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On some analogues of periodic problems for Laplace equation with an oblique derivative under boundary conditions

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Abstract. In this paper, we study solvability of new classes of nonlocal boundary value problems for the Laplace equation in a ball. The considered problems are multidimensional analogues (in the case of a ball) of classical periodic boundary value problems in rectangular regions. To study the main problem, first, for the Laplace equation, we consider an auxiliary boundary value problem with an oblique derivative. This problem generalizes the well-known Neumann problem and is conditionally solvable. The main problems are solved by reducing them to sequential solution of the Dirichlet problem and the problem with an oblique derivative. It is proved that in the case of periodic conditions, the problem is conditionally solvable; and in this case the exact condition for solvability of the considered problem is found. When boundary conditions are specified in the anti-periodic conditions form, the problem is certainly solvable. The obtained general results are illustrated with specific examples.

Keywords. Boundary value problems, Laplace equation, oblique derivative, periodic conditions, uniqueness of solution, existence of a solution.

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1 Introduction

Let $\Omega = \{x \in R^n : |x| < 1\}$ be a unit ball, $n \geq 2, \partial\Omega = \{x \in R^n : |x| = 1\}$ be a unit sphere. For any point $x \in \Omega$ we compare the point $x^* = (\alpha_1 x_1, \alpha_2 x_2, ..., \alpha_n x_n)$, where $\alpha_1 = -1$, and others $\alpha_j, \{j = 2, 3, ..., n\}$ take one of the values ± 1.

Denote

$$\partial\Omega_+ = \{x \in \partial\Omega : x_1 \geq 0\}, \partial\Omega_- = \{x \in \partial\Omega : x_1 \leq 0\},$$

$$\Gamma = \{x \in \partial\Omega : x_1 = 0\}.$$

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Further, let $a = (a_1, a_2, ..., a_n)$ be an arbitrary fixed point in the domain Ω.

Denote

$$\frac{\partial u}{\partial \ell_a}(x) = (x - a, \nabla u) \equiv \sum_{j=1}^{n} (x_j - a_j) \frac{\partial u}{\partial x_j}(x).$$

If $a = (0, 0, ..., 0) \equiv 0$, then

$$\frac{\partial u}{\partial \ell_0}(x) = \sum_{j=1}^{n} x_j \frac{\partial u}{\partial x_j}(x) \equiv r \frac{\partial u}{\partial r}(x), r = |x|.$$

Since for all $x \in \partial\Omega$, $r \frac{\partial u}{\partial r}(x)\big|_{\partial\Omega} = \frac{\partial u}{\partial v}(x)$, where ν is a normal vector to the sphere $\partial\Omega$, then in the case $a \equiv 0$ direction of the vector ℓ_a coincides with the direction of the normal vector ν.

Introduce an operator $I_S u(x) = u(x^*)$. We consider the following problems in the domain Ω.

Problem 1.1. Find a harmonic function $u(x) \in C^2(\Omega) \cap C^1(\bar{\Omega})$, satisfying the conditions

$$u(x) - u(x^*) = g_0(x), x \in \partial\Omega_+, \tag{1}$$

$$\frac{\partial u}{\partial \ell_a}(x) + \frac{\partial u}{\partial \ell_a}(x^*) = g_1(x), x \in \partial\Omega_+. \tag{2}$$

Problem 1.2. Find a harmonic function $u(x) \in C^2(\Omega) \cap C^1(\bar{\Omega})$, satisfying the conditions

$$u(x) + u(x^*) = g_0(x), x \in \partial\Omega_+, \tag{3}$$

$$\frac{\partial u}{\partial \ell_a}(x) - \frac{\partial u}{\partial \ell_a}(x^*) = g_1(x), x \in \partial\Omega_+. \tag{4}$$

We will call on Problem 1.1 as a periodic boundary value problem, and Problem 1.2 as an anti-periodic one. Note that Problems 1.1 and 1.2 for the case $a \equiv 0$ have been studied in [1–4].

If $x \in \Gamma$, then the point $x^* = (0, \alpha_2 x_2, ..., \alpha_n x_n)$ also belongs to Γ. Then from the boundary condition (1) it follows that

$$g_0(x) = u(x) - u(x^*)|_\Gamma = -[u(x^*) - u(x)]|_\Gamma = -g_0(x^*).$$

Therefore for the existence of a smooth solution to Problem 1.1 the following matching conditions are necessary:

$$g_0(0, x_2, ..., x_n) = -g_0(0, \alpha_2 x_2, ..., \alpha_n x_n), \tag{5}$$

$$\frac{\partial g_0}{\partial x_j}\left(0, x_2, ..., x_n\right) = -\frac{\partial g_0\left(0, \alpha_2 x_2, ..., \alpha_n x_n\right)}{\partial x_j}, j = 1, ...n, \tag{6}$$

$$g_1\left(0, x_2, ..., x_n\right) = g_1\left(0, \alpha_2 x_2, ..., \alpha_n x_n\right). \tag{7}$$

