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Exploring new types of Hardy-Hilbert integral inequalities

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Abstract. This article explores two new types of Hardy-Hilbert integral inequalities constructed using flexible kernel functions. These functions are versatile due to the interplay of two or more parameters, as well as their asymmetric structural properties. Our proofs establish sharp and consistent upper bounds.

Keywords. Hardy-Hilbert integral inequality, Hölder integral inequality, beta function, gamma function.

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1 Introduction

One of the most fundamental results in the theory of integral inequalities is the Hardy-Hilbert integral inequality. Its precise formulation is stated below. Let $p > 1$, $q = p/(p-1)$ and $f, g : (0, +\infty) \to (0, +\infty)$ be two functions such that

$$\int_0^{+\infty} f^p(x)dx < +\infty, \quad \int_0^{+\infty} g^q(x)dx < +\infty.$$

Then we have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1}{x+y} f(x)g(y)dxdy \leq \frac{\pi}{\sin(\pi/p)} \left[\int_0^{+\infty} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} g^q(x)dx\right]^{1/q}.$$

The constant factor $\pi/\sin(\pi/p)$ is optimal, meaning that it cannot be replaced by a smaller constant without invalidating the inequality; in other words, there exist functions f and g for which equality is approached arbitrarily closely. The technical details can be found in [5].

Over the years, numerous extensions and generalizations of this inequality have been established, including discrete and multidimensional analogues, versions with parameterized kernel functions, and weighted forms. They appear in operator theory, function spaces and Fourier analysis. A comprehensive overview of these developments is provided in the survey [3] and the monograph [14]. Recent contributions on various extensions and refinements can be found in [1,2,4,6–13].

In this article, we continue this line of research by investigating new types of Hardy-Hilbert integral inequalities. In particular, we focus on specific variations of the inequality of the following form:

$$\int_0^{+\infty}\int_0^{+\infty} k(x,y)f(x)g(y)dxdy \le \Upsilon \left[\int_0^{+\infty} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} g^q(x)dx\right]^{1/q}, \quad (1)$$

where $k : (0,+\infty)^2 \to (0,+\infty)$ denotes a novel or less-explored kernel function depending on two or three adjustable parameters. A distinctive feature of our approach is that the L^p and L^q norms of f and g, respectively, are intentionally considered in their unweighted form. Within this framework, we identify two main types of kernel functions, each generating its own family of Hardy-Hilbert type integral inequalities.

Kernel function of type I. The kernel function of type I is defined by

$$k(x,y) := \frac{x^\alpha y^\beta}{(x+y)^{\alpha+\beta+1}},$$

where α and β denote the parameters. This kernel function is homogeneous and exhibits an essentially asymmetric structure. Based on it, we establish a Hardy-Hilbert type integral inequality of the form given in Equation (1), and we show that the associated constant factor is sharp.

We then consider a natural extension of this kernel function, defined by

$$k(x,y) := \frac{x^\alpha y^\beta}{(x^\gamma+y^\gamma)^{(\alpha+\beta+1)/\gamma}},$$

where γ denotes an additional parameter. For this more general kernel function, we derive a corresponding Hardy-Hilbert type integral inequality still of the form in Equation (1) with an explicit and sharp constant factor.

The proofs are presented in detail and rely on minimal, direct arguments. They are based primarily on suitable changes of variables and an application of the Hölder integral inequality.

Kernel function of type II We then consider another type of kernel function, say of type II, defined by

$$k(x,y) := \frac{1}{x^\alpha + y^\alpha + \lambda},$$

where α and λ denote the parameters. This kernel function is inhomogeneous and symmetric. Based on this, a new Hardy-Hilbert-type integral inequality of the form in Equation (1) is then proved. We then turn to a natural asymmetric extension, defined by

$$k(x,y) := \frac{1}{x^{\alpha} + y^{\beta} + \lambda},$$

where β denotes an additional parameter. For this kernel function, we prove a corresponding Hardy-Hilbert type integral inequality still of the form in Equation (1) with an explicit constant factor.

The proofs are presented in full detail and follow a direct approach. They rely on exponential integral representations, suitable changes of variables and systematic applications of the Hölder integral inequality.

The remainder of the article is organized as follows: In Section 2, we present the Hardy-Hilbert type integral inequalities associated with the kernel function of type I, referred to as integral inequalities of type I. Section 3 is devoted to the corresponding results for the kernel function of type II, referred to as integral inequalities of type II. Finally, concluding remarks and perspectives are provided in Section 4.

2 Integral inequalities of type I

2.1 Main theorem

The first Hardy-Hilbert integral inequality based on the kernel function of type I of the form in Equation (1) is given below.

Theorem 2.1. *Let* $p > 1$, $q = p/(p-1)$, $\alpha > 1/p - 1$, $\beta > -1/p$ *and* $f, g : (0, +\infty) \to (0, +\infty)$ *be two functions such that*

$$\int_0^{+\infty} f^p(x)dx < +\infty, \quad \int_0^{+\infty} g^q(x)dx < +\infty.$$

Then we have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{x^{\alpha}y^{\beta}}{(x+y)^{\alpha+\beta+1}} f(x)g(y)dxdy$$

$$\leq B_{eta}\left(\alpha + 1 - \frac{1}{p}, \beta + \frac{1}{p}\right)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(x)dx\right]^{1/q},$$

where $B_{eta}(a,b)$ is the standard Beta function defined by

$$B_{eta}(a,b) := \int_0^1 t^{a-1}(1-t)^{b-1}dt,$$

with $a, b > 0$.

