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On $\rho-$statistical convergence of order α of sequences of function

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Abstract. In this study, by using definition of ρ-statistical convergence which was defined by Çakallı [5], we introduce the concepts of pointwise $w_{\rho}^{\alpha}(f)$-summability, pointwise ρ-statistical convergence of order α and uniform ρ-statistical convergence of order α sequences of real-valued functions. Also, we give some inclusion relations between these concepts.

Keywords. Statistical convergence, sequences of function, Cesàro summability.

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1 Introduction

In 1951, Steinhaus [32] and Fast [14] introduced the notion of statistical convergence and later in 1959, Schoenberg [27] reintroduced independently. Çakallı ([3–5]), Caserta et al. [6], Çolak [8], Connor [9], Et et al.[11], Fridy [16], Işık and Akbaş ([19–21]), Kolk [22], Mursaleen [23], Salat [26], Sakaoğlu Özgüç and Yurdakadim [25], Şengül et al. ([28–31]) and many others investigated some arguments related to this notion. The reader can refer to the monographs [15] and [24] on some new approaches and developments related to summability theory and its applications. Gökhan et al. ([17, 18]) introduced the definition of pointwise and uniform statistical convergence of sequences of real valued functions and Duman and Orhan [10] studied independently. Then, Çınar et al. [7] defined pointwise and uniform statistical convergence of order α for sequences of functions and pointwise λ and lacunary statistical convergence of order α for sequences of functions was introduced by Et et al. ([12, 13]).

Çolak [8] defined the α-density of a subset K of $\mathbb{N}$ as follows:

$$\delta_{\alpha}(K) = \lim_{n\to\infty} \frac{1}{n^{\alpha}} \left|\{ k \leq n : k \in K\}\right|$$

provided the limit exists, $\delta_{\alpha}(K)$ is said to be the α-density of a subset K, where α be a real number such that $0 < \alpha \leqslant 1$.

If $x = (x_k)$ is a sequence such that x_k satisfies property $P(k)$ for all k except a set of $\alpha-$density zero, then we say that x_k satisfies $P(k)$ for "almost all k according to α " and we denote this by "$a.a.k\ (\alpha)$ ".

A sequence $x = (x_k)$ is said to be strong ρ-convergent to ℓ if

$$\lim_{n\to\infty} \frac{1}{\rho_n} \sum_{k=1}^{n} |x_k - \ell| = 0,$$

holds [1].

A sequence $x = (x_k)$ is called $\rho-$statistically convergent to ℓ if

$$\lim_{n\to\infty} \frac{1}{\rho_n} |\{k \leq n : |x_k - \ell| \geq \varepsilon\}| = 0$$

for each $\varepsilon > 0$. We assume throughout here and after that $\rho = (\rho_n)$ is a non-decreasing sequence of positive real numbers tending to ∞ such that $\limsup_n \frac{\rho_n}{n} < \infty$, $\Delta\rho_n = O(1)$, and $\Delta\rho_n = \rho_{n+1} - \rho_n$ for each positive integer n. If $\rho = (\rho_n) = n$, then $\rho-$statistically convergence reduces to statistical convergence [5].

The $\rho-$density of order α of a set $K \subset \mathbb{N}$ is defined by

$$\delta_\rho^\alpha(K) = \lim_{n\to\infty} \frac{1}{\rho_n^\alpha} |\{\, k \leq n : k \in K\}|$$

provided this limit exists.

2 Main results

In this section we give the main results of this article.

Definition 2.1. Let $0 < \alpha \leqslant 1$. A sequence of functions $\{f_k\}$ is said to be strongly pointwise $\rho-$summable of order α (or pointwise $w_\rho^\alpha\,(f)\,-$summable), if there is a function f such that

$$\lim_{n\to\infty} \frac{1}{\rho_n^\alpha} \sum_{k=1, x\in E}^{n} |f_k(x) - f(x)| = 0.$$

In this case we write $w_\rho^\alpha\,(f) - \lim f_k(x) = f(x)$ on E.

