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Finite difference method for a nonlinear fractional Schrödinger equation with Neumann condition

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Abstract. In this paper, a special case of nonlinear fractional Schrödinger equation with Neumann boundary condition is considered. Finite difference method is implemented to solve the nonlinear fractional Schrödinger problem with Neumann boundary condition. Previous theoretical results for the abstract form of the nonlinear fractional Schrödinger equation are revisited to derive new applications of these theorems on the nonlinear fractional Schrödinger problems with Neumann boundary condition. Consequently, first and second order of accuracy difference schemes are constructed for the nonlinear fractional Schrödinger problem with Neumann boundary condition. Stability analysis show that the constructed difference schemes are stable. Stability theorems for the stability of the nonlinear fractional Schrödinger problem with Neumann boundary condition are presented. Additionally, applications of the new theoretical results are presented on a one dimensional nonlinear fractional Schrödinger problem and a multidimensional nonlinear fractional Schrödinger problem with Neumann boundary conditions. Numerical results are presented on one and multidimensional nonlinear fractional Schrödinger problems with Neumann boundary conditions and different orders of derivatives in fractional derivative term. Numerical results support the validity and applicability of the theoretical results. Numerical results present the convergence rates are appropriate with the theoretical findings and construction of the difference schemes for the nonlinear fractional Schrödinger problem with Neumann boundary condition.

Keywords. Fractional derivative, convergence, Neumann boundary condition.

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1 Introduction

In literature, fractional Schrödinger equations exist with a fractional derivative appearing in time, space or both time and space variables [1–4]. In this study, we consider a nonlinear fractional Schrödinger equation which is established by adding a nonlinear term to the linear form of Schrödinger equation as

$$\begin{cases} i\frac{du}{dt} + Au = \int\limits_0^t f\left(s, D_s^{\alpha} u(s)\right) ds, 0 < t < T, 0 < \alpha < 1, \\ u(0) = 0. \end{cases} \tag{1}$$

Analysis of various Schrödinger equations in an abstract form exists in literature widely [5]. Here, we will consider some applications of the theoretical results for abstract nonlinear fractional Schrödinger equation (1) in [6].

In [6], H_α is introduced as the Banach space consisting of all abstract functions $v(t)$ having a fractional derivative of order α, defined on $[0,T]$ with values in H for which the following norm is finite:

$$\| v \|_{H^\alpha} = \max_{0\leq t\leq T} \| D^\alpha_{0+} v(t) \|_H + \max_{0\leq t\leq T} \|v(t)\|_H . \tag{2}$$

Additionally, the theorem on existence and uniqueness of a bounded solution for the abstract problem (1) is given below.

Theorem 1.1. *We assume that the following hypotheses hold:*

(i) *For any* $t \in [0,T], u_0(t) \in H^\alpha$ *and*

$$\|D^\alpha_t u_0(t)\|_H \leq M. \tag{3}$$

(ii) *The function* $f : [0,T] \times H^\alpha \to H$ *is continuous, that is*

$$\|f(t, D^\alpha_t u(t))\|_H \leq \bar{M} \tag{4}$$

in $[0,T] \times H^\alpha$ *and Lipschitz condition is satisfied with respect to t uniformly*

$$\|f(t, D^\alpha_t u) - f(t, D^\alpha_t v)\|_H \leq L\|D^\alpha_t u - D^\alpha_t v\|_H, \tag{5}$$

where $L, M, \bar{M}$ *are positive constants. Then, a unique bounded solution* $u(t)$ *for problem* (1) *exists in* H^α.

Furthermore, first and second order of accuracy difference schemes are established for abstract problem (1) in [6]. Discretization through time variable in N steps starts with denoting a step size $\tau = T/N$ where $[0,T]$ is the time interval for problem (1). The set of grid points is given as

$$[0,T]_\tau = \{t_k = k\tau, 0 \leq k \leq N, N\tau = T\}, \tag{6}$$

where $u_k = u(t_k), 0 \leq k \leq N$ is the approximate solution and $u(t)$ is the analytical solution of problem (1).