Proof. For convenience, we set

$$\mathfrak{A} := \int_0^{+\infty}\int_0^{+\infty} \frac{x^\alpha y^\beta}{(x+y)^{\alpha+\beta+1}} f(x)g(y)dxdy. \tag{2}$$

To develop the integral, we consider the change of variables $t := x+y \in (0,+\infty)$ and $u := x/(x+y) \in (0,1)$. Thus, we have $x = ut$, $y = (1-u)t$ and the absolute value of the Jacobian determinant is t, so that $dxdy = tdudt$. The integral $\mathfrak{A}$ is simplified to

$$\begin{aligned}\mathfrak{A} &= \int_0^{+\infty}\int_0^{1} \frac{(ut)^\alpha[(1-u)t]^\beta}{t^{\alpha+\beta+1}} f(ut)g((1-u)t)tdudt \\ &= \int_0^{+\infty}\int_0^{1} u^\alpha(1-u)^\beta f(ut)g((1-u)t)dudt,\end{aligned}$$

and, by the Fubini-Tonelli integral theorem,

$$\mathfrak{A} = \int_0^1 u^\alpha(1-u)^\beta \mathfrak{B}(u)du, \tag{3}$$

where

$$\mathfrak{B}(u) := \int_0^{+\infty} f(ut)g((1-u)t)dt.$$

Applying the Hölder integral inequality and performing the changes of variables

$v := ut \in (0, +\infty)$ and $w := (1-u)t \in (0, +\infty)$, we get

$$\begin{aligned}\mathfrak{B}(u) &\leq \left[\int_0^{+\infty} f^p(ut)dt\right]^{1/p} \left[\int_0^{+\infty} g^q((1-u)t)dt\right]^{1/q} \\ &= \left[\int_0^{+\infty} f^p(v)u^{-1}dv\right]^{1/p} \left[\int_0^{+\infty} g^q(w)(1-u)^{-1}dw\right]^{1/q} \\ &= u^{-1/p}(1-u)^{-1/q} \left[\int_0^{+\infty} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} g^q(x)dx\right]^{1/q}.\end{aligned}$$

Using this, Equation (3) and $1 - 1/q = 1/p$, we obtain

$$\begin{aligned}\mathfrak{A} &\leq \int_0^1 u^\alpha (1-u)^\beta \left\{ u^{-1/p}(1-u)^{-1/q} \left[\int_0^{+\infty} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} g^q(x)dx\right]^{1/q} \right\} du \\ &= \left[\int_0^1 u^{\alpha-1/p}(1-u)^{\beta-1/q}du\right] \left[\int_0^{+\infty} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} g^q(x)dx\right]^{1/q} \\ &= B_{eta}\left(\alpha + 1 - \frac{1}{p}, \beta + \frac{1}{p}\right) \left[\int_0^{+\infty} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} g^q(x)dx\right]^{1/q}.\end{aligned}$$

Using the expression of $\mathfrak{A}$ in Equation (2), we arrive at

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty} \frac{x^\alpha y^\beta}{(x+y)^{\alpha+\beta+1}} f(x)g(y)dxdy \\ &\leq B_{eta}\left(\alpha + 1 - \frac{1}{p}, \beta + \frac{1}{p}\right) \left[\int_0^{+\infty} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} g^q(x)dx\right]^{1/q}.\end{aligned}$$

This completes the proof of the theorem. □

If we take $\alpha = 0$ and $\beta = 0$, using the known relation $B_{eta}(1-a, a) = \pi/\sin(\pi a)$, then Theorem 2.1 is reduced to the classical Hardy-Hilbert integral inequality, i.e.,

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1}{x+y} f(x)g(y)dxdy$$
$$\leq B_{eta}\left(1-\frac{1}{p},\frac{1}{p}\right)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(x)dx\right]^{1/q}$$
$$=\frac{\pi}{\sin(\pi/p)}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(x)dx\right]^{1/q}.$$

If we take $\beta = 1/q$, using the relation $B_{eta}(a,1) = 1/a$ and $1/p + 1/q = 1$, then Theorem 2.1 gives

$$\int_0^{+\infty}\int_0^{+\infty} \frac{x^{\alpha}y^{1/q}}{(x+y)^{\alpha+1/q+1}} f(x)g(y)dxdy$$
$$\leq B_{eta}\left(\alpha+1-\frac{1}{p},1\right)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(x)dx\right]^{1/q}$$
$$\leq \frac{1}{\alpha+1-1/p}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(x)dx\right]^{1/q}.$$

Similar examples can be found by varying the values of α and β. The next subsection is devoted to some complementary results relating to Theorem 2.1.

2.2 Complements

The theorem below clearly states the optimal nature of the constant factor determined in Theorem 2.1.

Theorem 2.2. *In the integral inequality framework of Theorem* 2.1*, the constant factor*

$$B_{eta}\left(\alpha+1-\frac{1}{p},\beta+\frac{1}{p}\right)$$

is optimal.

