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Some remarks on the links between the VIR and Gompertz diffusion models and the links between the CIR and Rayleigh diffusion models

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Abstract. The main purpose of this work is to establish a new links between the stochastic Cox-Ingersoll-Ross (CIR) model and the stochastic Rayleigh diffusion process (SRDP) and the links between the Vasicek Interest Rate (VIR) process and the stochastic Gompertz diffusion process (SGDP). These links focus on elementary stochastic calculus and Itô's calculus. Firstly, we prove that the square root of the CIR model is a SRDP. Secondly, we prove that the square of the SRDP is a CIR model. Thirdly, we prove that the exponential of the VIR model is a SGDP. Finally, we prove that the logarithm of the SGDP is a VIR model. New computations of the probability transition density function (PTDF) and the trend functions of the processes have quite simple formulations.

Keywords. Cox-Ingersoll-Ross model, Rayleigh diffusion process, Vasicek interest rate model, Stochastic Gompertz diffusion process..

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1 Introduction

Theoretical aspects of stochastic processes play a fundamental role in the theory and modeling of real phenomena such as financial, economic, biological, agronomic, medical, health and environmental problems, etc. Many physical, biological, and economic phenomena are either well approximated or reasonably modeled by stochastic differential equation (SDE). Typically, SDE contain a variable representing random white noise computed as the derivative of the Wiener process. In general, the SDE has the form:

$$dx(t) = a(t, x(t))dt + \sqrt{b(t, x(t))}dw(t); \quad x(t_0) = x_0 \tag{1}$$

with $\{w(t), t \in [t_0; T]\}$ is a Wiener process with an independent increment $w(t) - w(s)$ normally distributed with $E(w(t) - w(s)) = 0$ and $Var(w(t) - w(s)) = t - s$ for $t \geq s$, and x_0 is a fixed real positive and $a(t, x(t))$

is drift coefficient and $b(t, x(t))$ is diffusion coefficient. By using the properties of $a(t, x)$ and $b(t, x)$, it results that the equation (1) has a unique solution $\{x(t), t \in [t_0, T]\}$ which is diffusion process valued in $(0, \infty)$ with initial value x_0 (see, for instance [6]).

In the present study, two possibilities of equivalence between certain processes are studied. Four kinds of the point stochastic diffusion process are considered. The first is the CIR model, the second is the SRDP, the third is the VIR process and the last is the SGDP. By using a transformation in the diffusion process and applying Itô's calculus. Firstly, we show that the CIR is a SRDP and the SRDP is a CIR. The PTDF of a CIR is obtained using a PTDF of a SRDP and the PTDF of a SRDP is obtained using a PTDF of a CIR process, from which the trend functions of the processes are obtained by the properties of modified Bessel function of the first kind and Kummer function. Finally, we show that the VIR model is a SGDP and the SGDP is a VIR model. The PTDF of a VIR model is obtained using a PTDF of a SGDP and the PTDF of a SGDP is obtained using a PTDF of a VIR model, from which the trend functions of the processes are obtained.

The CIR model was originally introduced by John Carrington Cox et al. [1] in 1985. It has been applied in finance to describe the evolution of interest rates. It is also used by Heston [2] for the stochastic volatility model and by Duffie [3] for the default intensities in credit risk model. Zhu [12] proposed a generalization of the classical CIR model and the classical Hawkes process with exponential exciting function. Dyrting [13] tested the existing methods for evaluating the non-central χ^2-distribution for the CIR process and developed a new method based on a Bessel series representation. Guo [14] proves that for a model in which the historical stock price follows a CIR model, there is no equivalent martingale measure.

The SRDP, from its earliest formulation Rayleigh [10] has been widely exploited in physics. Giorno et al. [5] suggested a few remarks on the Rayleigh process. The SRDP have been studied in terms of specific theoretical considerations and trend analysis, and have been fruitfully applied to real cases in Gutiérrez et al. [4, 11].

