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Some notes on complex symmetric operators

Marcos S. Ferreira

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Abstract. In this paper we show that every conjugation C on the Hardy-Hilbert space H^2 is of type $C = T^* \mathcal{J} T$, where T is an unitary operator and $\mathcal{J} f(z) = \overline{f(\overline{z})}$ with $f \in H^2$. Moreover we prove some relations of complex symmetry between the operators T and $|T|$, where $T = U|T|$ is the polar decomposition of bounded operator $T \in \mathcal{L}(\mathcal{H})$ on the separable Hilbert space $\mathcal{H}$.

Keywords. Hardy space, Toeplitz operator, complex symmetric operator, Aluthge transform.

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1 Introduction

Let $\mathcal{L}(\mathcal{H})$ be the space of bounded linear operators on a separable Hilbert space $\mathcal{H}$. A *conjugation* C on $\mathcal{H}$ is an antilinear operator $C : \mathcal{H} \to \mathcal{H}$ such that $C^2 = I$ and $\langle Cf, Cg \rangle = \langle g, f \rangle$, for all $f, g \in \mathcal{H}$. An operator $T \in \mathcal{L}(H)$ is said to be *complex symmetric* if there exists a conjugation C on H such that $CT = T^*C$ (we will often say that T is C-symmetric). Complex symmetric operators generalize the concept of symmetric matrices of linear algebra. Indeed, it is well known ([5, Lemma 1]) that given a conjugation C, there exists an orthonormal basis $\{f_n\}_{n=0}^{\infty}$ for $\mathcal{H}$ such that $Cf_n = f_n$. Hence, if T is C-symmetric then

$$\langle Tf_n, f_m \rangle = \langle Cf_m, CTf_n \rangle = \langle f_m, T^*Cf_n \rangle = \langle Tf_m, f_n \rangle, \tag{1}$$

that is, T has a symmetric matrix representation. The converse result is also true. That is, if there is an orthonormal basis such that T has a symmetric matrix representation, then T is complex symmetric.

The complex symmetric operators class was initially addressed by Garcia and Putinar [5, 6] and includes the normal operators, Hankel operators and Volterra integration operators.

Now, let L^2 be the Hilbert space on the unit circle $\mathbb{T}$ and let L^∞ be the Banach space of all essentially bounded functions on $\mathbb{T}$. It is known that $\left\{e^{in\theta} : n \in \mathbb{Z}\right\}$

is an orthonormal basis for L^2. The *Hardy-Hilbert space*, denoted by H^2, consists of all analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on the unit disk $\mathbb{D}$ such that $\sum_{n=0}^{\infty} |a_n|^2 < \infty$. It is clear that $\{z^n : n = 0, 1, 2, \ldots\}$ is an orthonormal basis for H^2.

For each $\phi \in L^\infty$, the *Toeplitz operator* $T_\phi : H^2 \to H^2$ is defined by

$$T_\phi f = P(\phi f),$$

for each $f \in H^2$, where $P : L^2 \to H^2$ is the orthogonal projection. The concept of Toeplitz operators was initiated by Brown and Halmos [1] and generalizes the concept of Toeplitz matrices.

In [7], Guo and Zhu raised the question of characterizing complex symmetric Toeplitz operators on H^2 in the unit disk. In order to obtain such characterization, Ko and Lee [8] introduced the family of conjugations $C_\lambda : H^2 \to H^2$, given by

$$C_\lambda f(z) = \overline{f(\lambda \overline{z})}$$

with $\lambda \in \mathbb{T}$ and proved the following result:

Theorem 1.1. *Let $\phi(z) = \sum_{n=-\infty}^{\infty} \widehat{\phi}(n) z^n \in L^\infty$. Then T_ϕ is C_λ-symmetric if, and only if, $\widehat{\phi}(-n) = \lambda^n \widehat{\phi}(n)$, for all $n \in \mathbb{Z}$.*

2 Canonical conjugations

Our first objective in this paper is to study relations between an arbitrary conjugation C on H^2 and the conjugation $\mathcal{J} f(z) = \overline{f(\overline{z})}$. Once the conjugation $\mathcal{J}$ is a kind of canonical conjugation on H^2, we observe a close relationship between conjugations of H^2 and conjugation $\mathcal{J}$, namely:

Theorem 2.1. *If C is an conjugation on H^2, then exists an unitary operator $T : H^2 \to H^2$ such that $TC = \mathcal{J}T$.*