Proof. We proceed by contradiction. Suppose that there is a constant Ω such that

$$\int_0^{+\infty}\int_0^{+\infty} \frac{x^\alpha y^\beta}{(x+y)^{\alpha+\beta+1}} f(x)g(y)dxdy$$
$$\leq \Omega \left[\int_0^{+\infty} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} g^q(x)dx\right]^{1/q}, \tag{4}$$

with

$$\Omega < B_{eta}\left(\alpha+1-\frac{1}{p}, \beta+\frac{1}{p}\right). \tag{5}$$

For $R>1$, let us consider the following two functions:

$$f_R(x) := \frac{1}{[\ln(R)]^{1/p}} x^{-1/p}, \quad x \in (1,R)$$

and $f_R(x) := 0$ for any $x \notin (1,R)$, and

$$g_R(x) := \frac{1}{[\ln(R)]^{1/q}} x^{-1/q}, \quad x \in (1,R)$$

and $g_R(x) := 0$ for any $x \notin (1,R)$. Thus, they are defined on $(0,+\infty)$ for $R \to +\infty$, a limit that we will apply in the last step of the development. We have

$$\int_0^{+\infty} f_R^p(x)dx = \int_1^R \left\{\frac{1}{[\ln(R)]^{1/p}} x^{-1/p}\right\}^p dx$$
$$= \frac{1}{\ln(R)} \int_1^R x^{-1}dx = \frac{1}{\ln(R)}[\ln(R)-0] = 1$$

and

$$\int_0^{+\infty} g_R^q(x)dx = \int_1^R \left\{\frac{1}{[\ln(R)]^{1/q}} x^{-1/q}\right\}^q dx$$
$$= \frac{1}{\ln(R)} \int_1^R x^{-1}dx = \frac{1}{\ln(R)}[\ln(R)-0] = 1.$$

From Equation (4), it follows that

$$\Omega \geq \int_0^{+\infty}\int_0^{+\infty} \frac{x^\alpha y^\beta}{(x+y)^{\alpha+\beta+1}} f_R(x) g_R(y) dx dy. \tag{6}$$

Reusing the first steps of the proof of Theorem 2.1 up to Equation (3), we get

$$\int_0^{+\infty}\int_0^{+\infty} \frac{x^\alpha y^\beta}{(x+y)^{\alpha+\beta+1}} f_R(x) g_R(y) dx dy = \int_0^1 u^\alpha (1-u)^\beta \mathfrak{C}_R(u) du, \tag{7}$$

where

$$\mathfrak{C}_R(u) := \int_0^{+\infty} f_R(ut) g_R((1-u)t) dt.$$

Analyzing the support of the product function $f_R(ut)g_R((1-u)t)$ and using $1/p+1/q=1$, we get

$$\begin{aligned}
\mathfrak{C}_R(u) &= \int_{\max(u^{-1},(1-u)^{-1})}^{R\min(u^{-1},(1-u)^{-1})} f_R(ut) g_R((1-u)t) dt \\
&= \int_{\max(u^{-1},(1-u)^{-1})}^{R\min(u^{-1},(1-u)^{-1})} \frac{1}{[\ln(R)]^{1/p}} (ut)^{-1/p} \frac{1}{[\ln(R)]^{1/q}} [(1-u)t]^{-1/q} dt \\
&= \frac{1}{\ln(R)} u^{-1/p}(1-u)^{-1/q} \int_{\max(u^{-1},(1-u)^{-1})}^{R\min(u^{-1},(1-u)^{-1})} t^{-1} dt \\
&= \frac{1}{\ln(R)} u^{-1/p}(1-u)^{-1/q} \left\{\ln\left[R\min(u^{-1},(1-u)^{-1})\right] - \ln[\max(u^{-1},(1-u)^{-1})]\right\} \\
&= \frac{1}{\ln(R)} u^{-1/p}(1-u)^{-1/q} \left\{\ln(R) + \ln\left[\frac{\min(u^{-1},(1-u)^{-1})}{\max(u^{-1},(1-u)^{-1})}\right]\right\} \\
&= u^{-1/p}(1-u)^{-1/q} \left\{1 + \frac{1}{\ln(R)} \ln\left[\frac{\min(u^{-1},(1-u)^{-1})}{\max(u^{-1},(1-u)^{-1})}\right]\right\}.
\end{aligned}$$

Using this and Equation (7), we get

$$\int_0^{+\infty}\int_0^{+\infty} \frac{x^\alpha y^\beta}{(x+y)^{\alpha+\beta+1}} f_R(x) g_R(y) dx dy$$

$$= \int_0^1 u^\alpha (1-u)^\beta u^{-1/p}(1-u)^{-1/q} \left\{1 + \frac{1}{\ln(R)} \ln\left[\frac{\min(u^{-1},(1-u)^{-1})}{\max(u^{-1},(1-u)^{-1})}\right]\right\} du$$

$$= \int_0^1 u^{\alpha-1/p}(1-u)^{\beta-1/q} du$$

$$+ \frac{1}{\ln(R)} \int_0^1 u^{\alpha-1/p}(1-u)^{\beta-1/q} \ln\left[\frac{\min(u^{-1},(1-u)^{-1})}{\max(u^{-1},(1-u)^{-1})}\right] du$$

$$= B_{eta}\left(\alpha + 1 - \frac{1}{p}, \beta + \frac{1}{p}\right) + \frac{1}{\ln(R)}\mathfrak{D},$$

where

$$\mathfrak{D} := \int_0^1 u^{\alpha-1/p}(1-u)^{\beta-1/q} \ln\left[\frac{\min(u^{-1},(1-u)^{-1})}{\max(u^{-1},(1-u)^{-1})}\right] du.$$

Using standard integral convergence arguments based on $\alpha > 1/p - 1$ and $\beta > -1/p$, we have $\mathfrak{D} \in (-\infty, 0)$ (the logarithmic term being negative). Thus, we have

$$\begin{aligned}
&\lim_{R\to+\infty} \int_0^{+\infty}\int_0^{+\infty} \frac{x^\alpha y^\beta}{(x+y)^{\alpha+\beta+1}} f_R(x) g_R(y) dx dy \\
&= B_{eta}\left(\alpha + 1 - \frac{1}{p}, \beta + \frac{1}{p}\right) + \mathfrak{D} \lim_{R\to+\infty} \frac{1}{\ln(R)} \\
&= B_{eta}\left(\alpha + 1 - \frac{1}{p}, \beta + \frac{1}{p}\right),
\end{aligned}$$

so that, by Equation (6),

$$\mathbf{\Omega} \geq B_{eta}\left(\alpha + 1 - \frac{1}{p}, \beta + \frac{1}{p}\right).$$

This contradicts Equation (5). As a result, the mentioned constant factor is optimal. This completes the proof of the theorem. □

The significance of Theorem 2.1 is thus demonstrated.