Similar conditions are required for the existence of a smooth solution to Problem 1.2, namely,

$$g_0\left(0, x_2, ..., x_n\right) = g_0\left(0, \alpha_2 x_2, ..., \alpha_n x_n\right), \tag{8}$$

$$\frac{\partial g_0}{\partial x_j}\left(0, x_2, ..., x_n\right) = \frac{\partial g_0\left(0, \alpha_2 x_2, ..., \alpha_n x_n\right)}{\partial x_j}, j = 1, ...n, \tag{9}$$

$$g_1\left(0, x_2, ..., x_n\right) = -g_1\left(0, \alpha_2 x_2, ..., \alpha_n x_n\right). \tag{10}$$

2 Auxiliary statements

In this section we give some auxiliary statements related to properties of mapping $Sx = x^*$.

It is obvious that if $x \in \Omega$, or $x \in \partial\Omega$, then $x^* = Sx \in \Omega$, or $x^* = Sx \in \partial\Omega$. Let

$$\Lambda u\left(x\right) = \sum_{j=1}^{n} x_j \frac{\partial u(x)}{\partial x_j}, \frac{\partial u}{\partial a}\left(x\right) = \sum_{j=1}^{n} a_j \frac{\partial u}{\partial x_j}\left(x\right).$$

The following statement is true.

Lemma 2.1. *Let $u\left(x\right)$ be a smooth function in the domain Ω. Then the following equalities hold:*

$$\frac{\partial}{\partial a} I_S u\left(x\right) = \sum_{i=1}^{n} \alpha_i a_i I_S u_{x_i}\left(x\right), \tag{11}$$

$$\Lambda I_S u\left(x\right) = I_S \Lambda u\left(x\right), \tag{12}$$

$$\Delta I_S u\left(x\right) = I_S \Delta u\left(x\right). \tag{13}$$

Proof. Since

$$\frac{\partial}{\partial x_i} I_S u\left(x\right) = \frac{\partial}{\partial x_i} u\left(\alpha_1 x_1, ..., \alpha_n x_n\right) = \alpha_i I_S u_{x_i}\left(x\right),$$

then by the definition of the derivative $\frac{\partial u}{\partial a}(x)$, we get

$$\frac{\partial}{\partial a} I_S u\left(x\right) = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} I_S\left(x\right) = \sum_{i=1}^{n} \alpha_i a_i I_S u_{x_i}\left(x\right).$$

Further,

$$\Lambda I_S u(x) = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} u\left(\alpha_1 x_1, ..., \alpha_n x_n\right) = \sum_{i=1}^{n} \alpha_i x_i I_S u_{x_i}(x) = I_S \Lambda u(x).$$

Finally, equalities

$$\frac{\partial^2}{\partial x_i^2} I_S u(x) = \frac{\partial}{\partial x_i} \alpha_i I_S u_{x_i}(x) = \alpha_i \frac{\partial}{\partial x_i} I_S u_{x_i}(x) = \alpha_i^2 I_S u_{x_i x_i}(x) = I_S u_{x_i x_i}(x)$$

imply

$$\Delta I_S u(x) = \sum_{i=1}^{n} I_S u_{x_i x_i}(x) = I_S \sum_{i=1}^{n} u_{x_i x_i}(x) = I_S \Delta u(x).$$

□

Corollary 2.2. *If $u(x)$ is a harmonic function in the domain Ω, then functions $\Lambda I_S u\left(x\right)$ and $\frac{\partial}{\partial \ell_a} I_S u\left(x\right)$ are also harmonic in Ω.*

Further, for any function $u\left(x\right)$ given in the domain $\bar{\Omega}$ we denote

$$v(x) = \frac{u(x) + u(x^*)}{2}, w = \frac{u(x) - u(x^*)}{2}. \tag{14}$$

It is obvious that $u(x) = v(x) + w(x)$. Moreover, we have

$$v(x) = I_S v(x), w(x) = -I_S w(x), x \in \bar{\Omega}. \tag{15}$$

The following statement is true.

