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Local estimates for functionals rotationally invariant with respect to the gradient

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Abstract. This paper concerns minimization problems from Calculus of Variations rotationally invariant with respect to the gradient. Inspired by properties associated with results which are valid for elliptic partial differential equations, it presents some local estimates nearby non extremum points as well as nearby extremum points for these problems, generalizing some results obtained by Arrigo Cellina, Vladimir V. Goncharov and myself. As a consequence, some local estimates are obtained for the difference between the supremum and the infimum of any solution to the problem considered.

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1 Introduction

In variational setting, if we consider minimization problems involving functionals of the type $\int_\Omega F(x, u(x), \|\nabla u(x)\|)\,\mathrm{d}x$, with $\Omega \subset \mathbb{R}^n$ an open bounded set, it is possible to prove some local estimates nearby nonextremum points were proved in [4], in the particular case when F is invariant with respect to a compact convex subset of Ω. In this paper,these estimates are a tool to prove the *Strong Maximum Principle* under the conditions of. strict convexity and nonsmoothness of f at the origin, where $f(\|\cdot\|) = F(\cdot)$. The *Strong Maximum Principle* states that if any nonnegative solution $\bar{u}$ to the problem of minimizing the given functional is equal to zero on some interior point of Ω, then $\bar{u} \equiv 0$ on Ω.

In [6], in the particular case when F is the sum (difference) of a rotationally invariant function depending only on the gradient, and a function depending on u, local estimates nearby nonextremum, as well as local estimates nearby extremum points, were proven. The argument used in both cases was inspired in duality arguments of Convex Analysis ([7]).

The focus of this paper is to present and prove local estimates nearby non-extremum, as well as extremum points, and to obtain, as consequent result, a local bound for the distance between the infimum and the supremum of any solution inside a ball.

2 Preliminary results

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set, $f : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\}$ convex, lower semicontinuous, such that $f(0) = 0$. Assume that $\partial f^*(0) = \{0\}$ and $\partial f(0) = [0, a], a > 0$; that is, f is strictly convex and not smooth at the origin. Let us now also define $\phi(t) = sup\{\partial f(t) : t \in domf\}$. In this paper we consider the minimization problem

$$(P) \qquad \min\left\{\int_\Omega f(\|\nabla u(x)\|)\,\mathrm{d}x : u(\cdot) \in u^0(\cdot) + W_0^{1,1}(\Omega)\right\}.$$

Let us recall some facts from convex analysis (see [7] for more insights). If

$$f : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\}$$

is a convex function such that $f(0) = 0$, we have that the conjugate, or polar, function $f^* : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\}$, defined by the formula

$$f^*(p) = \sup_x\{\langle x, p\rangle - f(x)\}, \tag{1}$$

is convex and such that $f^*(0) = 0$. If $domf \neq \{0\}$, then $domf^* \neq \{0\}$.

Let us recall the comparison result presented and proved by A. Cellina in [2]:

Theorem 2.1 (Comparison Result). *Let $\Omega \subset A_{R_1,R_2}(\bar{x})$ be an open bounded set, where $A_{R_1,R_2}(\bar{x}) = \{x \in \mathbb{R}^n : R_2 < \|x - \bar{x}\| < R_2\}$. Consider $f : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\}$ convex, lower semicontinuous and such that $f(0) = 0$, $\partial f^*(0) = \{0\}$ and $\partial f(0) = [0, a], a > 0$. Consider also the differentiable function $R : [R_1, R_2] \to \mathbb{R}$ such that, for some $M > 0$,*

$$\frac{M}{r^{n-1}} \in \partial f(|R'(r)|) \text{ for every } r \in [R_1, R_2].$$

Let $\bar{u}$ be a continuous solution to (P).

(a) *If*

$$\bar{u} \leq R(\|x - \bar{x}\|) \text{ a.e. on } \partial A_{R_1,R_2}(\bar{x}),$$

then

$$\bar{u} \leq R(\|x - \bar{x}\|) \text{ a.e. on } A_{R_1,R_2}(\bar{x}).$$

(b) *If*

$$\bar{u} \geq R(\|x - \bar{x}\|) \text{ a.e. on } \partial A_{R_1,R_2}(\bar{x}),$$

then

$$\bar{u} \geq R(\|x - \bar{x}\|) \text{ a.e. on } A_{R_1,R_2}(\bar{x}).$$

3 Estimates nearby non extremum points

In this section we prove some local estimates nearby non extremum points. We start by recalling the result presented and proved in [4], where some local estimates are obtained near points far away from extremum ones, in the particular case when f is invariant with respect to the norm.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set. Consider $f : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\}$ convex, lower semicontinuous and such that $f(0) = 0$, $\partial f^*(0) = \{0\}$ and $\partial f(0) = [0, a], a > 0$. Let $\bar{u}$ be a continuous solution to (P). Let $\bar{x} \in \Omega$ and $R > 0$ be such that $B_R(\bar{x}) \subset \Omega$, and consider $\mu_1 = \inf_{B_R(\bar{x})} \bar{u}(x)$ and $\mu_2 = \sup_{B_R(\bar{x})} \bar{u}(x)$.*