The theorem below is a consequence of this theorem. It has the feature to involve only one function, f.

Theorem 2.3. *Let* $p > 1$, $q = p/(p-1)$, $\alpha > 1/p - 1$, $\beta > -1/p$ *and* $f : (0, +\infty) \to (0, +\infty)$ *be a function such that*

$$\int_0^{+\infty} f^p(x)dx < +\infty.$$

Then we have

$$\int_0^{+\infty} \left[\int_0^{+\infty} \frac{x^\alpha y^\beta}{(x+y)^{\alpha+\beta+1}} f(x)dx \right]^p dy$$
$$\leq B_{eta}^p \left(\alpha + 1 - \frac{1}{p}, \beta + \frac{1}{p} \right) \int_0^{+\infty} f^p(x)dx.$$

Proof. For convenience, we set

$$\mathfrak{E} := \int_0^{+\infty} \left[\int_0^{+\infty} \frac{x^\alpha y^\beta}{(x+y)^{\alpha+\beta+1}} f(x)dx \right]^p dy. \tag{8}$$

A decomposition of the integrand gives

$$\mathfrak{E} = \int_0^{+\infty} \left[\int_0^{+\infty} \frac{x^\alpha y^\beta}{(x+y)^{\alpha+\beta+1}} f(x)dx \right]^{p-1} \int_0^{+\infty} \frac{x^\alpha y^\beta}{(x+y)^{\alpha+\beta+1}} f(x)dxdy$$
$$= \int_0^{+\infty} \int_0^{+\infty} \frac{x^\alpha y^\beta}{(x+y)^{\alpha+\beta+1}} f(x) g_\dagger(y) dxdy, \tag{9}$$

where

$$g_\dagger(y) = \left[\int_0^{+\infty} \frac{x^\alpha y^\beta}{(x+y)^{\alpha+\beta+1}} f(x)dx \right]^{p-1}.$$

From Theorem 2.1 applied to f and $g_\dagger$, it follows that

$$\int_0^{+\infty}\int_0^{+\infty} \frac{x^\alpha y^\beta}{(x+y)^{\alpha+\beta+1}} f(x) g_\dagger(y)dxdy$$
$$\leq B_{eta}\left(\alpha+1-\frac{1}{p}, \beta+\frac{1}{p}\right)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g_\dagger^q(x)dx\right]^{1/q}. \quad (10)$$

Let us examine the last integral. Using $q(p-1)=p$ and the definition of $\mathfrak{E}$, we obtain

$$\int_0^{+\infty} g_\dagger^q(x)dx = \int_0^{+\infty}\left[\int_0^{+\infty} \frac{x^\alpha y^\beta}{(x+y)^{\alpha+\beta+1}} f(x)dx\right]^{q(p-1)} dx$$
$$= \int_0^{+\infty}\left[\int_0^{+\infty} \frac{x^\alpha y^\beta}{(x+y)^{\alpha+\beta+1}} f(x)dx\right]^{p} dx = \mathfrak{E}.$$

This and Equations (9) and (10) give

$$\mathfrak{E} \leq B_{eta}\left(\alpha+1-\frac{1}{p}, \beta+\frac{1}{p}\right)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p} \mathfrak{E}^{1/q}.$$

Using $1-1/q=1/p$, we obtain

$$\mathfrak{E}^{1/p} \leq B_{eta}\left(\alpha+1-\frac{1}{p}, \beta+\frac{1}{p}\right)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p},$$

so that, raising at the exponent p and using the definition of $\mathfrak{E}$, we have

$$\int_0^{+\infty}\left[\int_0^{+\infty} \frac{x^\alpha y^\beta}{(x+y)^{\alpha+\beta+1}} f(x)dx\right]^{p} dy$$
$$\leq B_{eta}^p\left(\alpha+1-\frac{1}{p}, \beta+\frac{1}{p}\right)\int_0^{+\infty} f^p(x)dx.$$

This ends the proof of the theorem. □

This result shows that the integral operator defined by

$$T(f)(y) := \int_0^{+\infty} \frac{x^\alpha y^\beta}{(x+y)^{\alpha+\beta+1}} f(x)dx$$

is continuous from L^p to L^p. This is an example of application of Theorem 2.1 in operator theory.

2.3 Generalization of the main theorem

The theorem below proposes a generalization of Theorem 2.1, with the addition of a new parameter, γ.