Lemma 2.3. *Let $u\left(x\right)$ be a smooth function in the domain $\bar{\Omega}$. Then*

$$\frac{\partial}{\partial \ell_a} v(x) + \frac{\partial}{\partial \ell_a} v(x^*) = 2 \sum_{i=1}^{n} \left(x_i - \frac{1 + \alpha_i}{2} a_i \right) v_{x_i}(x),\ x \in \bar{\Omega}\,, \tag{16}$$

$$\frac{\partial}{\partial \ell_a} v(x) - \frac{\partial}{\partial \ell_a} v(x^*) = \sum_{i=1}^{n} \left(1 - \alpha_i\right) a_i v_{x_i}(x), x \in \bar{\Omega}, \tag{17}$$

$$\frac{\partial}{\partial \ell_a} w\left(x\right) + \frac{\partial}{\partial \ell_a} w\left(x^*\right) = -\sum_{i=1}^{n} \left(1 - \alpha_i\right) a_i w_{x_i}\left(x\right),\ x \in \bar{\Omega}, \tag{18}$$

$$\frac{\partial}{\partial \ell_a} w(x) - \frac{\partial}{\partial \ell_a} w(x^*) = 2 \sum_{i=1}^{n} \left(x_i - \frac{1 + \alpha_i}{2} a_i \right) w_{x_i}(x),\ x \in \bar{\Omega}\,. \tag{19}$$

Proof. Since the operators Λ and I_S commute, then from the first equality of formula (15) for all $x \in \bar{\Omega}$ we have

$$\frac{\partial}{\partial \ell_a} v(x) + I_S \frac{\partial}{\partial \ell_a} v(x) = \frac{\partial}{\partial \ell_a} I_S v(x) + I_S \frac{\partial}{\partial \ell_a} v(x)$$

$$= \left(\Lambda - \frac{\partial}{\partial a}\right) I_S v(x) + I_S \left(\Lambda - \frac{\partial}{\partial a}\right) v(x) = 2 I_S \Lambda v(x) - I_S \frac{\partial}{\partial a} v(x) - \frac{\partial}{\partial a} I_S v(x)$$

$$= I_S \left(\sum_{i=1}^{n} 2 x_i v_{x_i}(x)\right) - I_S \left(\sum_{i=1}^{n} (1+\alpha_i) a_i v_{x_i}(x)\right)$$

$$= 2 I_S \left[\sum_{i=1}^{n} \left(x_i - \frac{1+\alpha_i}{2} a_i\right) v_{x_i}(x)\right].$$

Hence, for the function $v(x)$ we get

$$\frac{\partial}{\partial \ell_a} v(x) + I_S \frac{\partial}{\partial \ell_a} v(x) = 2 I_S \left[\sum_{i=1}^{n} \left(x_i - \frac{1+\alpha_i}{2} a_i\right) v_{x_i}(x)\right].$$

Applying the operator I_S to the both side of this equality, we obtain

$$\frac{\partial}{\partial \ell_a} v(x) + I_S \frac{\partial}{\partial \ell_a} v(x) = 2 \left[\sum_{i=1}^{n} \left(x_i - \frac{1+\alpha_i}{2} a_i\right) v_{x_i}(x)\right].$$

This yields (16). By the similar way, we get the equality

$$\frac{\partial}{\partial \ell_a} v(x) - I_S \frac{\partial}{\partial \ell_a} v(x) = -I_S \left(\sum_{i=1}^{n} (1-\alpha_i) a_i v_{x_i}(x)\right).$$

Then we have

$$\frac{\partial}{\partial \ell_a} v(x) - I_S \frac{\partial}{\partial \ell_a} v(x) = \sum_{i=1}^{n} (1-\alpha_i) a_i v_{x_i}(x).$$

Equality (17) is proved.

Now we prove the equalities (18) and (19). From the second equality of formula (15) for all $x \in \bar{\Omega}$ we get $\Lambda w(x) = -\Lambda I_S w(x) = -I_S \Lambda w(x)$ and $\frac{\partial}{\partial a} w(x) = -\sum_{i=1}^{n} a_i \alpha_i I_S w_{x_i}(x)$. Then

$$\frac{\partial}{\partial \ell_a} w(x) + I_S \frac{\partial}{\partial \ell_a} w(x) = \Lambda w(x) - \frac{\partial}{\partial a} w(x) + I_S \Lambda w(x) - I_S \frac{\partial}{\partial a} w(x)$$

$$= -I_S \Lambda w(x) + I_S \Lambda w(x) + \sum_{i=1}^{n} a_i \alpha_i I_S w_{x_i}(x) - \sum_{i=1}^{n} a_i I_S w_{x_i}(x)$$

$$= -I_S \left(\sum_{i=1}^{n} (1-\alpha_i) a_i w_{x_i}(x) \right).$$

Consequently,

$$\frac{\partial}{\partial \ell_a} w(x) + I_S \frac{\partial}{\partial \ell_a} w(x) = -I_S \left(\sum_{i=1}^{n} (1-\alpha_i) a_i w_{x_i}(x) \right).$$

Applying the operator I_S to the both side of this equality, we have

$$\frac{\partial}{\partial \ell_a} w(x) + I_S \frac{\partial}{\partial \ell_a} w(x) = -\sum_{i=1}^{n} (1-\alpha_i) a_i w_{x_i}(x).$$