The VIR model is an approach mathematical to modeling interest rate movements. It was introduced by Vasicek [15] in 1977. Basically, it is often employed in the valuation of interest rate futures and is occasionally used to price various hard-to-value bonds. Xiao et al. [16] proposed the Vasicek fractional model to illustrate the dynamics of the short interest rate.

The stochastic Gompertz diffusion has been discussed in terms of specific theoretical aspects and trend analysis, and has been used efficiently in real cases stochastic in Gutiérrez et al. [18–20]. Gompertz [21] introduced the curve to model the law of human mortality and formulated it as a double exponential. Thereafter, the curve has been modified several times and stated in various forms to ease its study. Gutiérrez-Jáimez et al. [17] presented a modified version of the Gompertz model with an application to random growth.

This paper is structured as follows: in the second section, we present some definitions and preliminaries. The third section, presents the result obtained such as the transformation of the CIR model into the SRDP, the PTDF, the trend function (TF) and conditional TF (CTF) of the CIR model are obtained. The fourth section presents a transformation of the SRDP into the CIR model; the PTDF, the TF and CTF of the SRDP are obtained. The fifth section presents a transformation of the VIR model into the SGDP, the PTDF, the TF and CTF of the VIR diffusion process are calculated. The sixth section given a transformation of the SGDP into the VIR model; the PTDF, the TF and CTF of the SGDP are obtained. Finally, a brief conclusion to this study.

2 Definitions and preliminaries

2.1 The CIR process

Definition 2.1. The stochastic CIR model is a diffusion process $\{x(t); t \in [t_0; T]\}$ with values in $(0; \infty)$, and with drift and diffusion coefficient:

$$a(t, x) = \kappa(\theta - x); \qquad b(t, x) = \sigma^2 x,$$

where κ is the mean reversion speed, θ is mean reversion parameter, and σ standard deviation that determines the volatility and κ, θ and σ are positive real parameters.

After replacing in equation (1), we obtain the SDE:

$$dx(t) = \kappa\left(\theta - x(t)\right)dt + \sigma\sqrt{x(t)}dw(t); \quad x\left(t_0\right) = x_0. \tag{2}$$

An examination of the boundary classification criteria demonstrates that $x(t)$ can reach zero if $\sigma^2 > 2\kappa\theta$. If $\sigma^2 \leq 2\kappa\theta$, the upward drift is large enough to make the origin inaccessible.

Let $\alpha = \kappa\theta$ and $\beta = -\kappa$. After substitution in equation (2), we obtain the following SDE:

$$dx(t) = \left(\alpha + \beta x(t)\right)dt + \sigma\sqrt{x(t)}dw(t); \quad x\left(t_0\right) = x_0. \tag{3}$$

2.2 The SRDP

Definition 2.2. The SRDP is a diffusion process is a diffusion process $\{x(t); t \in [t_0; T]\}$ with values in $(0; \infty)$, and with drift and diffusion coefficient:

$$a(t, x) = \frac{\alpha}{x} + \beta x; \qquad b(t, x) = c^2,$$

with $c > 0$, α and β are real parameters.

After replacing in equation (1), we obtain the SDE:

$$dx(t) = \left(\frac{\alpha}{x(t)} + \beta x(t)\right) dt + c dw(t); \quad x(t_0) = x_0. \tag{4}$$

2.3 The VIR model

Definition 2.3. The VIR model is a diffusion process $\{x(t); t \in [t_0; T]\}$ with values in $(0; \infty)$, and with drift and diffusion coefficient:

$$a(t,x) = \kappa(\theta - x); \qquad b(t,x) = \sigma^2,$$

with κ is speed of reversion of the mean, θ is long-term level of the mean, and ÏČ is standard deviation that determines the volatility and κ, θ and σ are real parameters

Let $\alpha = \kappa\theta$ and $\beta = -\kappa$. After replacing in equation (1), we obtain the SDE:

$$dx(t) = (\alpha + \beta x(t))\, dt + \sigma dw(t); \quad x(t_0) = x_0. \tag{5}$$

2.4 The SGDP

Definition 2.4. The SGDP is a diffusion process $\{x(t); t \in [t_0; T]\}$ with values in $(0; \infty)$,and with drift and diffusion coefficient:

$$a(t,x) = \alpha x + \beta x \log(x); \qquad b(t,x) = c^2 x^2,$$

with $c > 0$, α and β are real parameters.

