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About uniformly Lindelöf spaces

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Abstract. The idea of Lindelöfness is one of the most important in General Topology. There have been some variants in defining uniformly Lindelöfness of uniform spaces. For example, uniform A-Lindelöfness in the sense of L.V. Aparina [1], uniform B-Lindelöfness in the sense of A.A. Borubaev [2], uniform I-Lindelöfness in the sense of D.R. Isbell [5].

In this paper we propose a new approach to the definition of a uniform analogue of Lindelöfness. We introduce and study uniformly Lindelöf spaces.

Keywords. Finitely-additive open covering, locally finite uniform covering, $\aleph_0$-boundedness, uniformly perfect mappings.

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1 Introduction

Throughout this work all uniform spaces are defined in terms of coverings, are assumed to be Hausdorff and mappings are uniformly continuous [4].

For coverings α and β of the set X, the symbol $\alpha \succ \beta$ means that the covering α is a refinement of the covering β, i.e. for any $A \in \alpha$ there exists $B \in \beta$ such that $A \subset B$ and, for coverings α and β of a set X, we have: $\alpha \wedge \beta = \{A \bigcap B : A \in \alpha, B \in \beta\}$. The covering α is finitely-additive if $\alpha^{\angle} = \alpha$, where $\alpha^{\angle} = \{\bigcup \alpha_0 : \alpha_0 \subset \alpha \text{ is finite}\}$. $\alpha(x) = \bigcup St(\alpha, x)$, $\mathrm{St}(\alpha, x) = \{A \in \alpha : A \ni x\}$, $x \in X$, $\alpha(M) = \bigcup \mathrm{St}(\alpha, M)$, $\mathrm{St}(\alpha, M) = \{A \in \alpha : A \bigcap M \neq \emptyset\}$, $M \subset X$.

For a uniform space (X, U) by τ_U we understand the topology generated by the uniformity U; for the Tychonoff space X by U_X we understand the universal uniformity.

A uniform space (X, U) is called:

(1) uniformly A-Lindelöf, if for each open covering α exist a countable uniform covering $\beta = \{B_n : n \in N\}$ and $\gamma \in U$ such that $\beta \succ \alpha^{\angle}$ and $\gamma(\bar{B}_n) \subset B_{n+1}$ for any $n \in N$ [1];

(2) uniformly B-Lindelöf, if it is both uniformly B-paracompact and $\aleph_0$-bounded [2];

(3) uniformly I- Lindelöf, if every uniform covering has a locally finite uniform refinement [5];

(4) uniformly A-paracompact, if every open covering has a locally finite uniform refinement [1];

(5) strongly uniformly A-paracompact, if every open covering has a star finite uniform refinement [4];

(6) uniformly B-paracompact, if for each finitely-additive open covering λ of (X, U) there exists a sequence $\{\alpha_i : i \in N\} \subset U$ of uniform coverings such that the following condition is fulfilled: for each point $x \in X$ there exist number $i \in N$ and $L \in \lambda$ such that $\alpha_i(x) \subset L$ (U) [2];

(7) $\aleph_0$-bounded if the uniformity U has a base consisting of countable coverings [2];

(8) strongly uniformly locally compact if the uniformity of U contains a locally finite uniform covering such that the closure of each its element is compact [1];

(9) uniformly locally Lindelöf if the uniformity of U contains a uniform covering whose closure of each element is Lindelöf [3];

For covering properties of uniform spaces close to $\aleph_0$-boundedness and Lindelöfness see [7].

A uniformly continuous mapping $f : (X, U) \to (Y, V)$ of uniform space (X, U) onto a uniform space (Y, V) is called:

(1) precompact, if for each $\alpha \in U$ there exist a uniform covering $\beta \in V$ and finite uniform covering $\gamma \in U$, such that $f^{-1}\beta \wedge \gamma \succ \alpha$[2];

(2) uniformly perfect, if it is both precompact and perfect [2];

(3) uniformly open, if f maps each open uniform covering $\alpha \in U$ to an open uniform covering $f\alpha \in V$ [2].

2 Uniformly Lindelöf spaces

Let (X, U) be a uniform space.

Definition 2.1. A uniform space (X, U) is said to be uniformly Lindelöf, if it is both uniformly A-paracompact and $\aleph_0$-bounded.

Proposition 2.2. *If (X, U) is a uniformly Lindelöf space, then its topological space (X, τ_U) is Lindelöf. Conversely, if (X, τ) is Lindelöf topological space, then the uniform space (X, U_X) is uniformly Lindelöf.*

Proof. Let α be an arbitrary finitely-additive open covering of (X, τ_U). Then for covering α there exists a locally finite uniform covering β such that $\beta \succ \alpha$. Since the space (X, U) is $\aleph_0$-bounded the covering β contains a countable uniform covering β_0. Then for any $i \in N$ we have $B_i \subset \bigcup_{j=1}^{k} A_j$. The system $\{B_i \bigcap A_j\}$, $i = 1, 2, ..., n, j = 1, 2, ..., k$ forms a countable open covering which is refinement of α. Consequently, the space (X, τ_U) is Lindelöf.