Thus, (18) is true. Similarly,

$$\frac{\partial}{\partial \ell_a} w(x) - I_S \frac{\partial}{\partial \ell_a} w(x) = -\frac{\partial}{\partial \ell_a} I_S w(x) - I_S \frac{\partial}{\partial \ell_a} w(x) = -2\Lambda w(x)$$

$$+\frac{\partial}{\partial a} w(x) + I_S \frac{\partial}{\partial a} w(x) = -I_S \left(\sum_{i=1}^{n} 2x_i v_{x_i}(x) \right) - I_S \left(\sum_{i=1}^{n} (1+\alpha_i) a_i v_{x_i}(x) \right)$$

$$= -2I_S \left[\sum_{i=1}^{n} \left(x_i - \frac{1+\alpha_i}{2} a_i \right) v_{x_i}(x) \right],$$

i.e., the following equality holds:

$$\frac{\partial}{\partial \ell_a} w(x) - I_S \frac{\partial}{\partial \ell_a} w(x) = -2I_S \left[\sum_{i=1}^{n} \left(x_i - \frac{1+\alpha_i}{2} a_i \right) v_{x_i}(x) \right].$$

Hence, we get (19). □

Let $b_i = \frac{1}{2}(1+\alpha_i) a_i, i = 1, 2, ..., n$. Introduce a vector $b = (b_1, b_2, ..., b_n)$. If $a \in \Omega$, then $b \in \Omega$. Indeed, for any $i = 1, 2, ..., n$ we have the inequality $1 + \alpha_i \leq 2$. Therefore,

$$|b|^2 = \sum_{i=1}^{n} b_i^2 = \frac{1}{4} \sum_{i=1}^{n} (1+\alpha_i)^2 a_i^2 \leq \frac{1}{4} \left(\sum_{i=1}^{n} 4a_i^2 \right) = |a|^2 < 1 \Rightarrow b \in \boldsymbol{\Omega}.$$

Let

$$\frac{\partial u(x)}{\partial \ell_b} = (x_1 - b_1)\frac{\partial u(x)}{\partial x_1} + (x_2 - b_2)\frac{\partial u(x)}{\partial x_2} + ... + (x_n - b_n)\frac{\partial u(x)}{\partial x_n}.$$

Then equalities (16) and (19) can be rewritten as

$$(1 + I_S)\,\frac{\partial}{\partial \ell} v(x) = 2\frac{\partial v(x)}{\partial \ell_b}, \tag{20}$$

$$(1 + I_S)\,\frac{\partial}{\partial \ell} w(x) = 2\frac{\partial w(x)}{\partial l_b}. \tag{21}$$

3 On the generalized Neumann problem

We consider the following problem

$$\Delta v\,(x) = 0, x \in \Omega, \tag{22}$$

$$\frac{\partial v}{\partial \ell_a}\,(x) = h\,(x)\,, x \in \partial\Omega. \tag{23}$$

Solution of problem (22) -(23) is called a harmonic function $v\,(x)$ from the class $C^2\,(\Omega) \cap C\left(\bar{\Omega}\right)$ satisfying condition (23) in the classical sense.

Note that problem (22) - (23) in the case $n = 3, a = (0, 0, a_3)$ has been studied in [5]. If $a \in \Omega$, then the results, obtained in [5], carry over without changes for the general case. We give the main statement concerning to problem (22) -(23).

Theorem 3.1. *Let $a \in \Omega, h\,(x) \in C\,(\partial\Omega)$. Then for solvability of problem (22) - (23) it is necessary and sufficient that the following condition holds*

$$\int\limits_{\partial\Omega} \frac{1 - |a|^2}{|a - y|^n} h\,(y)\,ds_y = 0. \tag{24}$$

If a solution of the problem exists, then it is unique up to a constant term and is represented as

$$v\,(x) = \int\limits_0^1 t^{-1} w\,(a + t\,(x - a))\,dt,$$

where $w(x)$ is a solution of the following Dirichlet problem

$$\Delta w\,(x) = 0, x \in \Omega; w\,(x) = h\,(x)\,, x \in \partial\Omega, \tag{25}$$

moreover $w(a) = 0$.

4 Uniqueness of the solution of the main problem

We study uniqueness of a solution of Problem 1.1. The following statement is true.