Proof. Since C is an conjugation, there exists an orthonormal basis $\{f_n\}_{n=0}^{\infty}$ of H^2 such that $Cf_n = f_n$. Now, let $\{z^n\}_{n=0}^{\infty}$ the standard orthonormal basis of H^2 and the linear isomorphism $T : H^2 \to H^2$ given by

$$T\left(\sum_{n=0}^{\infty} a_n f_n\right) = \sum_{n=0}^{\infty} a_n z^n.$$

Note that $Tf_n = z^n$, for all $n \geq 0$, and therefore T is unitary. Now, for $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^2$, we get

$$\begin{aligned} \mathcal{J} f(z) &= \sum_{n=0}^{\infty} \overline{a_n} T(f_n) \\ &= T\left(\sum_{n=0}^{\infty} \overline{a_n} C f_n\right) \\ &= \left(TCT^{-1}\right) f(z), \end{aligned}$$

whence $\mathcal{J}T = TC$. □

The previous theorem says that all complex symmetric Toeplitz operator is unitarily equivalent to a $\mathcal{J}$-symmetric operator. Indeed, we have:

Remark 2.2. Let $T_\phi : H^2 \to H^2$ an Toeplitz operator. Observe that, if T_ϕ is C-symmetric, since the operator T of previous theorem is unitary, we have

$$\mathcal{J} = TCT^*,$$

therefore the operator $T_2 = TT_\phi T^*$ is $\mathcal{J}$-symmetric (see [5, p. 1291]). This shows that T_ϕ and T_2 are unitarily equivalent operators. Moreover, is obvious that, if T commutes with $\mathcal{J}$ or C, then $C = \mathcal{J}$.

The converse of the Theorem 2.1 is true and your proof is immediate.

Proposition 2.3. *If $T : H^2 \to H^2$ is an unitary operator, then $C = T^{-1}\mathcal{J}T$ is an conjugation on H^2.*

In short, the Theorem 2.1 and Proposition 2.3 tell us that:

Corollary 2.4. *If $T : H^2 \to H^2$ an linear isomorphism and $C = T^{-1}\mathcal{J}T$, then T is unitary if, and only if, C is a conjugation on H^2.*

We already know that every normal operator is complex symmetric and that the reciprocal in general is not true. However, for Toeplitz operators, by Theorem 1.1 we have that if T_ϕ is $\mathcal{J}$-symmetric, then T_ϕ is normal. Now note that if T_ϕ is normal not necessarily T_ϕ is $\mathcal{J}$-symmetric. In fact, if $\phi(z) = -\overline{z} + z$ then T_ϕ is normal, however is not $\mathcal{J}$-symmetric.

Matrix of operators $\mathcal{J}$-symmetric on H^2 are quite simple to determine. The proof of the next result is left to the reader.

Proposition 2.5. *Let $A \in \mathcal{L}\left(H^2\right)$. Then A is $\mathcal{J}$-symmetric if, and only if, the matrix of A with respect the canonical basis of H^2 is symmetric.*

3 Properties of complex symmetry

In the following, we present some properties of complex symmetry in Hilbert spaces. The first result gives us a way to get complex symmetric operators from another complex symmetric operator. First, we need some lemmas:

Lemma 3.1. *(*[6, Lemma 1]*) If C and J are conjugations on a Hilbert space $\mathcal{H}$, then $U = CJ$ is a unitary operator. Moreover, U is both C-symmetric and J-symmetric.*

Lemma 3.2. *(*[3, Lemma 2.2]*) If $U : \mathcal{H} \to \mathcal{H}$ is a unitary and complex symmetric operator with conjugation C, then UC is a conjugation.*

Proposition 3.3. *Let $T : \mathcal{H} \to \mathcal{H}$ an operator and C and J conjugations on $\mathcal{H}$. Then T is C-symmetric if, and only if, UT is UC-symmetric, where $U = CJ$.*

Proof. We already know that U is unitary and C and J-symmetric and that $UC = CJC$ is a conjugation, by Lemmas 3.1 and 3.2. Now since $U^* = U^{-1} = JC$ and T is C-symmetric, we have

$$UT\,(UC) = UTCU^* = UCT^*U^* = UC(UT)^*.$$

Reciprocally, suppose that $UC(UT)^* = UT(UC)$. Thus

$$\begin{aligned} CT^*U^* &= C(UT)^* \\ &= U^*UC(UT)^* \\ &= U^*UTUC \\ &= TUC \\ &= TCU^*, \end{aligned}$$

whence $CT^* = TC$. □

Lemma 3.4. *If $T : \mathcal{H} \to \mathcal{H}$ is both C-symmetric and J-symmetric, then T is both CJC-symmetric and JCJ-symmetric.*

Proof. By Lemma 3.1, we have that $U := CJ$ is unitary and C and J-symmetric. Hence, by Lemma 3.2, $UC = CJC$ is a conjugation on $\mathcal{H}$. Thus, since $CT = T^*C$ and $JT = T^*J$ we get

$$(CJC)\,T = C\,(TJ)\,C = T^*\,(CJC)\,,$$

and so T is CJC-symmetric. Analogous, we prove that T is JCJ-symmetric. □

Proposition 3.5. *If $T : \mathcal{H} \to \mathcal{H}$ is both C and J-symmetric, then TU is C-symmetric, where $U = CJ$.*