(a) *If $\bar{u}(\bar{x}) < \mu_2 - aR$, then there exists $\eta_2 > 0$ such that*

$$\bar{u}(x) \leq \mu_2 - \phi(\eta_2)(R - \|x - \bar{x}\|) \qquad \forall x \in B_R(\bar{x}).$$

(b) *If $\bar{u}(\bar{x}) > \mu_1 + aR$, then there exists $\eta_1 > 0$ such that*

$$\bar{u}(x) \geq \mu_1 + \phi(\eta_1)(R - \|x - \bar{x}\|) \qquad \forall x \in B_R(\bar{x}).$$

Proof. The proof of this result is an immediate application of Theorem 1 of [4] to the particular case when the functional considered in (P) is rotationally invariant. □

This theorem provides us with local estimates for solutions to (P) under some local conditions. It is an useful tool to prove local estimates near extremum points as we will see ahead in this paper, but it can be also seen as a result by itself: if we consider (P), a ball $B_R(\bar{x})$ contained in Ω and if its center point $\bar{x}$ is not an extremum point for $\bar{u}$ in $B_R(\bar{x})$ in the sense that the distance from $\bar{u}(\bar{x})$ to μ_1 or μ_2 is greater than aR, then we obtain an estimate for $\bar{u}$ on $B_R(\bar{x})$ through $\phi(\eta)(R - \|x - \bar{x}\|)$, for some $\eta > 0$. Now we can ask whether local estimates can be obtained if the distance from $\bar{u}(\bar{x})$ to μ_1 or μ_2 is smaller than aR. The answer is positive:

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set. Consider $f : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\}$ convex, lower semicontinuous and such that $f(0) = 0$, $\partial f^*(0) = \{0\}$ and $\partial f(0) = [0, a], a > 0$. Let $\bar{u}$ be a continuous solution to (P). Let $\bar{x} \in \Omega$ and $R > 0$ be such that $B_R(\bar{x}) \subset \Omega$, and consider $\mu_1 = \inf_{B_R(\bar{x})} \bar{u}(x)$ and $\mu_2 = \sup_{B_R(\bar{x})} \bar{u}(x)$*

(a) *If $\mu_2 - aR \leq \bar{u}(\bar{x}) < \mu_2$, then there exists $k_2 > \mu_2$ and $\eta_2 > 0$ such that*

$$\bar{u}(x) \leq k_2 - \phi(\eta_2)(R - \|x - \bar{x}\|) \qquad \forall x \in B_R(\bar{x}).$$

(b) *If* $\mu_1 < \bar{u}(\bar{x}) \le \mu_1 + aR$*, then there exists* $k_1 < \mu_1$ *an* $\eta_1 > 0$ *such that*

$$\bar{u}(x) \ge k_1 + \phi(\eta_1)(R - \|x - \bar{x}\|) \qquad \forall x \in B_R(\bar{x}).$$

Proof. Let us only prove (a), since (b) is analogous. Assume that $\mu_2 - aR \le \bar{u}(\bar{x}) < \mu_2$. This means that

$$\bar{u}(\bar{x}) = \mu_2 + m' - aR, \qquad \text{with } m' \in [0, aR[.$$

We have that, for $m \in]m', aR[$, $\bar{u}(x) \le \mu_2 + m \quad \forall x \in B_R(\bar{x})$, which is obvious by definition of μ_2, and

$$\bar{u}(\bar{x}) = \mu_2 + m' - aR < \mu_2 + m - aR.$$

By Theorem 3.1, we obtain, for some $\eta_2 > 0$ and for $k_2 \ge \mu_2 + m$,

$$\bar{u}(x) \le \mu_2 + m - \phi(\eta_2)(R - \|x - \bar{x}\|) \le k_2 - \phi(\eta_2)(R - \|x - \bar{x}\|) \qquad \forall x \in B_R(\bar{x}),$$

and the result is proved. □

In this theorem we proved some local estimates for solutions to (P) near nonextremum points not covered by Theorem 3.1.