Theorem 4.1. *If $a \in \Omega$ and a solution of Problem 1.1 exists, then it is unique up to a constant term.*

Proof. Let $u(x)$ be a solution of the homogenous problem 1.1. From the boundary condition (1) it follows that

$$u(x) = u(x^*), x \in \partial\Omega_+, u(x^*) = u(x), x \in \partial\Omega_-.$$

Consequently, $u(x) = u(x^*), x \in \partial\Omega$. Thus,

$$\frac{\partial}{\partial \ell_a} u(x) = \frac{\partial}{\partial \ell_a} I_S u(x), x \in \partial\Omega. \tag{26}$$

On the other hand, from the boundary condition (2) we obtain

$$\frac{\partial u(x)}{\partial \ell_a} = -I_S \left[\frac{\partial u(x)}{\partial \ell_a}\right], x \in \partial\Omega_+; I_S \left[\frac{\partial u(x)}{\partial \ell_a}\right] = -\frac{\partial u(x)}{\partial \ell_a}, x \in \partial\Omega_-.$$

Thus

$$\frac{\partial u(x)}{\partial \ell_a} = -I_S \left[\frac{\partial u(x)}{\partial \ell_a}\right], x \in \partial\Omega. \tag{27}$$

Then the equalities (26) and (27) imply

$$0 = I_S \frac{\partial}{\partial \ell_a} u(x) + \frac{\partial}{\partial \ell_a} I_S u(x), x \in \partial\Omega.$$

Further, using (15) for all $x \in \partial\Omega$, we have

$$I_S \frac{\partial}{\partial \ell_a} u(x) + \frac{\partial}{\partial \ell_a} I_S u(x) = 2 I_S \Lambda u(x) - \sum_{i=1}^{n} (1+\alpha_i) a_i I_S u_{x_i}(x)$$

$$= 2 I_S \sum_{i=1}^{n} \left[(x_i - b_i) u_{x_i}\right](x) = 2 I_S \frac{\partial u(x)}{\partial l_b}, x \in \partial\Omega.$$

Thus, for all $x \in \partial\Omega$ the following condition holds

$$I_S \frac{\partial u(x)}{\partial l_b} = 0 \Leftrightarrow \frac{\partial u(x)}{\partial l_b} = 0, x \in \partial\Omega. \tag{28}$$

Consequently, function $u(x)$ is a solution of the following problem

$$\Delta u(x) = 0, x \in \Omega; \frac{\partial u(x)}{\partial l_b} = 0, x \in \partial\Omega. \tag{29}$$

Since $b \in \Omega$, then by Theorem 1 the solution of problem (29) is the function $u(x) \equiv C, x \in \bar{\Omega}, C = const$. Theorem is proved. ◻

The following statement is proved similarly.

Theorem 4.2. *If $a \in \Omega$ and a solution of Problem 1.2 exists, then it is unique.*

5 Existence of the solution

In this section we study existence of solutions of Problems 1.1 and 1.2. Let $u(x)$ be a solution of Problem 1.1, and functions $v(x)$ and $w(x)$ be defined by (14). It is obvious that each of these functions is harmonic in the domain Ω. Further, the boundary condition (1) yields

$$w(x)|_{\partial\Omega_+} = \left.\frac{u(x)-u(x^*)}{2}\right|_{\partial\Omega_+} = \frac{1}{2}g_0(x),$$

$$w(x)\,|_{\partial\Omega_-} = -\left.\frac{u(x^*)-u(x)}{2}\right|_{x^*\in\partial\Omega_+} = -\frac{1}{2}g_0(x^*).$$

Introduce the function

$$2\tilde{g}_0(x) = \begin{cases} g_0(x), x \in \partial\Omega_+, \\ -g_0(x^*), x \in \partial\Omega_- \end{cases}.$$

Then $w(x)\,|_{\partial\Omega} = \tilde{g}_0\,(x)$. If the function $g_0\,(x)$ is smooth in the domain $\partial\Omega_+$, then by the matching conditions (5) - (6) the function $\tilde{g}_0\,(x)$ will have the same smoothness in the domain $\partial\Omega$. For example, if $g_0\,(x) \in C^{\lambda+1}\,(\partial\Omega_+)\,, 0<\lambda<1$, then $\tilde{g}_0\,(x) \in C^{\lambda+1}\,(\partial\Omega)$. Therefore, for the function $w(x)$ we get the following Dirichlet problem

$$\Delta w(x) = 0, x \in \Omega; w(x)\,|_{\partial\Omega} = \tilde{g}_0(x). \tag{30}$$

If $\tilde{g}_0\,(x) \in C^{\lambda+1}\,(\partial\Omega)$, then the solution of problem (30) exists, is unique and belongs to the class $C^{\lambda+1}(\bar{\Omega})$ (see e.g. [6]).