If we consider $\alpha = 1$ and $\rho_n = n$ for all $n \in \mathbb{N}$, then the strongly pointwise ρ-summable of order α reduces to the strong Cesàro summability. We denote the set of all strongly pointwise $\rho-$summable of order α sequence of functions by $w_\rho^\alpha\,(f)$.

Definition 2.2. Let $0 < \alpha \leqslant 1$. A sequence of functions $\{f_k\}$ is called pointwise $\rho-$statistically convergent of order α (or pointwise $S_\rho^\alpha(f)-$statistical convergence) to the function f on a set E, if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\rho_n^\alpha} \left|\{k \le n : |f_k(x) - f(x)| \ge \varepsilon, \quad \text{for every x} \in \text{E}\}\right| = 0$$

i.e., for every $x \in E$,

$$|f_k(x) - f(x)| < \varepsilon, \quad a.a.k(\alpha)$$

It is denoted by $S_\rho^\alpha(f) - \lim f_k(x) = f(x)$ on E. $S_\rho^\alpha(f) - \lim f_k(x) = f(x)$ means that for every $\delta > 0$ there is an integer M such that

$$\frac{1}{\rho_n^\alpha} \left|\{k \le n : |f_k(x) - f(x)| \ge \varepsilon, \quad \text{for every x} \in \text{E}\}\right| < \delta$$

for all $n > M(= M(\varepsilon, \delta, x))$ and for each $\varepsilon > 0$. The set of all pointwise $\rho-$statistically convergent sequences of function of order α is denoted by $S_\rho^\alpha(f)$.

The ρ-statistical convergence of order α for a sequence of functions is well defined for $0 < \alpha \leqslant 1$, but it is not well defined for $\alpha > 1$. Let $\{f_k\}$ be defined as follows and $\rho_n = n + 1$

$$f_k(x) = \begin{cases} 1, & \text{if} \quad k = 2n \\ x^k, & \text{if} \quad k \neq 2n \end{cases}$$

$n = 1, 2, 3, ...$and $x \in [0, 1/2]$. Both

$$\lim_{n\to\infty} \frac{1}{(n+1)^\alpha} \left|\{k \le n : |f_k(x) - 1| \ge \varepsilon, \quad \text{for every x} \in \text{E}\}\right| = \lim_{n\to\infty} \frac{n}{2(n+1)^\alpha} = 0$$

and

$$\lim_{n\to\infty} \frac{1}{(n+1)^\alpha} \left|\{k \le n : |f_k(x) - 0| \ge \varepsilon, \quad \text{for every x} \in \text{E}\}\right| = \lim_{n\to\infty} \frac{n}{2(n+1)^\alpha} = 0$$

for $\alpha > 1$. So $S_\rho^\alpha(f) - \lim f_k(x) = 1$ and $S_\rho^\alpha(f) - \lim f_k(x) = 0$, but this is impossible.

Definition 2.3. Let $0 < \alpha \leqslant 1$. A sequence of functions $\{f_k\}$ is called uniformly $S_\rho^u(f)$-statistically convergent of order α (or uniformly ρ-statistical convergence) to the function f on a set E, if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\rho_n^\alpha} \left|\{k \le n : |f_k(x) - f(x)| \ge \varepsilon, \quad \text{for all x} \in \text{E}\}\right| = 0$$

i.e., for all $x \in E$,

$$|f_k(x) - f(x)| < \varepsilon, \quad a.a.k(\alpha).$$

It is denoted by $S_\rho^u(f) - \lim f_k(x) = f(x)$ uniformly on E or $S_\rho^{u,\alpha}(f) - \lim f_k(x) = f(x)$ on E. The set of all uniformly $\rho-$statistically convergent sequences of function of order α is denoted by $S_\rho^{u,\alpha}(f)$.