Conversely, if (X, τ) is Lindelöf, then the system of all open coverings forms a base of universal uniformity U_X of the space (X, τ). It follows from this that (X, U_X) is uniformly Lindelöf. □

Proposition 2.3. *Any compact space is uniformly Lindelöf.*

Proof. Any compact uniform space is precompact and uniformly A-paracompact. Consequently, the uniform space (X, U) is uniformly Lindelöf.

Any uniformly Lindelöf space is uniformly A-paracompact. □

Proposition 2.4. *Any closed subspace of a uniformly Lindelöf space is uniformly Lindelöf.*

Proof. Let (M, U_M) be a subspace of (X, U). Note that space (M, U_M) is $\aleph_0$-bounded. Let α_M be an arbitrary finitely-additive open covering of the subspace (M, U_M). Then there exits a finitely-additive open family α of (X, U), such that $\alpha \wedge \{M\} = \alpha_M$. Assume $\beta = \{\alpha, X \setminus M\}$. Obviously the covering β is a finitely-additive open covering of (X, U). Then there exists a locally finite uniform covering $\lambda \in U$ such that $\lambda \succ \beta$. Evidently, $\lambda_M \in U_M$. Hence $\lambda_M \succ \alpha_M$. It is easy to check that λ_M is a locally finite uniform covering of (M, U_M). Thus, (M, U_M) is uniformly Lindelöf . □

Corollary 2.5. *Any uniformly Lindelöf space is uniformly locally Lindelöf.*

Corollary 2.6. *Any strongly uniformly locally compact and $\aleph_0$-bounded space is uniformly Lindelöf.*

Corollary 2.7. *The real space R with natural uniformity is uniformly Lindelöf.*

Proof. It is clear that R is $\aleph_0$-bounded. Let α be a finitely-additive open covering of R. Put $\beta = \{(n-1, n+1) : n = 0, \pm 1, \pm 2, ...\}$. Then β is a locally finite uniform covering of R. Since $[n-1, n+1]$ is compact, there exists a finite family $\{A_1, A_2, ..., A_n\}$, $A_i \in \alpha$, $i = 1, 2, ..., n$ such that $(n-1, n+1) \subset [n-1, n+1] \subset \bigcup_{i=1}^{n} A_i$. Therefore, $\beta \succ \alpha^{\angle}$. Hence, R is uniformly Lindelöf. □

Proposition 2.8. *Any uniformly Lindelöf space is uniformly B-Lindelöf and any uniformly B-Lindelöf space is uniformly I-Lindelöf.*

Proof. Let (X, U) be a uniformly Lindelöf space. As it is known, every uniformly A-paracompact space is uniformly B-paracompact [4]. Hence, the uniform space (X, U) is uniformly B-Lindelöf.

Let (X, U) be a uniformly B-Lindelöf space. Then by the definition of uniformly B-Lindelöfness [2] we have (X, U) is a uniformly I-Lindelöf. □

Corollary 2.9. *Any uniformly Lindelöf space is uniformly I-Lindelöf.*

Theorem 2.10. *Any uniformly Lindelöf space is strongly uniformly A-paracompact.*

Proof. Let (X, U) be a uniformly Lindelöf space. Then from the facts that the space (X, U) is strongly uniformly A-paracompact if and only if it is uniformly A-paracompact and (X, τ_U) is strongly paracompact, one concludes that b(X, U) is strongly uniformly A-paracompact. □

Proposition 2.11. *Any uniformly Lindelöf space is complete.*

Proof. Let (X, U) be a uniformly Lindelöf space. The completeness of (X, U) follows from the fact that every strongly uniformly A-paracompact space is complete [4]. □

Noncomplete separable uniform space need not be uniformly Lindelöf. For example, the space $(0, 1)$ is uniformly B-Lindelöf, but not uniformly Lindelöf.