Example 3.3. Consider $n = 1$, $f(t) = at$, and $\Omega =]0, 2[$. For each admissible solution to (P) with $u(0) = 0$ and $u(2) = 2$ we have, by Jensen's inequality,

$$\int_0^2 f(\|u'(x)\|)\, \mathrm{d}x \ge f\left(\left|\int_0^2 u'(x)\, \mathrm{d}x\right|\right)$$

$$= f(|u(2) - u(0)|) = f(2) = 2a = \int_0^2 f(\|\bar{u}'(x)\|)\, \mathrm{d}x,$$

where $\bar{u}(x) = x$, $x \in [0, 2]$. Like this, $\bar{u}$ is a solution to (P) with $u(0) = 0$ and $u(2) = 2$.
Consider now $\bar{x} = 1$, $R = \frac{1}{2}$ and $B_{\frac{1}{2}}(1) = \left]\frac{1}{2}, \frac{3}{2}\right[\subset \Omega =]0, 2[$. If $a = \frac{1}{2}$ we have that

$$1 = \bar{u}(\bar{x}) < \sup_{x \in \left]\frac{1}{2}, \frac{3}{2}\right[} \bar{u}(x) - aR = \frac{5}{4}.$$

So, by Theorem 3.1, we have that

$$\bar{u}(x) \le \frac{3}{2} - \phi(\eta_2)\left(\frac{1}{2} - |x - 1|\right) \qquad \forall x \in \left]\frac{1}{2}, \frac{3}{2}\right[,$$

where $\eta_2 > 0$. If $a = 1$, we have

$$1 = \bar{u}(\bar{x}) = \sup_{x \in \left]\frac{1}{2}, \frac{3}{2}\right[} \bar{u}(x) - aR$$

and by Theorem 3.2,

$$\bar{u}(x) \leq k_2 - \phi(\eta_2)\left(\frac{1}{2} - |x-1|\right) \qquad \forall x \in \left]\frac{1}{2}, \frac{3}{2}\right[,$$

where $k_2 > \frac{3}{2}$ and $\eta_2 > 0$.

Joining both Theorem 3.1 and Theorem 3.2, we obtain the result.

Corollary 3.4. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set. Consider $f : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\}$ convex, lower semicontinuous and such that $f(0) = 0$, $\partial f^*(0) = \{0\}$ and $\partial f(0) = [0, a], a > 0$. Let $\bar{u}$ be a continuous solution to (P). Let $\bar{x} \in \Omega$ and $R > 0$ be such that $B_R(\bar{x}) \subset \Omega$, and consider $\mu_1 = \inf_{B_R(\bar{x})} \bar{u}(x)$ and $\mu_2 = \sup_{B_R(\bar{x})} \bar{u}(x)$*

(a) *If $\bar{u}(\bar{x}) < \mu_2$, then there exists $k_2 \geq \mu_2$ and $\eta_2 > 0$ such that*

$$\bar{u}(x) \leq k_2 - \phi(\eta_2)(R - \|x - \bar{x}\|) \qquad \forall x \in B_R(\bar{x}).$$

(b) *If $\mu_1 < \bar{u}(\bar{x})$, then there exists $k_1 \leq \mu_1$ an $\eta_1 > 0$ such that*

$$\bar{u}(x) \geq k_1 + \phi(\eta_1)(R - \|x - \bar{x}\|) \qquad \forall x \in B_R(\bar{x}).$$

4 Estimates nearby extremum points

In this section we will prove some local estimates near extremum points, using Theorem 3.1, as presented in [4].

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set. Consider $f : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\}$ convex, lower semicontinuous and such that $f(0) = 0$, $\partial f^*(0) = \{0\}$ and $\partial f(0) = [0, a], a > 0$. Let $\bar{u}$ be a continuous solution to (P). Let $\bar{x} \in \Omega$ and $R > 0$ be such that $B_R(\bar{x}) \subset \Omega$, and consider $\mu_1 = \inf_{B_R(\bar{x})} \bar{u}(x)$ and $\mu_2 = \sup_{B_R(\bar{x})} \bar{u}(x)$*

(a) *If $\bar{u}(\bar{x}) = \mu_2$, then there exists $\eta_2 > 0$ such that*

$$\bar{u}(x) \geq \mu_2 - \phi(\eta_2)\|x - \bar{x}\| \qquad \forall x \in B_{\frac{R}{2}}(\bar{x}).$$