Theorem 2.4. *Let* $0 < \alpha \leqslant 1$ *and* $c \in \mathbb{R}$. *If* $S_\rho^\alpha(f) - \lim f_k(x) = f(x)$ *and* $S_\rho^\alpha(f) - \lim g_k(x) = g(x)$*, then, the following statements hold:*
(i) $S_\rho^\alpha(f) - \lim cf_k(x) = cf(x)$*,*
(ii) $S_\rho^\alpha(f) - \lim(f_k(x) + g_k(x)) = f(x) + g(x)$*.*

Proof. The proof of the theorem is obtained by using techniques Çınar, Karakaş and Et ([7], Theorem 3.2). □

It is easy to see that every convergent sequence of functions is statistically convergent of order $\alpha(0 < \alpha \leqslant 1)$, but the converse does not hold. For example the sequence $\{f_k\}$ defined by

$$f_k(x) = \begin{cases} 1, \quad \text{if} \quad k = n^2 \\ \frac{kx}{2+k^2x^2}, \quad \text{if} \quad k \neq n^2 \end{cases}$$

is statistically convergent of order α with $S^\alpha - \lim f_k(x) = 0$ for $\alpha > \frac{1}{2}$, but it is not convergent.

Theorem 2.5. *Let* $0 < \alpha \leqslant 1$. *If for each* $x \in E$, $S_\rho^\alpha(f) - \lim f_k(x) = f(x)$ *and* $S_\rho^\alpha(f) - \lim f_k(x) = g(x)$, *then* $f(x) = g(x)$.

Proof. Using the definition of pointwise $\rho-$statistically convergent of order α and classical techniques, we have the proof of theorem. □

Theorem 2.6. *Let* $0 < \alpha \leqslant 1$. *If* $\{g_k\}$ *is a convergent sequence of functions such that* $f_k = g_k$, $a.a.k\ (\alpha)$, *then* $\{f_k\}$ *is pointwise* $\rho-$*statistically convergent of order* α.

Proof. Suppose that for each $x \in E$

$$\frac{1}{\rho_n^\alpha} |\{k \leq n : f_k(x) \neq g_k(x)\}| = 0 \quad \text{and} \quad \lim_{k\to\infty} g_k(x) = f(x),$$

then for every $\varepsilon > 0$,

$$\{k \leq n : |f_k(x) - f(x)| \geq \varepsilon, \quad \text{for every x} \in \text{E}\}$$

$$\subseteq \{k \leq n : |g_k(x) - f(x)| \geq \varepsilon, \quad \text{for every x} \in \text{E}\} \cup \{k \leq n : f_k(x) \neq g_k(x)\}.$$

Therefore,

$$\frac{1}{\rho_n^\alpha} |\{k \leq n : |f_k(x) - f(x)| \geq \varepsilon, \quad \text{for every x} \in \text{E}\}| \tag{1}$$

$$\leqslant \frac{1}{\rho_n^\alpha} |\{k \leq n : |g_k(x) - f(x)| \geq \varepsilon, \quad \text{for every x} \in \text{E}\}| + \frac{1}{\rho_n^\alpha} |\{k \leq n : f_k(x) \neq g_k(x)\}|.$$

Since $\lim_{k\to\infty} g_k(x) = f(x)$ for each $x \in E$, the set

$$\{k \leq n : |g_k(x) - f(x)| \geq \varepsilon, \quad \text{for every x} \in \text{E}\}$$

contains finite number of integers and so

$$\frac{1}{\rho_n^\alpha} |\{k \leq n : |g_k(x) - f(x)| \geq \varepsilon, \quad \text{for every x} \in \text{E}\}| = 0.$$

Using inequality (1) we get for every $\varepsilon > 0$

$$\frac{1}{\rho_n^\alpha} |\{k \leq n : |f_k(x) - f(x)| \geq \varepsilon, \quad \text{for every x} \in \text{E}\}| = 0.$$

□

Theorem 2.7. *Let* $0 < \alpha \leqslant 1$. *If* $S_\rho^\alpha(f) - \lim f_k(x) = f(x)$, *then* $\{f_k\}$ *has a subsequence of function* $\{f_{k_i}\}$ *such that*

$$\lim_{i\to\infty} f_{k_i}(x) = f(x)$$

on E.