Theorem 2.4. *Let* $p > 1$, $q = p/(p-1)$, $\alpha > 1/p - 1$, $\beta > 1/q - 1$, $\gamma > 0$ *and* $f, g : (0, +\infty) \to (0, +\infty)$ *be two functions such that*

$$\int_0^{+\infty} f^p(x)dx < +\infty, \quad \int_0^{+\infty} g^q(x)dx < +\infty.$$

Then we have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{x^\alpha y^\beta}{(x^\gamma + y^\gamma)^{(\alpha+\beta+1)/\gamma}} f(x)g(y)dxdy$$

$$\leq \frac{1}{\gamma} B_{eta}\left(\frac{1}{\gamma}\left(\alpha + 1 - \frac{1}{p}\right), \frac{1}{\gamma}\left(\beta + \frac{1}{p}\right)\right) \left[\int_0^{+\infty} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} g^q(x)dx\right]^{1/q}.$$

Proof. The proof is similar to that of Theorem 2.1, but with more technical details. For convenience, we set

$$\mathfrak{F} := \int_0^{+\infty}\int_0^{+\infty} \frac{x^\alpha y^\beta}{(x^\gamma + y^\gamma)^{(\alpha+\beta+1)/\gamma}} f(x)g(y)dxdy. \tag{11}$$

To develop the integral, we consider the change of variables $t := (x^\gamma + y^\gamma)^{1/\gamma} \in (0, +\infty)$ and $u := x^\gamma/(x^\gamma + y^\gamma) \in (0, 1)$. Thus, we have $x = u^{1/\gamma}t$, $y = (1-u)^{1/\gamma}t$ and the absolute value of the Jacobian determinant is $(1/\gamma)tu^{1/\gamma-1}(1-u)^{1/\gamma-1}$, so that

$$dxdy = \frac{1}{\gamma} t u^{1/\gamma-1}(1-u)^{1/\gamma-1} dudt.$$

The integral $\mathfrak{F}$ becomes

$$\mathfrak{F} = \int_0^{+\infty}\int_0^1 \frac{(u^{1/\gamma}t)^{\alpha}[(1-u)^{1/\gamma}t]^{\beta}}{t^{\alpha+\beta+1}} f(u^{1/\gamma}t)g((1-u)^{1/\gamma}t)\frac{1}{\gamma}tu^{1/\gamma-1}(1-u)^{1/\gamma-1}dudt$$

$$= \frac{1}{\gamma}\int_0^{+\infty}\int_0^1 u^{(\alpha+1)/\gamma-1}(1-u)^{(\beta+1)/\gamma-1}f(u^{1/\gamma}t)g((1-u)^{1/\gamma}t)dudt.$$

By the Fubini-Tonelli integral theorem, we get

$$\mathfrak{F} = \frac{1}{\gamma}\int_0^1 u^{(\alpha+1)/\gamma-1}(1-u)^{(\beta+1)/\gamma-1}\mathfrak{G}(u)du, \tag{12}$$

where

$$\mathfrak{G}(u) := \int_0^{+\infty} f(u^{1/\gamma}t)g((1-u)^{1/\gamma}t)dt.$$

Applying the Hölder integral inequality and performing the changes of variables $v := u^{1/\gamma}t \in (0,+\infty)$ and $w := (1-u)^{1/\gamma}t \in (0,+\infty)$, we obtain

$$\mathfrak{G}(u) \leq \left[\int_0^{+\infty} f^p(u^{1/\gamma}t)dt\right]^{1/p}\left[\int_0^{+\infty} g^q((1-u)^{1/\gamma}t)dt\right]^{1/q}$$

$$= \left[\int_0^{+\infty} f^p(v)u^{-1/\gamma}dv\right]^{1/p}\left[\int_0^{+\infty} g^q(w)(1-u)^{-1/\gamma}dw\right]^{1/q}$$

$$= u^{-1/(\gamma p)}(1-u)^{-1/(\gamma q)}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(x)dx\right]^{1/q}.$$

Using this, Equation (12) and $1/p + 1/q = 1$, we establish

$$\mathfrak{F} \leq \frac{1}{\gamma}\int_0^1 u^{(\alpha+1)/\gamma-1}(1-u)^{(\beta+1)/\gamma-1}\times$$
$$\left\{u^{-1/(\gamma p)}(1-u)^{-1/(\gamma q)}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(x)dx\right]^{1/q}\right\}du$$
$$= \frac{1}{\gamma}\left[\int_0^1 u^{(\alpha+1-1/p)/\gamma-1}(1-u)^{(\beta+1/p)/\gamma-1}du\right]\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(x)dx\right]^{1/q}$$
$$= \frac{1}{\gamma}B_{eta}\left(\frac{1}{\gamma}\left(\alpha+1-\frac{1}{p}\right),\frac{1}{\gamma}\left(\beta+\frac{1}{p}\right)\right)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(x)dx\right]^{1/q}.$$

Using the expression of $\mathfrak{F}$ in Equation (11), we can write

$$\int_0^{+\infty}\int_0^{+\infty}\frac{x^\alpha y^\beta}{(x^\gamma+y^\gamma)^{(\alpha+\beta+1)/\gamma}}f(x)g(y)dxdy$$
$$\leq \frac{1}{\gamma}B_{eta}\left(\frac{1}{\gamma}\left(\alpha+1-\frac{1}{p}\right),\frac{1}{\gamma}\left(\beta+\frac{1}{p}\right)\right)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(x)dx\right]^{1/q}.$$

This completes the proof of the theorem. □

If we take $\alpha = 0$ and $\beta = 0$, then Theorem 2.4 gives

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(x^\gamma+y^\gamma)^{1/\gamma}}f(x)g(y)dxdy$$
$$\leq \frac{1}{\gamma}B_{eta}\left(\frac{1}{\gamma q},\frac{1}{\gamma p}\right)\left[\int_0^{+\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g^q(x)dx\right]^{1/q}.$$

If we take $\gamma = \alpha + \beta + 1$, then Theorem 2.4 gives

$$\int_0^{+\infty}\int_0^{+\infty} \frac{x^\alpha y^\beta}{x^{\alpha+\beta+1} + y^{\alpha+\beta+1}} f(x)g(y)dxdy$$
$$\leq \frac{1}{\alpha+\beta+1} B_{eta}\left(\frac{1}{\alpha+\beta+1}\left(\alpha+1-\frac{1}{p}\right), \frac{1}{\alpha+\beta+1}\left(\beta+\frac{1}{p}\right)\right)$$
$$\times \left[\int_0^{+\infty} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} g^q(x)dx\right]^{1/q}.$$

The rest of the article is devoted to the integral inequalities of type II.