(b) *If* $\bar{u}(\bar{x}) = \mu_1$, *then there exists* $\eta_1 > 0$ *such that*

$$\bar{u}(x) \leq \mu_1 + \phi(\eta_1)\|x - \bar{x}\| \qquad \forall x \in B_{\frac{R}{2}}(\bar{x}).$$

Proof. As the proof of (b) is similar to the proof of (a) with obvious modifications, we will prove only (a). (a) Assume, on the contrary, that

$$\exists \tilde{x} \in B_{\frac{R}{2}}(\bar{x}) : \quad \bar{u}(\tilde{x}) < \mu_2 - \phi(\eta_2)\|\tilde{x} - \bar{x}\| \qquad \forall \eta_2 > 0.$$

We have that $\tilde{x} \in B_{\frac{R}{2}}(\bar{x})$. This means that $2\|\tilde{x} - \bar{x}\| < R$. Let us choose $\epsilon > 0$ so small that

$$2\|\tilde{x} - \bar{x}\| + \epsilon < R \tag{2}$$

and

$$\bar{u}(\tilde{x}) < \mu_2 - \phi(\eta_2)(\|\tilde{x} - \bar{x}\| + \epsilon).$$

Set $\bar{R} = \|\tilde{x} - \bar{x}\| + \epsilon < R$. Like this, for every $\eta_2 > 0$,

$$\bar{u}(\tilde{x}) < \mu_2 - \phi(\eta_2)\bar{R} < \mu_2 - a\bar{R}. \tag{3}$$

By Theorem 3.1 - (a), we obtain that there exists $\eta > 0$ such that

$$\bar{u}(x) \leq \mu_2 - \phi(\eta)(\bar{R} - \|x - \tilde{x}\|) \tag{4}$$

for all $x \in B_{\bar{R}}(\tilde{x})$. Note that $B_{\bar{R}}(\tilde{x}) \subset B_R(\bar{x})$. In fact, considering $y \in B_{\bar{R}}(\tilde{x})$, we have

$$\begin{aligned} \|y - \bar{x}\| &\leq \|y - \tilde{x}\| + \|\tilde{x} - \bar{x}\| \\ &\leq \bar{R} + \|\tilde{x} - \bar{x}\| = 2\|\tilde{x} - \bar{x}\| + \epsilon < R, \end{aligned}$$

and so $y \in B_R(\bar{x})$.
This means that

$$\bar{u}(\bar{x}) \leq \mu_2 - \phi(\eta)(\bar{R} - \|x - \tilde{x}\|) \qquad \forall x \in B_R(\bar{x}).$$

In particular,

$$\mu_2 = \bar{u}(\bar{x}) \leq \mu_2 - \phi(\eta)\epsilon < \mu_2,$$

which is a contradiction. Then (a) is proved. □

Being $\bar{u}$ any solution to (P), if we consider any ball $B_R(\bar{x}) \subset \Omega$ such that $\bar{u}(\bar{x})$ an extremum point, by this result we obtain an estimate for $\bar{u}$ on half of the considered ball, $B_{\frac{R}{2}}(\bar{x})$.

Together with Corollary 3.4, if we consider $\bar{u}$ any solution for (P) and any ball $B_R(\bar{x})$, we obtain an estimate for $\bar{u}$ in $B_R(\bar{x})$ in the case when $\bar{x}$ is not an extremum point of $\bar{u}$ in $B_R(\bar{x})$ and if, instead, $\bar{x}$ is an extremum point of $\bar{u}$ in $B_R(\bar{x})$, then we obtain an estimate for $\bar{u}$ in $B_{\frac{R}{2}}(\bar{x})$.

5 A local bound for any solution to (P)

In this section we present finally a local bound on the difference between the infimum and the supremum of any solution $\bar{u}$ to (P)

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set. Consider $f : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\}$ convex, lower semicontinuous and such that $f(0) = 0$, $\partial f^*(0) = \{0\}$ and $\partial f(0) = [0, a], a > 0$. Let $\bar{u}$ be a continuous solution to (P), which is non constant by parts. Let $\bar{x} \in \Omega$ and $R > 0$ be such that $B_{2R}(\bar{x}) \subset \Omega$. We have that*

$$\left| \sup_{x \in B_{r(R)}(y_1)} \bar{u}(x) - \inf_{x \in B_{r(R)}(y_1)} \bar{u}(x) \right| \leq K\left(R, a, f^*\right),$$

where $r(R) \leq \frac{R}{2}$ and $y_1 \in B_R(\bar{x})$.