Proof. Proof of theorem is as an immediate consequence of Theorem 2.5. □

Theorem 2.8. *Let* $0 < \alpha \leqslant \beta \leqslant 1$. *Then* $S_\rho^\alpha(f) \subseteq S_\rho^\beta(f)$ *and this inclusion is strict for some* α *and* β *such that* $\alpha < \beta$.

Proof. The proof is similar to that of Theorem 3.6 in [7]. Therefore we omit it. □

Corollary 2.9. *Let* $0 < \alpha \leqslant 1$. *If a sequence of functions* $\{f_k\}$ *is pointwise* ρ*-statistically convergent of order* α*, to the function* f*, then it is pointwise* ρ*-statistically convergent to the function* f.

Theorem 2.10. *Let* $0 < \alpha \leqslant 1$. *If* $\rho_n \geq n$ *for all* $n \in \mathbb{N}$, *then* $S^{\alpha}(f) \subset S^{\alpha}_{\rho}(f)$.

Proof. If $S^{\alpha} - \lim f_k(x) = f(x)$ on E, then for every $\varepsilon > 0$ we have

$$\frac{1}{n^{\alpha}} \left|\{k \leq n : |f_k(x) - f(x)| \geq \varepsilon \quad \text{for every x} \in \text{E}\}\right|$$
$$= \frac{\rho_n^{\alpha}}{n^{\alpha}} \frac{1}{\rho_n^{\alpha}} \left|\{k \leq n : |f_k(x) - f(x)| \geq \varepsilon \quad \text{for every x} \in \text{E}\}\right|$$
$$\geqslant \frac{1}{\rho_n^{\alpha}} \left|\{k \leq n : |f_k(x) - f(x)| \geq \varepsilon \quad \text{for every x} \in \text{E}\}\right| .$$

This completes the proof of the theorem. □

Theorem 2.11. *Let* $\alpha \in (0, 1]$ *be any real number. A sequence of functions* $\{f_k\}$ *is pointwise* $\rho-$*statistically convergent of order* α, *to the function* f *if and only if there exists a subset* $K = \{k\} \subseteq \mathbb{N}$, *for each fixed* $x \in E$, $\delta^{\alpha}_{\rho}(K) = 1$ *and* $\lim_{k\to\infty} f_k(x) = f(x)$ *for each fixed* $x \in E$.

Proof. Let $S^{\alpha}_{\rho}(f) - \lim f_k(x) = f(x)$. For $r = 1, 2, ...$ put

$$K_r = \{k \in \mathbb{N} : |f_k(x) - f(x)| \geq \frac{1}{r}\}$$

and

$$K^*_r = \{k \in \mathbb{N} : |f_k(x) - f(x)| < \frac{1}{r}\}$$

for each fixed $x \in E$. Then, $\delta_{\rho}(K_r) = 0$ and

$$K^*_1 \supset K^*_2 \supset ... \supset K^*_i \supset K^*_{i+1} \supset ... \tag{2}$$

and

$$\delta^{\alpha}_{\rho}(K^*_r) = 1, \tag{3}$$

$r = 1, 2, ...$ for each fixed $x \in E$.