Further, since $v(x) = u(x) - w(x)$, then for all $x \in \bar{\Omega}$:

$$\frac{\partial}{\partial \ell_a}v(x) = \frac{\partial}{\partial \ell_a}u(x) - \frac{\partial}{\partial \ell_a}w(x) = \frac{\partial}{\partial \ell_a}u(x) + I_S\frac{\partial}{\partial \ell_a}u(x)$$

$$-I_S\frac{\partial}{\partial \ell_a}v(x) - I_S\frac{\partial}{\partial \ell_a}w(x) - \frac{\partial}{\partial \ell_a}w(x).$$

Hence

$$\frac{\partial}{\partial \ell_a}v(x) + I_S\frac{\partial}{\partial \ell_a}v(x) = (1+I_S)\,\frac{\partial}{\partial \ell_a}u(x) - (1+I_S)\,\frac{\partial}{\partial \ell_a}w(x).$$

Due to (16), we receive

$$\frac{\partial}{\partial \ell_a} v(x) + I_S \frac{\partial}{\partial \ell_a} v(x) = 2\frac{\partial v(x)}{\partial \ell_b}.$$

Further, if $x \in \partial\Omega_-$, then $x^* \in \partial\Omega_+$, and therefore, from the boundary condition (1) for the function $(1 + I_S)\frac{\partial}{\partial \ell_a} u(x)$ we get

$$(1 + I_S)\frac{\partial}{\partial \ell_a} u(x) = \begin{cases} g_0(x), x \in \partial\Omega_+, \\ g_0(x^*), x \in \partial\Omega_- \end{cases}.$$

By $\varphi(x)$ we denote a boundary value of the function $(1 + I_S)\frac{\partial}{\partial \ell_a} w(x)$. Then for $x \in \partial\Omega_+$ we obtain

$$\varphi(x) = \frac{\partial}{\partial \ell_a} w(x) + \frac{\partial}{\partial \ell_a} w(x^*)\Big|_{\partial\Omega_+}$$

and if $x \in \partial\Omega_-$, then $x^* \in \partial\Omega_+$ and

$$\varphi(x) = \frac{\partial}{\partial \ell_a} w(x) + \frac{\partial}{\partial \ell_a} w(x^*)\Big|_{\partial\Omega_-} = \frac{\partial}{\partial \ell_a} w(x^*) + \frac{\partial}{\partial \ell_a} w(x)\Big|_{x^* \in \partial\Omega_+} = \varphi(x^*).$$

Let

$$2\tilde{g}_1(x) = \begin{cases} g_1(x) - \varphi(x), x \in \partial\Omega_+, \\ g_1(x^*) - \varphi(x^*), x \in \partial\Omega_- \end{cases}.$$

Then the function $v(x)$ satisfies conditions of the following problem

$$\Delta v(x) = 0, x \in \Omega; \frac{\partial v}{\partial \ell_b}(x)\Big|_{\partial\Omega} = \tilde{g}_1(x). \tag{31}$$

We investigate smoothness of the function $\tilde{g}_1(x)$. If $g_1(x) \in C(\partial\Omega_+)$, then by the matching condition (7) the function $\tilde{g}_1(x) \in C(\partial\Omega)$. Further, since $b \in \Omega$, then by Theorem 1 for the existence of the solution of problem (31) it is necessary and sufficient that the following condition holds

$$\int\limits_{\partial\Omega} \frac{1 - |b|^2}{|b - y|^n} \tilde{g}_1(y)\, ds_y = 0. \tag{32}$$

Using presentation of the function $\tilde{g}_1(x)$, we find

$$\int\limits_{\partial\Omega} \frac{1-|b|^2}{|b-y|^n} \tilde{g}_1(y)\, ds_y = \int\limits_{\partial\Omega_+} \frac{1-|b|^2}{|b-y|^n} [g_1(y) - \varphi(y)]\, ds_y$$

$$+ \int\limits_{\partial\Omega_-} \frac{1-|b|^2}{|b-y|^n} [g_1(y^*) - \varphi(y^*)]\, ds_y.$$

After changing variables, the last integral can be written in the following form

$$\int\limits_{\partial\Omega_-} \frac{1-|b|^2}{|b-y|^n} [g_1(y^*) - \varphi(y^*)]\, ds_y = \int\limits_{\partial\Omega_+} \frac{1-|b|^2}{|b-y^*|^n} [g_1(y) - \varphi(y)]\, ds_y.$$

On the other hand by definition

$$b_j = \frac{(1+\alpha_j)}{2} a_j = \begin{cases} a_j, \alpha_j = 1 \\ 0, \alpha_j = -1 \end{cases}.$$

Thus,

$$\begin{aligned} (b_j - y_j)^2 &= \begin{cases} (a_j - y_j)^2, \alpha_j = 1 \\ y_j^2, \alpha_j = -1 \end{cases}, \\ (b_j - \alpha_j y_j)^2 &= \begin{cases} (a_j - y_j)^2, \alpha_j = 1 \\ (\alpha_j y_j)^2 = y_j^2, \alpha_j = -1 \end{cases}. \end{aligned}$$

Consequently, $|b-y|^n = |b-y^*|^n$. Therefore, (32) can be rewritten as

$$\int\limits_{\partial\Omega_+} \frac{1-|b|^2}{|b-y|^n} [g_1(y) - \varphi(y)]\, ds_y = 0,$$

it means that

$$\int\limits_{\partial\Omega_+} \frac{1-|b|^2}{|b-y|^n} g_1(y) ds_y = \int\limits_{\partial\Omega_+} \frac{1-|b|^2}{|b-y|^n} \varphi(y) ds_y. \tag{33}$$

Therefore, we have proved the following statement.

