpha < 1$, we have $(1-\alpha) > 0$. Hence, it follows that $S_b(q,q,fp) \leq 0$. However, by definition of S_b, we know that $S_b(q,q,fp) \geq 0$. Thus, $S_b(q,q,fp) = 0$. This implies that $fp = q = gp$. This shows that p is a coincidence point of f and g. Now, we claim that f and g have a unique coincidence point. Suppose, for the sake of contradiction, that there is an another coincidence point, say $r \neq p$, of f and g. Consider

$$\begin{aligned}
&S_b(gr,gr,gp) = S_b(fr,fr,fp) \\
&\leq \alpha \max\{S_b(gr,gr,fr), S_b(gr,gr,fr), S_b(gp,gp,fp)\} \\
&\quad - \phi\left(S_b(gr,gr,fr), S_b(gr,gr,fr)\right) - \psi\left(S_b(gr,gr,fr), S_b(gp,gp,fp)\right) \\
&= \alpha \max\{S_b(gr,gr,gr), S_b(gr,gr,gr), S_b(gp,gp,gp)\} \\
&\quad - \phi\left(S_b(gr,gr,gr), S_b(gr,gr,gr)\right) - \psi\left(S_b(gr,gr,gr), S_b(gp,gp,gp)\right) \\
&= \alpha \max\{0,0,0\} - \phi(0,0) - \psi(0,0) \\
&= 0.
\end{aligned}$$

Therefore, $S_b(gr,gr,gp) \leq 0$. However, we know that $S_b(gr,gr,gp) \geq 0$. Combining these inequalities, we obtain $S_b(gr,gr,gp) = 0$, which implies that $gr = gp$. Hence, f and g have a unique coincidence point. Furthermore, if f and g are weakly compatible, then Proposition 2.10 guarantees that f and g have a unique common fixed point in X. □

Example 3.5. Let $X = \left[0, \frac{1}{2}\right]$ and define the S_b-metric $S_b : X \times X \times X \to [0,\infty)$ by

$$S_b(x,y,z) = |x-y| + |y-z| + |z-x| \quad \text{for all } x,y,z \in X.$$

Now, define the mappings $f, g : X \to X$ by

$$f(x) = \frac{x}{8}, \quad g(x) = \frac{x}{2}, \quad \text{for all } \ x \in X,$$

and let

$$\phi(x,y,z) = \frac{x+y+z}{16}, \quad (x,y,z) \in [0,\infty)^3.$$

Then $f(X) = \left[0, \frac{1}{16}\right] \subseteq \left[0, \frac{1}{4}\right] = g(X)$ and $g(X)$ is a closed interval in $\mathbb{R}$ and hence complete. Now, for $x, y, z \in X$, we have

$$\begin{aligned} S_b(fx, fy, fz) &= S_b\left(\frac{x}{4}, \frac{y}{4}, \frac{z}{4}\right) \\ &= \left|\frac{x}{8} - \frac{y}{8}\right| + \left|\frac{y}{8} - \frac{z}{8}\right| + \left|\frac{z}{x} - \frac{x}{8}\right| \\ &= \frac{1}{8}\left\{|x-y| + |y-z| + |z-x|\right\}. \end{aligned} \tag{15}$$

Let us suppose that $x \geq y \geq z$. Then

$$|x-y| + |y-z| + |z-x| = (x-y) + (y-z) - (z-x) = 2(x-z).$$

Clearly, on $\left[0, \frac{1}{2}\right]$, $2(x-z)$ is maximum when $x = \frac{1}{2}$ and $z = 0$ and it is $2\left(\frac{1}{2} - 0\right) = 1$. That is, the maximum value of $|x-y| + |y-z| + |z-x|$ on $\left[0, \frac{1}{2}\right]$ is 1. Thus, equation (15) implies that $S_b(fx, fy, fz)$ attains its maximum value $\frac{1}{8}$ on $\left[0, \frac{1}{2}\right]$. Now,

$$\begin{aligned} S_b(gx, gx, fx) &= S_b\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{8}\right) \\ &= \left|\frac{x}{2} - \frac{x}{2}\right| + \left|\frac{x}{2} - \frac{x}{8}\right| + \left|\frac{x}{8} - \frac{x}{2}\right| \\ &= 2\left(\frac{x}{2} - \frac{x}{8}\right) \\ &= \frac{3x}{4}. \end{aligned}$$

In a similar manner, it can be shown that

$$S_b(gy, gy, fy) = \frac{3y}{4} \quad \text{and} \quad S_b(gz, gz, fz) = \frac{3z}{4}.$$

Therefore,

$$
\begin{aligned}
&\frac{1}{4}\left(S_b(gx,gx,fx)+S_b(gy,gy,fy)+S_b(gz,gz,fz)\right)\\
&-\phi\left(S_b(gx,gx,fx),S_b(gy,gy,fy),S_b(gz,gz,fz)\right)\\
&=\frac{1}{4}\left(\frac{3x}{4}+\frac{3y}{4}+\frac{3z}{4}\right)-\phi\left(\frac{3x}{4},\frac{3y}{4},\frac{3z}{4}\right)\\
&=\frac{3}{16}(x+y+z)-\frac{\left(\frac{3x}{4}+\frac{3y}{4}+\frac{3z}{4}\right)}{16}\\
&=\frac{9}{64}(x+y+z). \qquad (16)
\end{aligned}
$$