We can consider the SDE:

$$dx(t) = (\alpha x(t) + \beta x(t) \log(x(t)))\, dt + c x(t) dw(t); \quad x(t_0) = x_0. \tag{6}$$

In all that follows $\{x(t); t \in [t_0; T]\}$ denotes a diffusion process with values in $(0; \infty)$.

3 Transformation the CIR process into the SRDP

3.1 The first result

Theorem 3.1. *If $\{x(t); t \in [t_0; T]\}$ is a CIR process. Then, the diffusion process $\left\{\sqrt{x(t)}; t \in [t_0; T]\right\}$ is a SRDP.*

Proof. The CIR model (3):

$$dx(t) = (\alpha + \beta x(t))\, dt + \sigma \sqrt{x(t)} dw(t); \quad x(t_0) = x_0.$$

Hence, the equation (3) becomes:

$$\frac{dx(t)}{\sqrt{x(t)}} = \left(\frac{\alpha}{\sqrt{x(t)}} + \beta \sqrt{x(t)} \right) dt + \sigma dw(t); \quad x(t_0) = x_0.$$

By applying the Itô's lemma to $\sqrt{x(t)}$, we get

$$d\left(\sqrt{x(t)}\right) = \left(\frac{\left(\frac{\alpha}{2} - \frac{\sigma^2}{8} \right)}{\sqrt{x(t)}} + \frac{\beta}{2}\sqrt{x(t)} \right) dt + \frac{\sigma}{2} dw(t); \;\; x(t_0) = x_0. \quad (7)$$

Then, the equation (7) becomes:

$$d(y(t)) = \left(\frac{\gamma}{y(t)} + \delta y(t) \right) dt + c dw(t); \;\; y(t_0) = y_0. \quad (8)$$

with $y(t) = \sqrt{x(t)}$, $\gamma = \left(\frac{\alpha}{2} - \frac{\sigma^2}{8} \right)$, $\delta = \frac{\beta}{2}$, $c = \frac{\sigma}{2}$ and $y_0 = \sqrt{x_0}$. Note that if $\{x_t; t \geq 0\}$ be a diffusion process and g be a strictly monotone function with continuous second derivative g''. Then, $\{g(x_t)\}$ defines a diffusion process (see Theorem 2.1. p. 173, Karlin and Taylor [9]). The process $\{y(t), t \in [t_0; T]\}$ in the last equation is the SRDP with infinitesimal moments
$a(t, y) = \left(\frac{\alpha}{2} - \frac{\sigma^2}{8} \right) \frac{1}{y} + \frac{\beta}{2} y$; and $b(t, y) = \frac{\sigma^2}{4}$.

□

3.2 The PTDF of the CIR model using the SRDP

Let Φ and φ be respectively the cumulative probability distribution function (CPDF) and the TPDF of the SRDP, and F and f be respectively the CPDF and the TPDF of the CIR process. Then, we get:

$$\begin{aligned} f(x, t|y, s) =& \frac{dF(x, t|y, s)}{dx} = \frac{dP\left(x(t)|x(s) = y \leq x\right)}{dx} \\ =& \frac{dP\left(\sqrt{x(t)}|\sqrt{x(s)} = \sqrt{y} \leq \sqrt{x}\right)}{dx} = \frac{d\Phi\left(\sqrt{x}, t|\sqrt{y}, s\right)}{dx} \\ =& \frac{1}{2\sqrt{x}} \varphi\left(\sqrt{x}, t|\sqrt{y}, s\right). \end{aligned}$$