First order of accuracy DS for problem (1) is established as

$$\begin{cases} i\frac{u_k - u_{k-1}}{\tau} + Au_k = \tau \sum_{l=1}^{k} f(t_{l-1}, D^{1,\alpha}_\tau u_{l-1}), \\ t_l = l\tau, \ 1 \leq l \leq k \leq N, \ N\tau = T, \\ u_0 = 0 \end{cases} \tag{7}$$

where

$$D_\tau^{1,\alpha}u_k = \sum_{m=1}^{k} \frac{\Gamma(k-m-\alpha+1)}{\Gamma(1-\alpha)(k-m)!}\left(\frac{u_m - u_{m-1}}{\tau^\alpha}\right), 1 \le k \le N \tag{8}$$

and Crank-Nicolson method leads to second order of accuracy in time as

$$\begin{cases} i\frac{u_k - u_{k-1}}{\tau} + \frac{1}{2}Au_k + \frac{1}{2}Au_{k-1} = \tau\sum_{l=1}^{k} F_{l-1}\left(D_\tau^{2,\alpha}u_{l-1}\right), \\ t_l = l\tau,\ 1 \le l \le k \le N,\ N\tau = T, \\ u_0 = 0, \end{cases} \tag{9}$$

where [7]

$$D_\tau^{2,\alpha}u_k = \begin{cases} \frac{2\tau^{-\alpha}}{\Gamma(3-\alpha)}(u_1-u_0) + \tau^{1-\alpha}\frac{(-\alpha)}{\Gamma(3-\alpha)}u'(0),\ k=1, \\ \frac{2^{-\alpha}\tau^{-\alpha}}{\Gamma(3-\alpha)}\left\{(2+\alpha)\left(u_2 - 2u_1 + u_0\right)\right. \\ \left.+(4-2\alpha)\left(u_1-u_0\right)\right\}, k=2, \\ \sum_{m=1}^{k-1}\left\{\frac{\tau^{-\alpha}}{2\Gamma(2-\alpha)}\eta\left(k-m\right)\left(u_{m+1}-u_{m-1}\right)\right. \\ \left.+\frac{\zeta(k-m)}{(2-\alpha)}\frac{\left(u_{m+1}-2u_m+u_{m-1}\right)}{\tau^\alpha\Gamma(1-\alpha)}\right\} \\ +\frac{(k-m)\eta\left(k-m\right)}{(1-\alpha)}\frac{\left(u_{m+1}-2u_m+u_{m-1}\right)}{\tau^\alpha\Gamma(1-\alpha)} \\ +\frac{\tau^{-\alpha}(u_k-u_{k-2})}{2\Gamma(2-\alpha)} \\ +\frac{\tau^{-\alpha}(u_k-2u_{k-1}+u_{k-2})}{\Gamma(3-\alpha)},\ 3 \le k \le N, \end{cases} \tag{10}$$

$$\eta(r) = (r+1)^{1-\alpha} - r^{1-\alpha}, \tag{11}$$

$$\zeta(r) = r^{2-\alpha} - (r+1)^{2-\alpha} \tag{12}$$

and

$$\sum_{l=1}^{k} F_{l-1}\left(D^{2,\alpha}u_{l-1}\right) = \begin{cases} \frac{1}{2}\left(f(\frac{\tau}{2}, \frac{1}{2}\left(D_\tau^{2,\alpha}u_1 + D_\tau^{2,\alpha}u_0\right))\right. \\ \left. + f(0, D_\tau^{2,\alpha}u_0)\right), k = 1, \\ \sum\limits_{l=1}^{k-1} f(t_l, D_\tau^{2,\alpha}u_l) \\ +\frac{1}{2}f(t_{k-\frac{\tau}{2}}, \frac{1}{2}\left(D_\tau^{2,\alpha}u_k) + D_\tau^{2,\alpha}u_{k-1})\right) \\ +\frac{1}{2}f(0, D_\tau^{2,\alpha}u_0), 2 \le k \le N. \end{cases} \tag{13}$$

Furthermore, H_τ^α is introduced as the Banach space of all abstract mesh functions $v^\tau = \{v_k\}_{k=0}^N$ having a fractional difference derivative of order α, defined on $[0,T]_\tau$ with values in H for which the following norms are finite:

$$\| v^\tau \|_{H^\alpha} = \max_{0\le l\le N} \| D^{1,\alpha}v_l \|_H + \max_{0\le l\le N} \|v_l\|_H\,, \tag{14}$$

$$\| v^\tau \|_{H^\alpha} = \max_{0\le l\le N} \| D^{2,\alpha}v_l \|_H + \max_{0\le l\le N} \|v_l\|_H\,. \tag{15}$$

Stability theorem for difference schemes (34) and (35) is presented in [6] as follows:

Theorem 1.2. *Let the assumptions* (3)*,* (4) *and* (5) *be satisfied. Then, there exist unique solutions of DS* (7) *and DS* (9) *which are bounded in* H_τ^α *of uniformly with respect to* τ*.*