Proof. As $B_{2R}(\bar{x}) \subset \Omega$, let us consider $y_1, y_2 \in B_R(\bar{x})$ such that $\|y_1 - y_2\| = \epsilon$, for some $\epsilon > 0$, and let also be $r_1 \in]0, R]$ and $r_2 = r_1 - \epsilon$ such that $B_{r_2}(y_2) \subset B_{r_1}(y_1)$ and

$$\bar{u}(y_1) = \sup_{x \in B_{r_1}(y_1)} \bar{u}(x) = \sup_{x \in B_{r_2}(y_2)} \bar{u}(x) = \sup_{x \in B_{\frac{r_1}{2}}(y_1)} \bar{u}(x) = \sup_{x \in B_{\frac{r_2}{2}}(y_2)} \bar{u}(x)$$

and also such that

$$\inf_{x \in B_{r_1}(y_1)} \bar{u}(x) = \inf_{x \in B_{r_2}(y_2)} \bar{u}(x).$$

(i) As $\bar{u}(y_1) = \sup_{x \in B_{r_1}(y_1)} \bar{u}(x)$, by Theorem 4.1 - (a), there exists $\eta_2 > 0$ such that

$$\bar{u}(y_1) - \phi(\eta_2)\|x - y_1\| \leq \bar{u}(x) \leq \bar{u}(y_1) \qquad \forall x \in B_{\frac{r_1}{2}}(y_1).$$

As

$$\bar{u}(y_1) = \sup_{x \in B_{r_1}(y_1)} \bar{u}(x) \geq \sup_{x \in B_{\frac{r_1}{2}}(y_1)} \bar{u}(x),$$

we obtain

$$\sup_{x \in B_{\frac{r_1}{2}}(y_1)} \bar{u}(x) \leq \bar{u}(x) + \phi(\eta_2)\|x - y_1\| \leq \bar{u}(x) + \phi(\eta_2).\frac{r_1}{2} \forall x \in B_{\frac{r_1}{2}}(y_1).$$

By continuity of $\bar{u}$, we have

$$\sup_{x \in B_{\frac{r_1}{2}}(y_1)} \bar{u}(x) \leq \inf_{x \in B_{\frac{r_1}{2}}(y_1)} \bar{u}(x) + \phi(\eta_2)\frac{r_1}{2}. \tag{5}$$

(ii) As $\bar{u}$ is continuous and it isn't constant by parts, we have that

$$\bar{u}(y_1) - ar_2 \leq \bar{u}(y_2) < \bar{u}(y_1).$$

By Theorem 3.2, there exists $k_2 > \bar{u}(y_1)$ and there exists $\eta_1 > 0$ such that

$$\bar{u}(x) \leq k_2 - \phi(\eta_1)(r_2 - \|x - y_2\|) \qquad \forall x \in B_{r_2}(y_2).$$

In particular, as $k_2 = \bar{u}(y_1) + m$, with $m \in]0, ar_2[$ and $B_{\frac{r_1}{2}}(y_1) \subset B_{r_2}(y_2)$,

$$\bar{u}(x) \leq \bar{u}(y_1) + m - \phi(\eta_1)(r_2 - \frac{r_1}{2}) \qquad \forall x \in B_{\frac{r_1}{2}}(y_1).$$

As $r_2 = r_1 - \epsilon$, and $m < ar_2 = ar_1 - a\epsilon$,

$$\bar{u}(x) \leq \bar{u}(y_1) + a\frac{r_1}{2} + (\phi(\eta_1) - a)\epsilon \qquad \forall x \in B_{\frac{r_1}{2}}(y_1).$$

Then we obtain

$$\inf_{x \in B_{\frac{r_1}{2}}(y_1)} \bar{u}(x) \leq \sup_{x \in B_{\frac{r_1}{2}}(y_1)} \bar{u}(x) + a\frac{r_1}{2} + (\phi(\eta_1) - a)\epsilon. \tag{6}$$

By (5) and (6), we have that

$$-a\frac{r_1}{2} - (\phi(\eta_1) - a)\epsilon \leq \sup_{x \in B_{\frac{r_1}{2}}(y_1)} \bar{u}(x) - \inf_{x \in B_{\frac{r_1}{2}}(y_1)} \bar{u}(x) \leq \phi(\eta_2)\frac{r_1}{2}.$$

Setting $\frac{r_1}{2} = r(R)$ and

$$K(f^*, a, r(R)) = \max\left(\phi(\eta_2)\frac{r_1}{2}, a\frac{r_1}{2} + (\phi(\eta_1) - a)\epsilon\right),$$

the result is proved. □

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