Theorem 5.1. *Let* $|a| < 1, g_0(x) \in C^{\lambda+1}(\partial\Omega_+), g_1(x) \in C^{\lambda}(\partial\Omega_+), 0 < \lambda < 1$. *Then Problem 1.1 is solvable if and only if condition* (33) *holds. If a solution exists, then it is unique up to a constant term.*

Example 5.2. Suppose that in Problem 1.1: $g_0(x) \equiv 1$. Then $w(x) = 1, w_{x_j}(x) = 0$ and condition (33) can be rewritten in the form

$$\int\limits_{\partial\Omega_+} \frac{1-|b|^2}{|b-y|^n} g_1(y) ds_y = 0.$$

We turn to study the existence of a solution of Problem 1.2. The following statement is true.

Theorem 5.3. *Let* $|a| < 1, g_0(x) \in C^{\lambda+1}(\partial\Omega_+), g_1(x) \in C^{\lambda}(\partial\Omega_+), 0 < \lambda < 1$. *Then a solution of Problem 1.2 exists and is unique.*

Proof. Let $u(x)$ be a solution of Problem 1.2, and functions $v(x)$ and $w(x)$ are defined by (14). From (3) it follows that

$$v(x)|_{\partial\Omega_+} = \left.\frac{u(x)+u(x^*)}{2}\right|_{\partial\Omega_+} = \frac{1}{2} g_0(x),$$

$$v(x)\,|_{\partial\Omega_-} = \left.\frac{u(x^*)+u(x)}{2}\right|_{x^*\in\partial\Omega_+} = \frac{1}{2} g_0(x^*).$$

We introduce the function

$$2\tilde{g}_0(x) = \begin{cases} g_0(x), x \in \partial\Omega_+, \\ g_0(x^*), x \in \partial\Omega_- \end{cases}.$$

Then $w(x)\,|_{\partial\Omega} = \tilde{g}_0(x)$. If the function $g_0(x)$ is smooth in the domain $\partial\Omega_+$, then by the matching conditions (8) - (9) the function $\tilde{g}_0(x)$ will have the same smoothness in the domain $\partial\Omega$. Therefore, for the function $v(x)$ we get the following Dirichlet problem

$$\Delta v(x) = 0, x \in \Omega; v(x)\,|_{\partial\Omega} = \tilde{g}_0(x). \tag{34}$$

If $\tilde{g}_0(x) \in C^{\lambda+1}(\partial\Omega)$, then a solution of problem (34) exists, is unique and belongs to the class $C^{\lambda+1}\left(\bar{\Omega}\right)$. Further, since $w(x) = u(x) - v(x)$, then for any $x \in \bar{\Omega}$ we have

$$\frac{\partial}{\partial\ell_a} w(x) - I_S \frac{\partial}{\partial\ell_a} w(x) = (1-I_S)\frac{\partial}{\partial\ell_a} u(x) - (1-I_S)\frac{\partial}{\partial\ell_a} v(x).$$

Due to (19), we obtain

$$(1+I_S)\frac{\partial}{\partial\ell_a} w(x) = 2\frac{\partial w(x)}{\partial\ell_b}.$$

Further, from the boundary condition (3) we get

$$(1 - I_S)\,\frac{\partial}{\partial \ell_a} u(x) = \begin{cases} g_1(x), x \in \partial\Omega_+, \\ -g_1(x^*), x \in \partial\Omega_- \end{cases}.$$

If by $\psi(x)$ we denote a boundary value of the function $(1 - I_S)\,\frac{\partial}{\partial \ell_a} v(x)$, then for $x \in \partial\Omega_+$ we find

$$\psi(x) = \frac{\partial}{\partial \ell_a} v(x) - \frac{\partial}{\partial \ell_a} v(x^*)\Big|_{\partial\Omega_+}$$

and if $x \in \partial\Omega_-$, then $x^* \in \partial\Omega_+$, therefore

$$\begin{aligned} \psi(x) &= \frac{\partial}{\partial \ell_a} v(x) + \frac{\partial}{\partial \ell_a} v(x^*)\Big|_{\partial\Omega_-} \\ &= -\left[\frac{\partial}{\partial \ell_a} v(x^*) - \frac{\partial}{\partial \ell_a} v(x)\right]\Big|_{x^* \in \partial\Omega_+} = -\psi(x^*). \end{aligned}$$