3 Integral inequalities of type II

3.1 Main theorem

The first Hardy-Hilbert integral inequality based on the kernel function of type II of the form in Equation (1) is given below.

Theorem 3.1. *Let* $p > 1$, $q = p/(p-1)$, $\alpha > 1$, $\lambda > 0$ *and* $f, g : (0, +\infty) \to (0, +\infty)$ *be two functions such that*

$$\int_0^{+\infty} f^p(x)dx < +\infty, \quad \int_0^{+\infty} g^q(x)dx < +\infty.$$

Then we have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1}{x^\alpha + y^\alpha + \lambda} f(x)g(y)dxdy$$
$$\leq \frac{\pi}{\alpha \sin(\pi/\alpha)} p^{-1/(\alpha p)} q^{-1/(\alpha q)} \lambda^{1/\alpha - 1} \left[\int_0^{+\infty} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} g^q(x)dx\right]^{1/q}.$$

Proof. For convenience, we set

$$\mathfrak{H} := \int_0^{+\infty}\int_0^{+\infty} \frac{1}{x^\alpha + y^\alpha + \lambda} f(x)g(y)dxdy.$$

Using a basic integral result on the exponential function, we can write

$$\frac{1}{x^\alpha + y^\alpha + \lambda} = \int_0^{+\infty} e^{-(x^\alpha + y^\alpha + \lambda)t} dt.$$

From this and the Fubini-Tonelli integral theorem, it follows that

$$\begin{aligned}\mathfrak{H} &= \int_0^{+\infty} \int_0^{+\infty} \left[\int_0^{+\infty} e^{-(x^\alpha + y^\alpha + \lambda)t} dt \right] f(x)g(y) dx dy \\ &= \int_0^{+\infty} \left[\int_0^{+\infty} f(x) e^{-tx^\alpha} dx \right] \left[\int_0^{+\infty} g(y) e^{-ty^\alpha} dy \right] e^{-\lambda t} dt. \qquad (13)\end{aligned}$$

Applying the Hölder integral inequality, performing the change of variables $u := tqx^\alpha$ and using the standard Gamma function defined by

$$G_{am}(a) := \int_0^{+\infty} t^{a-1} e^{-t} dt,$$

with $a > 0$, we have

$$\begin{aligned}&\int_0^{+\infty} f(x) e^{-tx^\alpha} dx \leq \left[\int_0^{+\infty} f^p(x) dx \right]^{1/p} \left[\int_0^{+\infty} e^{-tqx^\alpha} dx \right]^{1/q} \\ &= \left[\int_0^{+\infty} f^p(x) dx \right]^{1/p} \left[(tq)^{-1/\alpha} \frac{1}{\alpha} \int_0^{+\infty} u^{1/\alpha - 1} e^{-u} du \right]^{1/q} \\ &= \left[\frac{1}{\alpha} G_{am}\left(\frac{1}{\alpha} \right) \right]^{1/q} q^{-1/(\alpha q)} \left[\int_0^{+\infty} f^p(x) dx \right]^{1/p} t^{-1/(\alpha q)}. \qquad (14)\end{aligned}$$

Similarly but with the change of variables $v := tpy^{\alpha}$, we obtain

$$\int_{0}^{+\infty} g(y)e^{-ty^{\alpha}}dy \leq \left[\int_{0}^{+\infty} g^{q}(y)dy\right]^{1/q}\left[\int_{0}^{+\infty} e^{-tpy^{\alpha}}dy\right]^{1/p}$$

$$= \left[\int_{0}^{+\infty} g^{q}(y)dy\right]^{1/q}\left[(tp)^{-1/\alpha}\frac{1}{\alpha}\int_{0}^{+\infty} v^{1/\alpha-1}e^{-v}dv\right]^{1/p}$$

$$= \left[\frac{1}{\alpha}G_{am}\left(\frac{1}{\alpha}\right)\right]^{1/p} p^{-1/(\alpha p)}\left[\int_{0}^{+\infty} g^{q}(y)dy\right]^{1/q} t^{-1/(\alpha p)}. \tag{15}$$