Finally, the TPDF of the CIR model $x(t)$, given $x(s)$ for $s < t$ is

$$f\left(x,t|y,s\right) = \frac{2\beta}{\sigma^2\left(e^{\beta(t-s)}-1\right)}\exp\left(\frac{-2\beta\left(x+ye^{\beta(t-s)}\right)}{\sigma^2\left(e^{\beta(t-s)}-1\right)}\right)\left(\frac{x}{ye^{\beta(t-s)}}\right)^{\frac{q}{2}} \times I_q\left(\frac{4\beta\sqrt{xy}e^{\frac{\beta(t-s)}{2}}}{\sigma^2\left(e^{\beta(t-s)}-1\right)}\right), \tag{9}$$

where I_q denotes the modified Bessel function of the first kind and $q = \dfrac{2\alpha}{\sigma^2} - 1$. Finally, let $v(t,s) = x(t)|x(s) = x_s$ for $s < t$, we remark that $v(t,s)$ follows a non-central chi-square distribution: $v(t,s) \sim \zeta\chi^2_d(\lambda)$ with degree of freedom $d = \dfrac{4\alpha}{\sigma^2}$, $\zeta = \frac{\sigma^2\left(e^{\beta(t-s)}-1\right)}{4\beta}$ and non centrality parameter $\lambda = \dfrac{4\beta x_s e^{\beta(t-s)}}{\sigma^2\left(e^{\beta(t-s)}-1\right)}$.

3.3 TF of the CIR process

The CTF of the CIR is:

$$E\left(x(t)|x(s) = x_s\right) = \int_0^\infty xf\left(x,t|x_s,s\right)dx.$$

Then, we get

$$E\left(x(t)|x(s) = x_s\right) = \frac{2\beta x_s^{\frac{-q}{2}}}{\sigma^2\left(e^{\beta(t-s)}-1\right)}\exp\left(\frac{-2\beta x_s e^{\beta(t-s)}}{\sigma^2\left(e^{\beta(t-s)}-1\right)} - \frac{q\beta}{2}(t-s)\right) \times \int_0^\infty x^{\frac{q}{2}+1}\exp\left(\frac{-2\beta x}{\sigma^2\left(e^{\beta(t-s)}-1\right)}\right)I_q\left(\frac{4\beta\sqrt{x}\sqrt{x_s}e^{\frac{\beta(t-s)}{2}}}{\sigma^2\left(e^{\beta(t-s)}-1\right)}\right)dx.$$

By using the relations Gradshteyn and Ryzhik [[7], 6.643],

$$\int_0^\infty e^{-\lambda y}y^{\mu-\frac{1}{2}}I_{2\nu}\left(2\xi\sqrt{y}\right)dy = \frac{\Gamma\left(\mu+\nu+\frac{1}{2}\right)}{\Gamma\left(2\nu+1\right)}\xi^{-1}\lambda^{-\mu}\exp\left(\frac{\xi^2}{2\lambda}\right)M_{-\mu,\nu}\left(\frac{\xi^2}{\lambda}\right),$$

where $\mu + \nu + \dfrac{1}{2} > 0$ and $M_{-\mu,\nu}$ is a Whittaker function Sepanier and Oldham [[8], p. 477:48-13.1],

$$M_{\mu,\nu}(x) = x^{\nu+\frac{1}{2}}e^{\frac{-x}{2}}K\left(\nu-\mu+\frac{1}{2}, 2\nu+1, x\right),$$

with K is the Kummer function, the CTF of the process leads to

$$E\left(x(t)|x(s)=x_s\right)=\frac{\Gamma\left(q+2\right)}{\Gamma\left(q+1\right)}\left(\frac{2\beta}{\sigma^2\left(e^{\beta(t-s)}-1\right)}\right)^{-1}\exp\left(\frac{-2\beta x_s e^{\beta(t-s)}}{\sigma^2\left(e^{\beta(t-s)}-1\right)}\right)$$
$$\times K\left(q+2,q+1,\frac{2\beta x_s e^{\beta(t-s)}}{\sigma^2\left(e^{\beta(t-s)}-1\right)}\right).$$