Now, we have to show that for $k \in K^*_r$, $\{f_k\}$ is convergent to f. Suppose that $\{f_k\}$ is not convergent to f. Therefore, there is $\varepsilon > 0$ such that

$$|f_k(x) - f(x)| \geqslant \varepsilon$$

for infinitely many terms and some $x \in E$. Let

$$K^*_{\varepsilon} = \{k \in \mathbb{N} : |f_k(x) - f(x)| < \varepsilon\}$$

and $\varepsilon > \frac{1}{r}$, $(r = 1, 2, ...)$. Then, $\delta_\rho^\alpha(K_\varepsilon^*) = 0$ and by (2) $K_r^* \subset (K_\varepsilon^*)$. Hence, $\delta_\rho^\alpha(K_r^*) = 0$ which contradicts (3). Therefore, $\{f_k\}$ is convergent to f.

Conversely, suppose that there exists a subset $K = \{k\} \subset \mathbb{N}$ for each fixed $x \in E$ such that $\delta_\rho^\alpha(K) = 1$ and $\lim_{k\to\infty} f_k(x) = f(x)$, i.e., there exist an $N(x, \varepsilon)$ such that for each fixed $x \in E$ and each $\varepsilon > 0$, $k \geqslant N$ implies $|f_k(x) - f(x)| < \varepsilon$. Now,

$$K_\varepsilon = \{k \in \mathbb{N} : |f_k(x) - f(x)| \geq \varepsilon\} \subseteq \mathbb{N} - \{k_{N+1}, k_{N+2}, ...\}$$

for each fixed $x \in E$. Therefore, $\delta_\rho^\alpha(K_\varepsilon) \leqslant 1 - 1 = 0$ for each fixed $x \in E$. Hence, $\{f_k\}$ is pointwise statistically convergent of order α to f. □

Theorem 2.12. *Let* $0 < \alpha \leqslant 1$. *Let* f *and* $\{f_k\}$ *for all* $k = 1, 2, ...$ *be continuous functions on* $E = [a, b] \subset \mathbb{R}$. *Then,* $S_\rho^{u,\alpha}(f) - \lim f_k(x) = f(x)$ *on* A *if and only if* $c_k \to 0\left(S_\rho^\alpha\right)$, *where* $c_k = \max_{x\in E} |f_k(x) - f(x)|$.

Proof. Assume that $\{f_k\}$, $\rho-$uniformly statistically convergent of order α to f on E. Since f and $\{f_k\}$ are continuous functions on E, so $|f_k(x) - f(x)|$ is continuous on E, for each $k \in \mathbb{N}$. It has absolute maximum value at some point $x_k \in E$, i.e., there exists $x_1, x_2, ... \in E$ such that $c_1 = |f_1(x_1) - f(x_1)|, c_2 = |f_2(x_2) - f(x_2)|, ...,$ etc. Hence, we may write $c_k = |f_k(x_k) - f(x_k)|$, $k = 1, 2, ...$ From the definition of uniform $\rho-$statistical convergence of order α, we have, for every $\varepsilon > 0$,

$$|f_k(x) - f(x)| < \varepsilon \quad a.a.k(\alpha).$$

Thus, $c_k \to 0\left(S_\rho^\alpha\right)$.

Now, we suppose that $c_k \to 0\left(S_\rho^\alpha\right)$. We let following set

$$B(\varepsilon) = \{k \leq n : \max_{x\in E} |f_k(x) - f(x)| \geqslant \varepsilon\},$$

for $\varepsilon > 0$. Then, by hypothesis we have $\delta(B(\varepsilon)) = 0$. Since for $\varepsilon > 0$

$$\max_{x\in E} |f_k(x) - f(x)| \geqslant |f_k(x) - f(x)| \geqslant \varepsilon$$

we have

$$\{k \leq n : |f_k(x) - f(x)| \geq \varepsilon, \quad \text{for all x} \in \text{E}\} \subseteq B(\varepsilon)$$

and so

$$\frac{1}{\rho_n^\alpha} |\{k \leq n : |f_k(x) - f(x)| \geq \varepsilon, \quad \text{for all x} \in \text{E}\}| = 0.$$

This completes the proof. □

Theorem 2.13. *Let* $0 < \alpha \leqslant \beta \leqslant 1$. *Then,* $w_\rho^\alpha(f) \subseteq w_\rho^\beta(f)$.