In the present paper, a mixed boundary value problem for the m-dimensional nonlinear fractional Schrödinger equation is considered with Neumann boundary conditions:

$$\begin{cases} i\frac{\partial u}{\partial t} - \sum_{r=1}^m (a_r(x)u_{x_r})_{x_r} = \int\limits_0^t f\left(s, D_s^\alpha u(s,x)\right) ds, \\ 0 < t < T, x = (x_1, \cdots, x_m) \in \Omega, \\ u(0,x) = 0,\ x \in \overline{\Omega}, \\ \partial u/\partial \overrightarrow{n} = 0,\ x \in S. \end{cases} \tag{16}$$

Here $a_r(x),\ x \in \Omega$ is a smooth function and $a_r(x) \ge a > 0$ for $1 \le r \le m$ where a is a constant. $\overrightarrow{n}$ denotes the normal to the boundary $S = \partial\Omega$.

As an application of Theorem 1.1, multidimensional problem with Neumann condition (16) is considered in the present work. Additionally, first and second order of accuracy difference schemes are established for problem (16). Stability of the constructed difference schemes are investigated in the light of Theorem 1.2. Numerical experiments are carried out on a mixed boundary value problem for a one-dimensional nonlinear fractional Schrödinger equation to show the effectiveness and applicability of the theoretical results.

2 Discretization of the problem

In order to discretize problem (16) with respect to space variable, we define the grid sets:

$$\begin{gathered}\overline{\Omega}_h=\{x_r=(h_1r_1,...,h_mr_m),r=(r_1,...,r_m),\\ 0\le r_j\le N_j,h_jN_j=1,j=1,...,m,\}\\ \Omega_h=\overline{\Omega}_h\cap\Omega,S_h=\overline{\Omega}_h\cap S.\end{gathered}\tag{17}$$

Then, the spatial derivative operator in problem (16) is replaced with the difference operator:

$$A_h^x u^h=-\sum_{r=1}^{m}(a_r(x)u^h_{\overline{x_r}})_{x_r,j_r}\tag{18}$$

acting in the space of grid functions $u^h(x)$, satisfying the conditions $D^hu^h(x)=0$ for all $x\in S_h$. Here, $D^hu^h(x)$ is an approximation of $\partial u/\partial\overrightarrow{n}$ which is formulated according to the desired order of accuracy [8]. Since A_h^x is a self-adjoint positive definite operator in $L_2(\overline{\Omega}_h)$ [9], the following initial value problem is achieved

$$\begin{cases} i\frac{\partial u^h(t,x)}{\partial t}+A_h^xu^h(t,x)=\int\limits_0^t f^h\left(s,D_s^\alpha u^h(s,x)\right)ds,\\ 0<t<T,x\in\overline{\Omega}_h,\\ u(0,x)=0,x\in\overline{\Omega}_h.\end{cases}\tag{19}$$

In order to discretize equation (19) with respect to time, we use approximation formulas (8) and (10) for fractional time derivative.

Implementing a similar approach with the one in [5] for approximation of parabolic equations, we get the first and second orders of accuracy difference schemes for problem (19) respectively as

$$\begin{cases} i\frac{u_k^h-u_{k-1}^h}{\tau}+A_h^xu_k^h=\sum_{l=1}^k f(t_{l-1},x,D_\tau^{1,\alpha}u_{l-1}^h)\tau,\ x\in\overline{\Omega}_h,\\ u_0^h(x)=0,\ x\in\overline{\Omega}_h\end{cases}\tag{20}$$

and

$$\begin{cases} i\frac{u_k^h-u_{k-1}^h}{\tau}+\frac{1}{2}A_h^x u_k^h+\frac{1}{2}A_h^x u_{k-1}^h=\tau\sum\limits_{l=1}^{k}F_{l-1}^h\left(D_\tau^{2,\alpha}u_{l-1}^h,x\right), \\ x\in\overline{\Omega}_h, t_k=k\tau,\ 1\le k\le N,\ N\tau=1, \\ u_0^h(x)=0,\ x\in\overline{\Omega}_h, \end{cases} \tag{21}$$

where

$$\sum_{l=1}^{k}F_{l-1}^h\left(D_\tau^{2,\alpha}u_{l-1}^h,x\right)=\begin{cases} \frac{1}{2}\left(f(\frac{\tau}{2},x,\frac{1}{2}\left(D_\tau^{2,\alpha}u_1^h+D_\tau^{2,\alpha}u_0^h\right))\right. \\ \left.+f(0,x,D_\tau^{2,\alpha}u_0^h)\right),k=1, \\ \sum\limits_{l=1}^{k-1}f(t_l,x,D_\tau^{2,\alpha}u_l^h) \\ +\frac{1}{2}f(t_{k-\frac{\tau}{2}},x,\frac{1}{2}\left(D_\tau^{2,\alpha}u_k^h+D_\tau^{2,\alpha}u_{k-1}^h\right)) \\ \\ +\frac{1}{2}f(0,x,D_\tau^{2,\alpha}u_0^h),2\le k\le N. \end{cases} \tag{22}$$