Proof. In fact, once T is both C-symmetric and J-symmetric, we have by Lemma 3.4 that T is CJC-symmetric and so

$$(TU)\,C = T\,(CJC) = CU^*T^* = C\,(TU)^*\,.$$

□

Proposition 3.6. *An operator $T : \mathcal{H} \to \mathcal{H}$ is C-symmetric if, and only if, $\mathcal{J}T^*C = (C\mathcal{J})^*T$, where $\mathcal{J}(\sum_{n=0}^{\infty}\alpha_n f_n) = \sum_{n=0}^{\infty}\overline{\alpha_n} f_n$ and $\{f_n\}_{n=0}^{\infty}$ is a orthonormal basis for $\mathcal{H}$.*

Proof. We already know that $U = C\mathcal{J}$ is unitary and both C and $\mathcal{J}$-symmetric. Now, note that

$$\mathcal{J}T^*C = (C\mathcal{J})^*T \Leftrightarrow UT^*C = CU^*T.$$

First see that if T is C-symmetric, then $UT^*C = U(CT) = (CU^*)T$. Reciprocally, we have

$$\begin{aligned} CT^* &= CU^*(UT^*C)C \\ &= CU^*(CU^*T)C \\ &= (UCCU^*)TC \\ &= TC. \end{aligned}$$

□

We end this section with two properties of complex symmetry which we leave the proof to the reader.

Proposition 3.7. *Let $U : \mathcal{H} \to \mathcal{H}$ an unitary operator J-symmetric. If T is an operator such that $UT^* = TU$ then:*
(i) $JT^* = T^*J \Leftrightarrow T$ is $UJ-$symmetric.
(ii) $UJT = TJU^* \Leftrightarrow T$ is $J-$symmetric.

4 Complex symmetry of Aluthge and Duggal transforms

Recall that the polar decomposition of an operator $T : \mathcal{H} \to \mathcal{H}$ is uniquely expressed by $T = U\,|T|$, where $|T| = \sqrt{T^*T}$ is a positive operator and U is a partial isometry such that $Ker(U) = Ker\,|U|$ and U maps cl($Ran\,|T|$) onto cl($Ran(T)$). In this case, the Aluthge and Duggal Transforms are given, respectively, by $\widetilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$ and $\widehat{T} = |T| U$.

We already known that the Aluthge transform of a complex symmetric operator is also complex symmetric (see [4, Theorem 1]). In this section we study relations between complex symmetry of T and $|T|$ with relation the conjugations C and J, as well as the operators $\widetilde{T}$ and $\widehat{T}$.

Proposition 4.1. *If T is complex symmetric, then $|T|$ is also complex symmetric.*

Proof. If $CT = T^*C$, we have by Remark of [4, Lemma 1] that $T = CJ|T|$, where J commutes with $|T|$. Thus, once that CJ is a unitary operator, follows that

$$J|T| = C(CJ|T|) = |T|^* (CJ)^* C = |T|^* J.$$

□

Corollary 4.2. *If T is complex symmetric, then $|T|$ is self-adjoint.*

Proposition 4.3. *Let C and J conjugations on $\mathcal{H}$ such that $T = CJ|T|$. If $|T|$ is C-symmetric, then T is also C-symmetric.*

Proof. First, let's show that $|T|$ is J-symmetric. In fact, see that

$$J(JC|T|) = C|T| = |T|^* C = (|T|^* CJ)J,$$

and so $JC|T|$ is J-symmetric. Thus, by Proposition 3.3, $|T|$ is J-symmetric. Therefore, it is enough to see that:

$$\begin{aligned} CT &= C(CJ|T|) \\ &= |T|^* J \\ &= (|T|^* JC)C \\ &= (CJ|T|)^* C \\ &= T^* C. \end{aligned}$$

□

Corollary 4.4. *Let $T = CJ|T|$. If $|T|$ is C-symmetric, then $\widehat{T} = T$.*

Corollary 4.5. *Let $T = CJ|T|$. Then $|T|$ is C-symmetric if, and only if, $\widehat{T}$ is J-symmetric.*

Proposition 4.6. *Let* $T = CJ\,|T|$*. If* $C\,|T| = |T|^*\,C$ *and* $CJ = JC$*, then* T *is* J*-symmetric.*

Proof. In fact, we have that

$$\begin{aligned} JT &= J(CJ\,|T|) \\ &= C\,|T| \\ &= |T|^*\,JJC \\ &= |T|^*\,JCJ \\ &= (CJ\,|T|)^*J \\ &= T^*J. \end{aligned}$$

□

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Author information

Marcos S. Ferreira, Departamento de Ciências Exatas e Tecnológicas, Universidade Estadual de Santa Cruz, Ilhéus, Bahia, Brasil.
E-mail: msferreira@uesc.br