Proof. The proof is clear. To show that the inclusion is strict for some α and β such that $\alpha < \beta$, we define the sequence $\{f_k\}$ by

$$f_k(x) = \begin{cases} \frac{kx}{1+kx}, & \text{if} \quad k = n^2 \\ 0, & \text{if} \quad k \neq n^2, \end{cases} \quad x \in [1,2]$$

and consider $\rho_n = n$. Then,

$$\frac{1}{\rho_n^\alpha} \sum_{k=1, x \in A}^{n} |f_k(x) - f(x)| \leqslant \frac{\sqrt{n}}{n^\beta}$$

since $1/(n^{\beta-\frac{1}{2}}) \to 0$ as $n \to \infty$, then $w_\rho^\beta(f) - \lim f_k(x) = 0$, i.e., the sequence $\{f_k\}$ is strongly pointwise $\rho-$summable of order α for $\frac{1}{2} < \beta \leqslant 1$, but since

$$\frac{\sqrt{n}}{2n^\alpha} \leqslant \frac{1}{\rho_n^\alpha} \sum_{k=1, x \in A}^{n} |f_k(x) - f(x)|$$

and $\sqrt{n}/2n^\alpha \to \infty, n \to \infty$, the sequence $\{f_k\}$ is not strongly pointwise $\rho-$summable of order α for $0 < \alpha < \frac{1}{2}$. □

Corollary 2.14. *Let* $0 < \alpha \leqslant \beta \leqslant 1$. *Then, the following statements hold:*

(i) *if* $\alpha = \beta$, *then* $w_\rho^\alpha(f) = w_\rho^\beta(f)$;

(ii) $w_\rho^\alpha(f) \subseteq w_\rho(f)$ *for each* $\alpha \in (0, 1]$.

Theorem 2.15. *Let* $0 < \alpha \leqslant 1$. *If a sequence of functions* $\{f_k\}$ *is pointwise* $w_\rho^\alpha(f)$ *−summable to the function* f, *then it is pointwise* $\rho-$*statistical convergent of order* α *to the function* f.

Proof. The proof is similar to that of Theorem 3.12 in [7]. Therefore we omit it. □

Definition 2.16. Let $0 < \alpha \leqslant 1$. A sequence of functions $\{f_k\}$ is said to be pointwise $\rho-$statistically Cauchy sequence of order α provided that for every $\varepsilon > 0$, there exist a number $N = N(\varepsilon, x)$ such that

$$\frac{1}{\rho_n^\alpha} \left|\{k \leq n : |f_k(x) - f_N(x)| \geq \varepsilon, \quad \text{for every x} \in \text{E}\}\right| = 0.$$

Theorem 2.17. *Let* $0 < \alpha \leqslant 1$ *and let* $\{f_k\}$ *be a pointwise* $\rho-$*statistically Cauchy sequence of order* α *of sequence of functions. Then there exists a pointwise convergent sequence of functions* $\{g_k\}$ *such that* $f_k = g_k$ *for* $a.a.k\ (\alpha)$.

Proof. The proof is similar to that of Theorem 3.4 in [7]. Therefore we omit it. □

Theorem 2.18. *Let* $0 < \alpha \leqslant 1$. *The sequence* $\{f_k\}$ *is pointwise* $\rho-$*statistically convergent of order* α *if and only if* $\{f_k\}$ *is a pointwise* $\rho-$*statistically Cauchy sequence of order* α.

Proof. The first part of the proof is easy. Now we assume that $\{f_k\}$ is a pointwise $\rho-$statistically Cauchy sequence of order α of functions. By Theorem 2.17 there exists a pointwise convergent sequence $\{g_k\}$ such that $f_k = g_k$ for $a.a.k\ (\alpha)$. By Theorem 2.12 we have $S_\rho^\alpha(f) - \lim f_k(x) = f(x)$ for each $x \in E$. □

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