Before presenting the stability theorem on DSs (28) and (49), we define $C\left([0,T]_\tau,H\right)$ as the Banach space of the mesh functions v^τ defined on $[0,T]_\tau$ with values in H, equipped with the norm

$$\parallel v^\tau \parallel_{C([0,T]_\tau,H)}=\max_{0\le l\le N}\|v_l\|_H . \tag{23}$$

Theorem 2.1. *There exist unique solutions for DSs (28) and (49) which are bounded in $C^{(\alpha)}\left([0,T]_\tau,H\right)$ uniformly with respect to τ and h.*

The proof of Theorem 2.1 is established on Theorem 1.1, Theorem 1.2 and symmetry properties of the operator A_h^x which is specified by formula (18).

3 Applications on one and multidimensional problems

In the present part, we consider a one-dimensional and a multidimensional problem to apply the theoretical results in the previous section. The following one-dimensional problem is considered:

$$\begin{cases} i\frac{\partial u}{\partial t}-(a(x)u_x(t,x))_x+\delta u(t,x)=\int\limits_0^t f\left(s,D_s^\alpha u(s,x)\right)ds, \\ 0<t<T,0<x<1, \\ u(0,x)=0,\ x\in[0,1], \\ u_x(t,0)=u_x(t,1)=0,0\le t\le T \end{cases} \tag{24}$$

where $\delta > 0$. Firstly, the grid set for spatial discretization through one dimension is defined as:

$$[0,1]_h = \{x : x_r = rh, 0 \le r \le M, Mh = 1\}. \tag{25}$$

To the operator A originated by problem (24), we assign A_h^x difference operator as:

$$A_h^x \varphi^h(x) = \{-(a(x)\varphi_{\overline{x}})_{x,r} + \delta\varphi_r\}_1^{M-1}, \tag{26}$$

performing in the space of functions $\varphi^h(x) = \{\varphi^r\}_0^M$ with the property $\varphi^1 - \varphi^0 = 0$ and $\varphi^M - \varphi^{M-1} = 0$. As the first step, we perform a substitution with A_h^x and we get

$$\begin{cases} i\frac{du^h(t,x)}{dt} + A_h^x u^h(t,x) = \int_0^t f^h(s,x,D_s^\alpha u^h(s,x))ds, \\ 0 < t < 1,\ x \in [0,1]_h, \\ u^h(0,x) = 0,\ x \in [0,1]_h. \end{cases} \tag{27}$$

Secondly, we replace problem (27) by DSs (7) and (9) as

$$\begin{cases} i\frac{u_k^h - u_{k-1}^h}{\tau} + A_h^x u_k^h = \sum_{l=1}^k f(t_{l-1}, x, D_\tau^{1,\alpha} u_{l-1}^h)\tau,\ x \in [0,1]_h, \\ t_k = k\tau,\ 1 \le k \le N,\ N\tau = 1, \\ u_0^h(x) = 0,\ x \in [0,1]_h \end{cases} \tag{28}$$

and

$$\begin{cases} i\frac{u_k^h - u_{k-1}^h}{\tau} + \frac{1}{2}A_h^x u_k^h + \frac{1}{2}A_h^x u_{k-1}^h = \tau\sum\limits_{l=1}^k F_{l-1}^h\left(D_\tau^{2,\alpha} u_{l-1}^h, x\right), \\ x \in [0,1]_h, t_k = k\tau,\ 1 \le k \le N,\ N\tau = 1, \\ u_0^h(x) = 0,\ x \in [0,1]_h, \end{cases} \tag{29}$$

where $\sum\limits_{l=1}^k F_{l-1}^h\left(D_\tau^{2,\alpha} u_{l-1}^h, x\right)$ is defined by formula (22).

Theorem 3.1. *There exist unique solutions for DSs* (28) *and* (49) *which are bounded in* $C^{(\alpha)}([0,T]_\tau, L_2[0,1]_h)$ *uniformly with respect to* τ *and* h.

The proof of Theorem 3.1 is established on base of Theorem 1.1, Theorem 1.2 and symmetry properties of the operator A_h^x which is specified by formula (26).