Therefore, by the definition of the Kummer function K (for details see Appendix the equation (25)), we have

$$K(q+2,q+1,z)=e^z+\frac{\Gamma\left(q+1\right)}{\Gamma\left(q+2\right)}ze^z,$$

with Γ is the Gamma function and $z=\dfrac{2\beta x_s e^{\beta(t-s)}}{\sigma^2\left(e^{\beta(t-s)}-1\right)}$ and $q=\dfrac{2\alpha}{\sigma^2}-1$. Finally, by the last formula and the formula

$$\Gamma\left(q+2\right)=(q+1)\Gamma\left(q+1\right)=\frac{2\alpha}{\sigma^2}\Gamma\left(q+1\right).$$

We deduce that the CTF of the model is

$$E\left(x(t)|x(s)=x_s\right)=\frac{\alpha}{\beta}\left(e^{\beta(t-s)}-1\right)+x_s e^{\beta(t-s)}. \tag{10}$$

Finally, from (10) and by the condition $P\left(x\left(t_0\right)=x_0\right)=1$, the TF of CIR is given by

$$E(x(t))=\frac{\alpha}{\beta}\left(e^{\beta t}-1\right)+x_0e^{\beta t}. \tag{11}$$

Remark 3.2. We can also study the asymptotic behavior in time of the TF of CIR model, if $\beta<0$, thus obtaining

$$\lim_{t\to\infty}E(x(t))=\frac{-\alpha}{\beta}.$$

4 Transformation the SRDP into the CIR process

4.1 The second result

Theorem 4.1. *If $\{x(t);t\in[t_0;T]\}$ is a SRDP. Then, the stochastic diffusion process $\left\{x(t)^2;t\in[t_0;T]\right\}$ is a CIR model.*

Proof. The SRDP (4):

$$dx(t)=\left(\frac{\alpha}{x(t)}+\beta x(t)\right)dt+cdw(t);\quad x\left(t_0\right)=x_0.$$

Hence, the equation (4) takes the form:

$$x(t)dx(t)=\left(\alpha+\beta x(t)^2\right)dt+cx(t)dw(t);\quad x\left(t_0\right)=x_0.$$

By using the Itô's lemma to $x(t)^2$, we get the equation

$$d\left(x(t)^2\right)=\left(2\alpha+c^2+2\beta x(t)^2\right)dt+2cx(t)dw(t);\quad x\left(t_0\right)=x_0. \tag{12}$$

Then, the equation (12) becomes:

$$dy(t)=\left(\gamma+\delta y(t)\right)dt+\sigma\sqrt{y(t)}dw(t);\quad y\left(t_0\right)=y_0, \tag{13}$$

where $y(t)=x(t)^2$, $\gamma=2\alpha+c^2$, $\delta=2\beta$, $\sigma=2c$ and $y_0=x_0^2$. The diffusion process $\{y(t);t\in[t_0;T]\}$, in the last equation is the CIR model with infinitesimal moments $a(t,y)=2\alpha+c^2+2\beta y$; and $b(t,y)=4c^2$. □

4.2 The PTDF of the SRDP using the CIR model

The PTDF of the SRDP is:

$$\begin{aligned}\varphi(x,t|y,s)=&\frac{d\Phi(x,t|y,s)}{dx}=\frac{dP\left(x(t)|x(s)=y\leq x\right)}{dx}\\=&\frac{dP\left(x(t)^2|x(s)^2=y^2\leq x^2\right)}{dx}=\frac{dF\left(x^2,t|y^2,s\right)}{dx}\\=&2xf\left(x^2,t|y^2,s\right).\end{aligned}$$

Finally, the PTDF of the SRDP model $x(t)$, given $x(s)$ for $s<t$ is

$$\begin{aligned}\varphi(x,t|y,s)=&\frac{2\beta y^{-q}x^{q+1}}{c^2\left(e^{2\beta(t-s)}-1\right)}\exp\left\{-\frac{\beta\left(x^2+y^2e^{2\beta(t-s)}\right)}{c^2\left(e^{2\beta(t-s)}-1\right)}-q\beta(t-s)\right\}\\&\times I_q\left(\frac{2\beta xye^{\beta(t-s)}}{c^2\left(e^{2\beta(t-s)}-1\right)}\right),\end{aligned} \tag{14}$$

for $\alpha>\dfrac{-c^2}{2}$ and with the zero-flux condition and $q=\dfrac{\alpha}{c^2}-\dfrac{1}{2}$. Finally, let $v(t,s)=x^2(t)|x^2(s)=x_s^2$ for $s<t$, we remark that $v(t,s)\sim\zeta\chi_d^2(\lambda)$ with degree of freedom $k=\dfrac{2\alpha}{c^2}+1$, $\zeta=\dfrac{c^2\left(e^{2\beta(t-s)}-1\right)}{2\beta}$ and $\lambda=\dfrac{2\beta x_s^2e^{2\beta(t-s)}}{c^2\left(e^{2\beta(t-s)}-1\right)}$.