Lemma 2.12. *Let $f : (X, U) \to (Y, V)$ be a precompact mapping of a uniform space (X, U) to a uniform space (Y, V). If a space (Y, V) is $\aleph_0$-bounded, then (X, U) is also $\aleph_0$-bounded.*

Proof. Let $f : (X, U) \to (Y, V)$ be a precompact mapping of a uniform space (X, U) to a uniform space (Y, V) and $\alpha \in U$ be an arbitrary uniform covering of X. Then by virtue of the precompactness of f there exist such a countable covering $\beta \in V$ and a finite covering $\gamma \in U$ that $f^{-1}\beta \wedge \gamma \succ f^{-1}\alpha$. But the covering $f^{-1}\beta \wedge \gamma$ is countable. Therefore the space (X, U) is $\aleph_0$-bounded. □

Lemma 2.13. *Let $f : (X, U) \to (Y, V)$ be a uniformly continuous mapping of a uniform space (X, U) and (Y, V). If β is a locally finite uniform covering of the space (Y, V), then $f^{-1}\beta$ is a locally finite uniform covering of the space (X, U).*

Proof. By the conditions of the lemma for each point $x \in X$ there exists a neighborhood O_x such that O_x meets at most a finite number of elements of the covering β, i.e. exist elements $B_i \in \beta$, $i = 1, 2, ..., n$, such $O_x \subset \bigcup_{i=1}^{n} B_i$. As the mapping

f is uniformly continuous, the covering $f^{-1}\beta$ is uniform, i.e. $f^{-1}\beta \in U$. Hence $f^{-1}O_x \subset \bigcup_{i=1}^{n} f^{-1}B_i$,$f^{-1}B_i \in f^{-1}\beta$. So, $f^{-1}\beta$ is a locally finite uniform covering of (X, U). □

Lemma 2.14. *Let $f : (X, U) \to (Y, V)$ be a uniformly continuous mapping of a uniform space (X, U) to uniform space (Y, V). If a space (X, U) is $\aleph_0$-bounded, then (Y, V) is also $\aleph_0$-bounded.*

Proof. Let $\beta \in V$ be an arbitrary uniform covering. Then $f^{-1}\beta \in U$. According to Proposition 1.1.6. [2, p. 40] the covering $f^{-1}\beta$ has a countable subcovering $f^{-1}\beta_0$. Then β_0 is a countable subcovering of β. Therefore the space (Y, V) is $\aleph_0$-bounded. □

Theorem 2.15. *Let $f : (X, U) \to (Y, V)$ be a uniformly perfect mapping of a uniform space (X, U) onto a uniformly Lindelöf space (Y, V). Then (X, U) is uniformly Lindelöf.*

Proof. According to Lemma 12 the space (X, U) is $\aleph_0$-bounded. Let the uniform space (Y, V) be uniformly A-paracompact and α be an arbitrary finitely-additive open covering of (X, U). Then $\{f^{-1}y : y \in Y\} \succ \alpha$. Since the mapping f is closed, it follows that $\lambda = \{f^{\#}A : A \in \alpha\}$ is a finitely-additive open covering of (Y, V), $f^{\#}A = Y \setminus f(X \setminus A)$. By virtue of the Lindelöfness of the space (Y, V) there exist a locally finite covering $\beta \in V$ such that $\beta \succ \lambda$. Then $f^{-1}\beta \succ \lambda$ and according to Lemma 13 the covering $f^{-1}\beta$ is locally finite. So, (X, U) is uniformly Lindelöf. □

Proposition 2.16. *Let $f : (X, U) \to (Y, V)$ be a uniformly open mapping of a uniform space (X, U) onto a uniform space (Y, V). If (X, U) is a uniformly Lindelöf space, then (Y, V) is also uniformly Lindelöf.*

Proof. Let f be a uniformly open mapping of a uniform space (X, U) onto a uniform space (Y, V) and α be an arbitrary finitely-additive open covering of Y. Then $f^{-1}\alpha$ is a finitely-additive open covering of the space (X, U) and due to the uniform A-paracompactness where is a locally finite uniformly open covering $\beta \in U$ of $f^{-1}\alpha$. Since f is open, $f(\beta) \succ \alpha$. Then (Y, V) is uniformly A-paracompact. Note that (Y, V) is $\aleph_0$-bounded. So, the space (Y, V) is uniformly Lindelöf. □

Bibliography

[1] L.V. Aparina, Uniform Lindelöf spaces, *Trudy Mosk. Math. Obshestva* **57** (1996), 3-15 (in Russian).

[2] A.A. Borubaev, *Uniform Spaces and Uniformly Continuous Mappings*, Ilim, Frunze, 1990 (in Russian).

[3] D. Buhagiar and B. Pasynkov, On uniform paracompactness, *Czech Math. J.* **46(121)** (1996), 577-586.

[4] R. Engelking, *General Topology*, 2nd edition, vol. 6 of Sigma Series in Pure Mathematics, Heldermann Verlag, Berlin, 1989.

[5] J.R. Isbell, *Uniform Spaces*, American Mathematical Society, Providence, 1964.

[6] B.E. Kanetov, *Some Classes of Uniform Spaces and Uniformly Continuous Mappings*, Bishkek, 2013 (in Russian).

[7] L.D.R. Kočinac, Selection principles in uniform spaces, *Note Mat.* **22(2)** (2003/2004), 127–139.

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