As a more generalized form of the previous problem, we consider an m-dimensional Schrödinger problem with Neumann condition in boundary:

$$\begin{cases} i\frac{\partial u}{\partial t} - \sum_{r=1}^{m}(a_r(x)u_{x_r})_{x_r} = \int\limits_0^t f\left(s, D_s^\alpha u(s,x)\right)ds, \\ 0 < t < T, x = (x_1, \cdots, x_m) \in \Omega, \\ u(0,x) = 0,\ x \in \overline{\Omega}, \\ \partial u/\partial \overrightarrow{n} = 0,\ x \in S, \end{cases} \tag{30}$$

where $a_r(x),\ x \in \Omega$ is a smooth function and $a_r(x) \geq a > 0$ for $1 \leq r \leq m$. Here, a is a constant and $S = \partial\Omega$ is the boundary of the region. For spatial discretization, we define the grid sets as:

$$\begin{gathered} \overline{\Omega}_h = \{x_r = (h_1 r_1, ..., h_m r_m), r = (r_1, ..., r_m), \\ 0 \leq r_j \leq N_j, h_j N_j = 1, j = 1, ..., m\}, \\ \Omega_h = \overline{\Omega}_h \cap \Omega, S_h = \overline{\Omega}_h \cap S. \end{gathered} \tag{31}$$

To the operator A originated by problem (30), we assign the difference operator A_h^x by the formula

$$A_h^x u^h(x) = -\sum_{r=1}^{m}\left(a_r(x)u_{\bar{x}_r}^h\right)_{x_r, j_r} \tag{32}$$

which is a self-adjoint positive definite operator $L_2(\overline{\Omega}_h)$ performing in the space of functions $u^h(x)$, satisfying the conditions $D^h u^h(x) = 0$ ($\forall\ x \in S_h$). Performing a substitution with A_h^x, we arrive at the initial value problem

$$\begin{cases} i\frac{du^h(t,x)}{dt} + A_h^x u^h(t,x) = \int_0^t f^h(s, x, D_s^\alpha u^h(s,x))ds, \\ 0 < t < 1,\ x \in \overline{\Omega}_h, \\ u^h(0,x) = 0,\ x \in \overline{\Omega}_h. \end{cases} \tag{33}$$

We replace problem (30) by the following DSs:

$$\begin{cases} i\frac{u_k^h - u_{k-1}^h}{\tau} + A_h^x u_k^h = \sum_{l=1}^{k} f(t_{l-1}, x, D_\tau^{1,\alpha} u_{l-1}^h)\tau,\ x \in \overline{\Omega}_h, \\ t_k = k\tau,\ 1 \leq k \leq N,\ N\tau = 1, \\ u_0^h(x) = 0,\ x \in \overline{\Omega}_h \end{cases} \tag{34}$$

and

$$\begin{cases} i\frac{u_k^h - u_{k-1}^h}{\tau} + \frac{1}{2}A_h^x u_k^h + \frac{1}{2}A_h^x u_{k-1}^h = \tau\sum\limits_{l=1}^{k} F_{l-1}\left(D_\tau^{2,\alpha} u_{l-1}\right),\ x \in \overline{\Omega}_h, \\ t_k = k\tau,\ 1 \leq k \leq N,\ N\tau = 1, \\ u_0^h(x) = 0,\ x \in \overline{\Omega}_h, \end{cases} \tag{35}$$

where $\sum\limits_{l=1}^{k} F_{l-1}^{h}\left(D_{\tau}^{2,\alpha} u_{l-1}^{h}, x\right)$ is defined by formula (22).

Theorem 3.2. *There exist unique solutions for DSs (34) and (35) which are bounded in $C^{(\alpha)}\left([0,T]_{\tau}, L_2(\overline{\Omega}_h)\right)$ uniformly with respect to τ and h.*

The proof of Theorems 3.2 is established on base of Theorem 1.1, Theorem 1.2 and symmetry properties of the difference operator A_h^x specified by formula (32) and the following theorem in [10]:

Theorem 3.3. *The solutions of the elliptic difference problem*

$$\begin{cases} A_h^x u^h(x) = w^h(x),\ x \in \Omega_h, \\ D^h u^h(x) = 0,\ x \in S_h \end{cases}$$

satisfy the following coercivity inequality:

$$\sum_{r=1}^{m} \left\| u_{x_r \overline{x}_r, j_r}^{h} \right\|_{L_2(\overline{\Omega}_h)} \leq M \left\| w^h \right\|_{L_2(\overline{\Omega}_h)}.$$