The maximum value of $\frac{9}{64}(x+y+z)$ on $\left[0,\frac{1}{2}\right]$ is $\frac{27}{128}$. Therefore, from (15) and (16) we have

$$
\begin{aligned}
S_b(fx,fy,fz)\leq{}&\frac{1}{4}\left(S_b(gx,gx,fx)+S_b(gy,gy,fy)+S_b(gz,gz,fz)\right)\\
&-\phi\left(S_b(gx,gx,fx),S_b(gy,gy,fy),S_b(gz,gz,fz)\right). \qquad (17)
\end{aligned}
$$

Therefore, by Theorem 3.1, the mappings f and g have a unique coincidence point in X, which is $x=0$. Moreover, since $fg(0)=g(0)=0=f(0)=gf(0)$, the mappings f and g are weakly compatible. Therefore, by Theorem 3.1 f and g have a unique common fixed point $x=0\in X$.

4 Conclusion

In this paper, we have established some common fixed point theorems for a pair of weakly compatible mappings within the framework of S_b-metric spaces. Our results extend and generalize several existing fixed point theorems. By introducing generalized contractive conditions for weakly compatible self mappings, we have shown that such mappings admit a unique common fixed point.

Bibliography

[1] M. Abbas and D. Doric, Common fixed point theorem for four mappings satisfying generalized weak contractive condition, *Filomat* **24(2)** (2010), 1–10.

[2] M. Abbas, I. Z. Chemac and A. Razani, Existence of common fixed point for b-metric rational type contraction, *Filomat* **30(6)** (2016), 1413–1429.

[3] M. Abbas and G. Jungck, Common fixed point results for noncommuting mappings without continuity in cone metric spaces, *J. Math. Anal. Appl.* **341(1)** (2008), 416–420.

[4] H. Aydi, M. Bota and S. Moradi, A common fixed points for weak ϕ-contractions on b-metric spaces, *Fixed Point Theory* **13** (2012), 337–346.

[5] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, *Fund. Math.* **3** (1922), 133–181.

[6] S. Czerwik, Contraction mappings in b-metric spaces, *Acta Math. Inform. Univ. Ostraviensis* **1(1)** (1993), 5–11.

[7] M. Boriceanu, M. Bota and A. Petrusel, Multivalued fractals in b-metric spaces, *Cent. Eur. J. Math.* **8(2)** (2010), 367–377.

[8] M. Iqbal, A. Batool, O. Ege and M. De la Sen, Fixed point of almost contraction in b-metric spaces, *J. Math.* **2020(1)** (2020), 1–6: 3218134.

[9] M. Iqbal, A. Batool, O. Ege and M. De la Sen, Fixed point of generalized weak contraction in b-metric spaces, *J. Funct. Spaces* **2021(1)** (2021), 2042162.

[10] G. Jungck and B. E. Rhoades, Fixed points for set valued functions without continuity, *Indian J. Pure Appl. Math.* **29(3)** (1998), 227–238.

[11] A. Latif, V. Parvane, P. Salimi and A. E. Al-Mazrooei, Various Suzuki type theorems in b-metric spaces, *J. Nonlinear Sci. Appl.* **8(4)** (2015), 363–377.

[12] M. Mlaiki, Common fixed points in complex S-metric space, *Adv. Fixed Point Theory* **4** (2014), 509–524.

[13] N. Mlaiki, α-ψ-Contractive Mapping on S-Metric Space, *Math. Sci. Lett.* **4** (2015), 9–12.

[14] K. Prudhvi, Fixed Point Theorems in S-Metric Spaces, *Univ. J. Comput. Math.* **3** (2015), 19–21.

[15] S. Sedghi, N. Shobe and A. Aliouche, A generalization of fixed point theorems in S-metric spaces, *Mat. Vesnik* **64(3)** (2012), 258–266.

[16] A. Singh and N. Hooda, Coupled Fixed Point Theorems in S-metric Spaces, *Inter. J. Math. Stat. Invent.* **2** (2014), 33–39.

[17] P. S. Sharma and C. T. Aage, On contractive mappings satisfying Φ-maps in S_b-metric spaces, *Electron. J. Math. Anal. Appl.* (Accepted).

[18] W. Shatanawi, A. Pitea and R. Lazovic, Contraction conditions using comparison functions on b-metric spaces, *Fixed Point Theory Appl.* **2014** (2014), 135.

[19] N. Souayah and N. Mlaiki, A fixed point theorem in S_b-metric spaces, *J. Math. Comput. Sci.* **16** (2016), 131–139.

[20] M. B. Zada, M. Sarwar and P. Kumam, Fixed point results for rational type contraction in b-metric spaces, *Int. J. Anal. Appl.* **16(6)** (2018), 904–920.

Received November 17, 2025; revised December 3, 2025; accepted December 27, 2025.

Author information

Pramodkumar Sohanlal Sharma, Department of Mathematics, Kavayitri Bahinabai Chaudhari North Maharashtra University, Jalgaon, India.
E-mail: pssair1@gmail.com

Chintaman Tukaram Aage, Department of Mathematics, Kavayitri Bahinabai Chaudhari North Maharashtra University, Jalgaon, India.
E-mail: caage17@gmail.com