4.3 TF of the SRDP

The CTF of the model is:

$$E(x(t)|x(s)=x_s)=\int_0^\infty x\varphi(x,t|x_s,s)dx.$$

Then, we have

$$E\left(x(t)|x(s)\right.=x_s\left.\right)=\frac{2\beta x_s^{-q}}{c^2\left(e^{2\beta(t-s)}-1\right)}\exp\left(\frac{-\beta x_s^2 e^{2\beta(t-s)}}{c^2\left(e^{2\beta(t-s)}-1\right)}-q\beta\left(t-s\right)\right)$$
$$\times\int_0^\infty x^{q+2}\exp\left(\frac{-\beta x^2}{c^2\left(e^{2\beta(t-s)}-1\right)}\right)I_q\left(\frac{2\beta x x_s e^{\beta(t-s)}}{c^2\left(e^{2\beta(t-s)}-1\right)}\right)dx.$$

By using the Gradshteyn and Ryzhik relations [[7], 6.643] and by applying the change of variable $y=x^2$,

$$\int_0^\infty e^{-\lambda y}y^{\mu-\frac{1}{2}}I_{2\nu}\left(2\xi\sqrt{y}\right)dy=\frac{\Gamma\left(\mu+\nu+\frac{1}{2}\right)}{\Gamma\left(2\nu+1\right)}\xi^{-1}\lambda^{-\mu}\exp\left(\frac{\xi^2}{2\lambda}\right)M_{-\mu,\nu}\left(\frac{\xi^2}{\lambda}\right),$$

where $\mu+\nu+\dfrac{1}{2}>0$. Then, the CTF of the process leads to

$$E\left(x(t)|x(s)=x_s\right)=\frac{\Gamma\left(q+\frac{3}{2}\right)}{\Gamma\left(q+1\right)}\left(\frac{\beta}{c^2\left(e^{2\beta(t-s)}-1\right)}\right)^{\frac{-1}{2}}\exp\left(\frac{-\beta x_s^2 e^{2\beta(t-s)}}{c^2\left(e^{2\beta(t-s)}-1\right)}\right)$$
$$\times K\left(q+\frac{3}{2},q+1,\frac{\beta x_s^2 e^{2\beta(t-s)}}{c^2\left(e^{2\beta(t-s)}-1\right)}\right).$$

Then, by the Kummer transformation $K(a,b,z)=e^zK(b-a,b,-z)$ (for details see Appendix the equation (26)), we have

$$K\left(q+\frac{3}{2},q+1,\frac{\beta x_s^2 e^{2\beta(t-s)}}{c^2\left(e^{2\beta(t-s)}-1\right)}\right)=\exp\left(\frac{\beta x_s^2 e^{2\beta(t-s)}}{c^2\left(e^{2\beta(t-s)}-1\right)}\right)$$
$$\times K\left(\frac{-1}{2},q+1,\frac{-\beta x_s^2 e^{2\beta(t-s)}}{c^2\left(e^{2\beta(t-s)}-1\right)}\right).$$

Finally, we deduce that the CTF of the SRDP is

$$E\left(x(t)|x(s)=x_s\right)=\frac{\Gamma\left(q+\frac{3}{2}\right)}{\Gamma\left(q+1\right)}\left(\frac{\beta}{c^2\left(e^{2\beta(t-s)}-1\right)}\right)^{\frac{-1}{2}}$$
$$\times K\left(\frac{-1}{2},q+1,\frac{-\beta x_s^2 e^{2\beta(t-s)}}{c^2\left(e^{2\beta(t-s)}-1\right)}\right). \quad (15)$$