4 Numerical analysis

We consider the following one-dimensional nonlinear FSDE with the exact solution $u = t^2 \cos(x)$:

$$\begin{aligned} & i\frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} = \int_0^t f(s,x,D^{\alpha}u(s,x))ds, \\ & f(t,x,D^{\alpha}u) = \left(\sin(D^{\alpha}u(t,x) - \tfrac{t^{2-\alpha}\cos x}{\Gamma(3-\alpha)}) + 2\,(i+t)\cos x\right), \\ & 0 < x < \pi, 0 < t < 1, \\ & u(0,x) = 0, 0 < x < \pi, \\ & u_x(t,0) = u_x(t,\pi) = 0, 0 < t < 1. \end{aligned} \tag{36}$$

Here, we implement iterated forms of first and second order of difference schemes (34) and (35) on a one dimensional nonlinear fractional Schrödinger problem as in [11] due to nonlinearity. Fully discrete difference schemes are achieved by implementation of central difference method for discretization of spatial operator. Here, we consider $\tau = 1/N$ and $h = \pi/M$.

The fully discrete difference scheme for (36) can be established by method (34) as

$$\begin{aligned}
&i\frac{u_k^{n,(p+1)}-u_{k-1}^{n,(p+1)}}{\tau}-\frac{u_k^{n+1,(p+1)}-2u_k^{n,(p+1)}+u_k^{n-1,(p+1)}}{h^2}=g_k^{n,(p)},\\
&1\leq n\leq M-1, 1\leq k\leq N,\\
&u_0^{n,(p+1)}=0, 1\leq n\leq M-1,\\
&u_k^{1,(p+1)}-u_k^{0,(p+1)}=u_k^{M,(p+1)}-u_k^{M-1,(p+1)}=0,\\
&1\leq k\leq N, 0\leq p\leq p_{max}-1.
\end{aligned} \tag{37}$$

Here, p_{max} is the maximum number of iterations needed to achieve desired minimum error desired for each step of iteration.

$$u_k^{n,(0)}=0, 0\leq k\leq N, 0\leq n\leq M \tag{38}$$

is considered as starting vector for the iteration of this experimental problem. Right hand side of the main equation is defined with implementing classical Riemann sum for integration as:

$$g_k^{n,(p)}=\tau\sum_{l=1}^{k}f^{(1)}\left(t_{l-1},x_n,D_\tau^{1,\alpha}u_{l-1}^{n,(p)}\right), \tag{39}$$

where

$$f^{(1)}(t_k,x_n,D_\tau^{1,\alpha}u_k^n)=\sin(D_\tau^{1,\alpha}u_k^n-\frac{2t_k^{2-\alpha}\cos x_n}{\Gamma(3-\alpha)})+2(i+t_k)\cos x_n. \tag{40}$$

Here, $D_\tau^{\alpha,1}$ is first order accurate formula for fractional derivative (8). Fully discrete difference scheme (37) leads to the following matrix equation:

$$AU^{n+1,(p+1)}+BU^{n,(p+1)}+CU^{n-1,(p+1)}=G^{n,(p)}, \tag{41}$$

where $G^{n,(0)}$ is computed by considering $U^{n,(0)}=\overrightarrow{0}$ as the initial guess for iteration. Furthermore, matrices can be described as follows:

$$\begin{aligned}
&A=[a_{i,j}]_{N\times N},\ \text{where}\\
&a_{i,i}=-1/(h^2)\ \text{for}\ 2\leq i\leq N,\\
&a_{i,j}=0\ \text{else};
\end{aligned} \tag{42}$$

$$\begin{aligned}
&B=[b_{i,j}]_{N\times N},\ \text{where}\ b_{1,1}=1,\\
&b_{i,i-1}=-i/\tau, b_{i,i}=i/\tau+2/(h^2)\ \text{for}\ 2\leq i\leq N,\\
&b_{i,j}=0\ \text{else};
\end{aligned} \tag{43}$$

$$U^{n,(p+1)} = [u_0^{n,(p+1)}, u_1^{n,(p+1)}, ..., u_N^{n,(p+1)}]^T, 1 \leq k \leq N, 0 \leq p \leq p_{max}, \quad (44)$$

$$G^{n,(p)} = [0, g_1^{n,(p)}, ..., g_N^{n,(p)}]^T, 1 \leq k \leq N, 0 \leq p \leq p_{max} \quad (45)$$

and $C = A$. Matrix equation (41) is solved with a modified Gauss elimination method applied in [6].