Combining Equations (13), (14) and (15), using $1/p + 1/q = 1$ and making the change of variables $w := \lambda t$, we get

$$\mathfrak{H} \leq \int_{0}^{+\infty}\left[\frac{1}{\alpha}G_{am}\left(\frac{1}{\alpha}\right)\right]^{1/q} q^{-1/(\alpha q)}\left[\int_{0}^{+\infty} f^{p}(x)dx\right]^{1/p} t^{-1/(\alpha q)}$$

$$\times\left[\frac{1}{\alpha}G_{am}\left(\frac{1}{\alpha}\right)\right]^{1/p} p^{-1/(\alpha p)}\left[\int_{0}^{+\infty} g^{q}(y)dy\right]^{1/q} t^{-1/(\alpha p)}e^{-\lambda t}dt$$

$$= \frac{1}{\alpha}G_{am}\left(\frac{1}{\alpha}\right)p^{-1/(\alpha p)}q^{-1/(\alpha q)}\left[\int_{0}^{+\infty} t^{-1/\alpha}e^{-\lambda t}dt\right]\left[\int_{0}^{+\infty} f^{p}(x)dx\right]^{1/p}\left[\int_{0}^{+\infty} g^{q}(x)dx\right]^{1/q}$$

$$= \frac{1}{\alpha}G_{am}\left(\frac{1}{\alpha}\right)p^{-1/(\alpha p)}q^{-1/(\alpha q)}\left[\lambda^{1/\alpha-1}\int_{0}^{+\infty} w^{-1/\alpha}e^{-w}dw\right]\left[\int_{0}^{+\infty} f^{p}(x)dx\right]^{1/p}\left[\int_{0}^{+\infty} g^{q}(x)dx\right]^{1/q}$$

$$= \frac{1}{\alpha}G_{am}\left(\frac{1}{\alpha}\right)p^{-1/(\alpha p)}q^{-1/(\alpha q)}\left[\lambda^{1/\alpha-1}G_{am}\left(1-\frac{1}{\alpha}\right)\right]\left[\int_{0}^{+\infty} f^{p}(x)dx\right]^{1/p}\left[\int_{0}^{+\infty} g^{q}(x)dx\right]^{1/q}.$$

Using the relation $G_{am}(a)G_{am}(1-a) = \pi/\sin(\pi a)$ and the definition of $\mathfrak{H}$, we obtain

$$\int_{0}^{+\infty}\int_{0}^{+\infty}\frac{1}{x^{\alpha}+y^{\alpha}+\lambda}f(x)g(y)dxdy$$

$$\leq \frac{\pi}{\alpha\sin(\pi/\alpha)}p^{-1/(\alpha p)}q^{-1/(\alpha q)}\lambda^{1/\alpha-1}\left[\int_{0}^{+\infty} f^{p}(x)dx\right]^{1/p}\left[\int_{0}^{+\infty} g^{q}(x)dx\right]^{1/q}.$$

This completes the proof of the theorem. □

If we take $\alpha = 2$ and $\lambda = 1$, Theorem 3.1 gives

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1}{x^2+y^2+1} f(x)g(y)dxdy$$
$$\leq \frac{\pi}{2} p^{-1/(2p)} q^{-1/(2q)} \left[\int_0^{+\infty} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} g^q(x)dx\right]^{1/q}.$$

To the best of our knowledge, this represents a new Hardy-Hilbert type integral inequality in the literature.

The theorem below is a one-parameter generalization of Theorem 3.1, making the kernel function asymmetric at a price of a more sophisticated constant factor.

Theorem 3.2. *Let* $p > 1$*,* $q = p/(p-1)$*,* $\alpha > 0$*,* $\beta > 0$*,* $\lambda > 0$ *such that*

$$\frac{1}{\beta p} + \frac{1}{\alpha q} < 1$$

and $f, g : (0, +\infty) \to (0, +\infty)$ *be two functions such that*

$$\int_0^{+\infty} f^p(x)dx < +\infty, \quad \int_0^{+\infty} g^q(x)dx < +\infty.$$

Then we have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1}{x^\alpha+y^\beta+\lambda} f(x)g(y)dxdy$$
$$\leq \left[\frac{1}{\alpha} G_{am}\left(\frac{1}{\alpha}\right)\right]^{1/q} \left[\frac{1}{\beta} G_{am}\left(\frac{1}{\beta}\right)\right]^{1/p} G_{am}\left(1 - \frac{1}{\beta p} - \frac{1}{\alpha q}\right) p^{-1/(\beta p)} q^{-1/(\alpha q)}$$
$$\times \lambda^{1/(\beta p)+1/(\alpha q)-1} \left[\int_0^{+\infty} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} g^q(x)dx\right]^{1/q}.$$

Proof. The proof is similar to that of Theorem 3.1, but with more technical details. For convenience, we set

$$\mathfrak{I} := \int_0^{+\infty}\int_0^{+\infty} \frac{1}{x^\alpha+y^\beta+\lambda} f(x)g(y)dxdy.$$

Using a basic integral result on the exponential function, we can write

$$\frac{1}{x^{\alpha}+y^{\beta}+\lambda}=\int_{0}^{+\infty} e^{-(x^{\alpha}+y^{\beta}+\lambda)t}dt.$$

From this and the Fubini-Tonelli integral theorem, it follows that

$$\begin{aligned}\mathfrak{I} &= \int_{0}^{+\infty}\int_{0}^{+\infty}\left[\int_{0}^{+\infty} e^{-(x^{\alpha}+y^{\beta}+\lambda)t}dt\right] f(x)g(y)dxdy \\ &= \int_{0}^{+\infty}\left[\int_{0}^{+\infty} f(x)e^{-tx^{\alpha}}dx\right]\left[\int_{0}^{+\infty} g(y)e^{-ty^{\beta}}dy\right]e^{-\lambda t}dt. \end{aligned} \tag{16}$$