Let

$$2\tilde{g}_1(x) = \begin{cases} g_1(x) - \psi(x), x \in \partial\Omega_+, \\ -g_1(x^*) + \psi(x^*), x \in \partial\Omega_- \end{cases}.$$

Then, the function $w(x)$ satisfies conditions of the problem

$$\Delta w(x) = 0, x \in \Omega; \ \frac{\partial w}{\partial \ell_b}(x)\Big|_{\partial\Omega} = \tilde{g}_1\,(x)\,. \tag{35}$$

If $g_1\,(x) \in C\,(\partial\Omega_+)$, then by the matching condition (9) the function $\tilde{g}_1\,(x) \in C\,(\partial\Omega)$. Further, since $b \in \Omega$, then by Theorem 3.1 for the existence of the solution of problem (35) it is necessary and sufficient that the following condition holds

$$\int_{\partial\Omega} \frac{1 - |b|^2}{|b - y|^n} \tilde{g}_1\,(y)\,ds_y = 0. \tag{36}$$

By using presentation of the function $\tilde{g}_1(x)$, we find

$$\int_{\partial\Omega} \frac{1 - |b|^2}{|b - y|^n} \tilde{g}_1\,(y)\,ds_y = \int_{\partial\Omega_+} \frac{1 - |b|^2}{|b - y|^n}\,[g_1(y) - \psi(y)]\,ds_y$$

$$+ \int_{\partial\Omega_-} \frac{1 - |b|^2}{|b - y|^n}\,[-g_1(y^*) + \psi(y^*)]\,ds_y.$$

After changing the variables the last integral can be written in the following form

$$\int\limits_{\partial\Omega_-} \frac{1-|b|^2}{|b-y|^n}\left[-g_1(y^*)+\psi(y^*)\right] ds_y = \int\limits_{\partial\Omega_+} \frac{1-|b|^2}{|b-y^*|^n}\left[-g_1(y)+\psi(y)\right] ds_y.$$

Then condition (32) can be rewritten as follows

$$\int\limits_{\partial\Omega} \frac{1-|b|^2}{|b-y|^n}\tilde{g}_1\left(y\right) ds_y = \int\limits_{\partial\Omega_+} \left[\frac{1-|b|^2}{|b-y|^n} - \frac{1-|b|^2}{|b-y^*|^n}\right]\left[g_1(y)-\psi(y)\right] ds_y = 0.$$

Since $|b-y|^n = |b-y^*|^n$, then this condition always holds. The solution of problem (35) is unique up to the constant value C. Since $w(x) = -w(x^*)$, then $C = 0$. □

Now we consider a case when solvability condition of Problem 1.1 can be simplified. The following statement is true.

Theorem 5.4. *Suppose that conditions of Theorem* 5.1 *hold and the coefficients* a_j *vanish, if in the mapping* $Sx = x^*$ *the parameters* α_j *take values* -1. *Then solvability condition of Problem 1.1 can be rewritten in the following form*

$$\int\limits_{\partial\Omega_+} \frac{1-|a|^2}{|a-y|^n} g_1\left(y\right) ds_y = 0. \tag{37}$$

Proof. Before, in the proof of Theorem 5.3 we have proved that if $\alpha_j = 1$, then the equality $b_j = a_j$ holds, and if $\alpha_j = -1$, then $b_j = 0$. Assume that in the mapping $Sx = x^*$ for some index j_0 we have the equality $\alpha_{j_0} = -1$. Then by condition of the theorem $a_{j_0} = 0$, and therefore $b_{j_0} = 0 = a_{j_0}$. Consequently, when conditions of the theorem hold we have $|b-y|^n = |a-y|^n = |a-y^*|^n$. Moreover, in this case from equality (18) it follows that

$$\varphi(x) = \frac{\partial}{\partial \ell} w(x) + \frac{\partial}{\partial \ell} w(x^*) = -\sum_{i=1}^{n} \left(1-\alpha_i\right) a_i w_{x_i}\left(x\right) = 0.$$

Then the solvability condition of Problem 1.1 can be rewritten as (37). □

Corollary 5.5. *If* $a = 0$, *then solvability condition of Problem 1.1 can be rewritten in the following form*

$$\int\limits_{\partial\Omega_+} g_1\left(x\right) ds_x = 0.$$

This condition (the case $a = 0$*) has been obtained in [1].*

Example 5.6. Suppose that in Problem 1.1 $g_0(x) \equiv 1$ and $x^* = -x$. Then $w(x) = 1, \varphi(x) = 0$ and $b_j = 0$. In this case by Theorem 5.1 it is possible to rewrite condition (33) as follows

$$\int\limits_{\partial\Omega_+} g_1(y)ds_y = 0.$$

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