Similarly, fully discrete second order accurate difference scheme for (36) is established by method (35) as

$$\begin{cases} i\frac{u_k^{n,(p+1)}-u_{k-1}^{n,(p+1)}}{\tau} - \frac{u_k^{n+1,(p+1)}-2u_k^{n,(p+1)}+u_k^{n-1,(p+1)}}{2h^2} \\ -\frac{u_{k-1}^{n+1,(p+1)}-2u_{k-1}^{n,(p+1)}+u_{k-1}^{n-1,(p+1)}}{2h^2} = q_k^{n,(p)}, \\ 1 \leq n \leq M-1, 1 \leq k \leq N, \\ u_0^{n,(p+1)} = 0, 1 \leq n \leq M-1, \\ -3u_k^{0,(p+1)} + 4u_k^{1,(p+1)} - u_k^{2,(p+1)} = 0, \\ -3u_k^{M,(p+1)} + 4u_k^{M-1,(p+1)} - u_k^{M-2,(p+1)} = 0, \\ 1 \leq k \leq N, 0 \leq p \leq p_{max} - 1. \end{cases} \quad (46)$$

Here, p_{max} is the maximum number of iterations needed to achieve desired minimum error desired for each step of iteration.

$$u_k^{n,(0)} = 0, 0 \leq k \leq N, 0 \leq n \leq M \quad (47)$$

is considered as starting vector for the iteration of this experimental problem. Right hand side of the main equation is defined as:

$$q_k^{n,(p)} = \tau \sum_{l=1}^{k} F_{l-1}\left(D_\tau^{2,\alpha} u_{l-1}^{n,(p)}\right), \quad (48)$$

where $\sum_{l=1}^{k} F_{l-1}\left(D_\tau^{2,\alpha} u_{l-1}\right)$ is defined by formula (22) and

$$f(t_k, x_n, D_\tau^{2,\alpha} u_k^n) = \sin(D_\tau^{2,\alpha} u_k^n - \frac{2t_k^{2-\alpha}}{\Gamma(3-\alpha)} \cos x_n) + 2(i + t_k)\cos x_n. \quad (49)$$

Here, $D_\tau^{\alpha,2}$ is first order accurate formula for fractional derivative (10). Fully discrete difference scheme (46) leads to the following matrix equation:

$$AU^{n+1,(p+1)} + BU^{n,(p+1)} + CU^{n-1,(p+1)} = Q^{n,(p)}, \quad (50)$$

where $Q^{n,(0)}$ is computed by considering $U^{n,(0)} = \overrightarrow{0}$. Furthermore, matrices can be described as follows:

$$A = [a_{i,j}]_{N\times N}, \text{ where}$$

$$a_{i,i} = -1/(2h^2), a_{i,i-1} = -1/(2h^2) \text{ for } 2 \le i \le N, \quad (51)$$
$$a_{i,j} = 0 \text{ else;}$$

$$B = [b_{i,j}]_{NxN}, \text{ where } b_{1,1} = 1,$$
$$b_{i,i-1} = -i/\tau, b_{i,i} = i/\tau + 1/(h^2) \text{ for } 2 \le i \le N, \quad (52)$$
$$b_{i,j} = 0 \text{ else;}$$

$$U^{n,(p+1)} = [u_0^{n,(p+1)}, u_1^{n,(p+1)}, ..., u_N^{n,(p+1)}]^T, 1 \le k \le N, 0 \le p \le p_{max} - 1, \quad (53)$$

$$Q^{n,(p)} = [0, q_1^{n,(p)}, ..., q_N^{n,(p)}]^T, 1 \le k \le N, 0 \le p \le p_{max} - 1 \quad (54)$$

and $C = A$. Matrix equation (50) is solved with the same modified Gauss elimination method in [6]. Throughout the experiments, iterations terminate when the maximum absolute error between each iteration becomes less than 10^{-7} at an iteration step.

Table 1. Errors of solution for first order difference scheme (34) for problem (36), where $\alpha = 0.50$ when $h = 0.002\pi$

N	E_1	C_1	E_2	C_2
20	4.70×10^{-2}	0.99	5.99×10^{-2}	0.98
40	2.36×10^{-2}	1.00	3.04×10^{-2}	0.97
80	1.18×10^{-2}	0.95	1.55×10^{-2}	0.94
160	6.11×10^{-3}	-	8.10×10^{-3}	-

L_2 error-(E_1) and maximum error-(E_2) of first order difference scheme for problem (36) are reported in Table 1 and Table 2 for $\alpha = 0.5$ and $\alpha = 0.75$ respectively. Also, estimated rates of convergence are demonstrated in Table 1 and Table 2 for the time variable. A similar experiment is carried out for second order of accuracy difference scheme of problem (36) which is obtained by implementation of difference scheme (35) on problem (36). Numerical results are presented in Table 3 and Table 4 for $\alpha = 0.5$ and $\alpha = 0.75$ respectively. Here, it is useful to note that estimated rates of convergence are demonstrated in the tables for errors E_1 and E_2 as C_1 and C_2 respectively.