Applying the Hölder integral inequality and performing the change of variables $u := tqx^{\alpha}$, we have

$$\begin{aligned}\int_{0}^{+\infty} f(x)e^{-tx^{\alpha}}dx &\leq \left[\int_{0}^{+\infty} f^{p}(x)dx\right]^{1/p}\left[\int_{0}^{+\infty} e^{-tqx^{\alpha}}dx\right]^{1/q} \\ &= \left[\int_{0}^{+\infty} f^{p}(x)dx\right]^{1/p}\left[(tq)^{-1/\alpha}\frac{1}{\alpha}\int_{0}^{+\infty} u^{1/\alpha-1}e^{-u}du\right]^{1/q} \\ &= \left[\frac{1}{\alpha}G_{am}\left(\frac{1}{\alpha}\right)\right]^{1/q} q^{-1/(\alpha q)}\left[\int_{0}^{+\infty} f^{p}(x)dx\right]^{1/p} t^{-1/(\alpha q)}. \end{aligned} \tag{17}$$

Similarly but with the change of variables $v := tpy^{\beta}$, we obtain

$$\begin{aligned}\int_{0}^{+\infty} g(y)e^{-ty^{\beta}}dy &\leq \left[\int_{0}^{+\infty} g^{q}(y)dy\right]^{1/q}\left[\int_{0}^{+\infty} e^{-tpy^{\beta}}dy\right]^{1/p} \\ &= \left[\int_{0}^{+\infty} g^{q}(y)dy\right]^{1/q}\left[(tp)^{-1/\beta}\frac{1}{\beta}\int_{0}^{+\infty} v^{1/\beta-1}e^{-v}dv\right]^{1/p} \\ &= \left[\frac{1}{\beta}G_{am}\left(\frac{1}{\beta}\right)\right]^{1/p} p^{-1/(\beta p)}\left[\int_{0}^{+\infty} g^{q}(y)dy\right]^{1/q} t^{-1/(\beta p)}. \end{aligned} \tag{18}$$

Combining Equations (16), (17) and (18), using $1/p + 1/q = 1$ and making the change of variables $w := \lambda t$, we get

$$\mathfrak{I} \leq \int_0^{+\infty} \left[\frac{1}{\alpha} G_{am}\left(\frac{1}{\alpha}\right)\right]^{1/q} q^{-1/(\alpha q)} \left[\int_0^{+\infty} f^p(x)dx\right]^{1/p} t^{-1/(\alpha q)}$$

$$\times \left[\frac{1}{\beta} G_{am}\left(\frac{1}{\beta}\right)\right]^{1/p} p^{-1/(\beta p)} \left[\int_0^{+\infty} g^q(y)dy\right]^{1/q} t^{-1/(\beta p)} e^{-\lambda t} dt$$

$$= \left[\frac{1}{\alpha} G_{am}\left(\frac{1}{\alpha}\right)\right]^{1/q} \left[\frac{1}{\beta} G_{am}\left(\frac{1}{\beta}\right)\right]^{1/p} p^{-1/(\beta p)} q^{-1/(\alpha q)} \left[\int_0^{+\infty} t^{-1/(\beta p)-1/(\alpha q)} e^{-\lambda t} dt\right]$$

$$\times \left[\int_0^{+\infty} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} g^q(x)dx\right]^{1/q}$$

$$= \left[\frac{1}{\alpha} G_{am}\left(\frac{1}{\alpha}\right)\right]^{1/q} \left[\frac{1}{\beta} G_{am}\left(\frac{1}{\beta}\right)\right]^{1/p} p^{-1/(\beta p)} q^{-1/(\alpha q)}$$

$$\times \left[\lambda^{1/(\beta p)+1/(\alpha q)-1} \int_0^{+\infty} w^{-1/(\beta p)-1/(\alpha q)} e^{-w} dw\right] \left[\int_0^{+\infty} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} g^q(x)dx\right]^{1/q}$$

$$= \left[\frac{1}{\alpha} G_{am}\left(\frac{1}{\alpha}\right)\right]^{1/q} \left[\frac{1}{\beta} G_{am}\left(\frac{1}{\beta}\right)\right]^{1/p} p^{-1/(\beta p)} q^{-1/(\alpha q)}$$

$$\times \left[\lambda^{1/(\beta p)+1/(\alpha q)-1} G_{am}\left(1 - \frac{1}{\beta p} - \frac{1}{\alpha q}\right)\right] \left[\int_0^{+\infty} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} g^q(x)dx\right]^{1/q}.$$

Using the definition of $\mathfrak{I}$ and slightly arranging the upper bound, we obtain

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1}{x^\alpha + y^\beta + \lambda} f(x)g(y)dxdy$$

$$\leq \left[\frac{1}{\alpha} G_{am}\left(\frac{1}{\alpha}\right)\right]^{1/q} \left[\frac{1}{\beta} G_{am}\left(\frac{1}{\beta}\right)\right]^{1/p} G_{am}\left(1 - \frac{1}{\beta p} - \frac{1}{\alpha q}\right) p^{-1/(\beta p)} q^{-1/(\alpha q)}$$

$$\times \lambda^{1/(\beta p)+1/(\alpha q)-1} \left[\int_0^{+\infty} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} g^q(x)dx\right]^{1/q}.$$

This completes the proof of the theorem. □

If we take $\alpha = \beta$, Theorem 3.2 is reduced to Theorem 3.1.

This result is one of the rare proposing a Hardy-Hilbert type integral inequality based on an asymmetric inhomogeneous kernel function and of the form in Equation (1).

4 Conclusion

In conclusion, the results of this article extend the classical Hardy-Hilbert integral inequality to a broad class of flexible kernel functions depending on two or three adjustable parameters, often exhibiting asymmetric structures. For each inequality, explicit and sharp constant factors are established. Future directions include multidimensional generalizations, weighted formulations, and applications to operator theory, harmonic analysis, and mathematical physics.

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