Table 2. Errors of solution for first order difference scheme (34) for problem (36), where $\alpha = 0.75$ when $h = 0.002\pi$

N	E_1	C_1	E_2	C_2
20	4.71×10^{-2}	1.00	5.99×10^{-2}	0.97
40	2.35×10^{-2}	0.99	3.06×10^{-2}	0.97
80	1.18×10^{-2}	0.96	1.56×10^{-2}	0.95
160	6.10×10^{-3}	-	8.10×10^{-3}	-

Numerical results support the convergence of solutions of the constructed first and second orders of accuracy difference schemes to exact solution of problem (36).

Table 3. Errors of solution for second order difference scheme (35) for problem (36), where $\alpha = 0.50$ when $h = 0.002\pi$

N	E_1	C_1	E_2	C_2
20	5.99×10^{-4}	1.99	7.53×10^{-4}	1.99
40	1.51×10^{-4}	2.00	1.90×10^{-4}	2.02
80	3.75×10^{-5}	2.08	4.69×10^{-5}	2.09
160	8.83×10^{-6}	-	1.10×10^{-5}	-

Table 4. Errors of solution for second order difference scheme (35) for problem (36), where $\alpha = 0.75$ when $h = 0.002\pi$

N	E_1	C_1	E_2	C_2
20	6.09×10^{-4}	1.99	7.64×10^{-4}	1.99
40	1.53×10^{-4}	2.02	1.92×10^{-4}	2.02
80	3.77×10^{-5}	2.09	4.73×10^{-5}	2.11
160	8.80×10^{-6}	-	1.09×10^{-5}	-

5 Conclusion

First and second orders of accuracy difference schemes are constructed for a mixed type of nonlinear fractional Schrödinger problem with Neumann boundary condition. Stability theorems are presented for constructed difference schemes. Numerical results support the applicability of the theoretical findings.

Bibliography

[1] M. Fei, N. Wang and C. Huang, A second-order implicit difference scheme for the nonlinear time-space fractional Schrödinger equation, *Appl. Numer. Math.* **153** (2020) 399–411.

[2] H. Emamirad and A. Rougirel, Time-fractional Schrödinger equation, *J. Evol. Equ.* **20(1)** (2020) 279–293.

[3] P. Gorka, H. Prado and D. J. Pons, The asymptotic behavior of the time fractional Schrödinger equation on Hilbert space, *J. Math. Phys.* **61(3)** (2020) Article Number: 031501.

[4] M. Zheng, F. Liu and Z. Jin, The global analysis on the spectral collocation method for time fractional Schrödinger equation, *Appl. Math. Comput.* **365** (2020) Article Number: UNSP 124689.

[5] A. Ashyralyev and P. E. Sobolevskii, *New Difference Schemes for Partial Differential Equations*, vol. 148 of Oper. Theory Adv. Appl., Birkhäuser, Basel, Boston, Berlin, 2004.

[6] A. Ashyralyev and B. Hicdurmaz, Multidimensional problems for nonlinear fractional Schrödinger differential and difference equations, *Math. Methods Appl. Sci.* **2019** (2019) 1–21, DOI:10.1002/mma.5866.

[7] A. Ashyralyev, N. Emirov and Z. Cakir, Well-posedness of fractional parabolic differential and difference equations with Dirichlet-Neumann conditions, *Electron. J. Differential Equations* **97** (2014) 1–17.

[8] C. Ashyralyyev and M. Dedeturk, Approximation of the inverse elliptic problem with mixed boundary value conditions and overdetermination, *Bound. Value Probl.* **2015** (2015) Article number:51, DOI=10.1186/s13661-015-0312-x.

[9] S. G. Krein, *Linear Differential Equations in Banach Space*, Nauka, Moskow, 1966.

[10] P. E. Sobolevskii, *Difference Methods for the Approximate Solution of Differential Equations*, Izdat. Voronezh. Gosud. Univ., Voronezh, 1975 (in Russian).

[11] B. Hicdurmaz, On the numerical analysis of a nonlinear fractional Schrödinger equation with Neumann boundary condition, *AIP Conference Proceedings* **2183** (2019), 070022, DOI: 10.1063/1.5136184.

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