From (15) and by the condition $P\left(x\left(t_0\right)=x_0\right)=1$, the TF of the SRDP is given by

$$\begin{aligned} E(x(t)) = & \frac{\Gamma\left(q+\frac{3}{2}\right)}{\Gamma(q+1)}\left(\frac{\beta}{c^2\left(e^{2\beta(t-t_0)}-1\right)}\right)^{\frac{-1}{2}} \\ & \times K\left(\frac{-1}{2}, q+1, \frac{-\beta x_0^2}{c^2\left(1-e^{2\beta(t-t_0)}\right)}\right). \end{aligned} \tag{16}$$

Remark 4.2. We can also analyze the asymptotic behavior in time of the TF of the SRDP and using the formula $K(a,b,0)=1$ (for details see Appendix the equation (24)), if $\beta<0$, thus obtaining

$$\lim_{t\to\infty} E(x(t)) = \frac{\Gamma\left(q+\frac{3}{2}\right)}{\Gamma(q+1)}\left(-\frac{\beta}{c^2}\right)^{\frac{-1}{2}}.$$

5 Transformation the VIR model into the SGDP

5.1 The third result

Theorem 5.1. *If* $\{x(t); t\in[t_0;T]\}$ *is a VIR model. Then, the diffusion process* $\left\{e^{x(t)}; t\in[t_0;T]\right\}$ *is a SGDP.*

Proof. The VIR model (5):

$$dx(t)=(\alpha+\beta x(t))\,dt+\sigma dw(t);\quad x\left(t_0\right)=x_0.$$

By applying the Itô's lemma to $e^{(x(t)}$, we get

$$d\left(e^{x(t)}\right)=\left(\left(\alpha+\frac{\sigma^2}{2}\right)e^{x(t)}+\beta x(t)e^{x(t)}\right)dt+\sigma e^{x(t)}dw(t);\quad x\left(t_0\right)=x_0. \tag{17}$$

Then, the equation (17) becomes:

$$d\left(y(t)\right)=\left(\left(\alpha+\frac{\sigma^2}{2}\right)y(t)+\beta y(t)\log\left(y(t)\right)\right)dt+\sigma y(t)dw(t);\quad y\left(t_0\right)=y_0.$$

where $y(t)=e^{x(t)}$ and $y_0=e^{x_0}$. The process $\{y(t); t\in[t_0;T]\}$ in the last equation is the SGDP with infinitesimal moments $a(t,y)=\left(\alpha+\frac{\sigma^2}{2}\right)y+\beta y\log\left(y\right)$; and $b(t,y)=\sigma^2y^2$. □

5.2 The PTDF of the VIR model using the SGDP

Let H and h be respectively the CPDF and the PTDF of the SGDP, and let G and g be respectively the CPDF and the PTDF of the VIR process. Then, we have:

$$\begin{aligned} g(x,t|y,s) =& \frac{dG(x,t|y,s)}{dx} = \frac{dP(x(t)|x(s)=y\leq x)}{dx} \\ =& \frac{dP\left(e^{x(t)}|e^{x(s)}=e^{y}\leq e^{x}\right)}{dx} = \frac{dH\left(e^{x},t|e^{y},s\right)}{dx} \\ =& e^{x}h\left(e^{x},t|e^{y},s\right) \end{aligned}$$

Finally, the PTDF of the VIR model $x(t)$, given $x(s)$ for $t>s$ is

$$g\left(x,t|y,s\right) = \frac{1}{\sqrt{2\pi\sigma^{2}\upsilon^{2}(s,t)}} \exp\left(\frac{-\left[x-\mu(s,t,y)\right]^{2}}{2\sigma^{2}\upsilon^{2}(s,t)}\right), \tag{18}$$

where $\mu(s,t,y) = e^{\beta(t-s)}y + \frac{\alpha}{\beta}\left(e^{\beta(t-s)}-1\right)$ and $\upsilon^{2}(s,t) = \frac{1}{2\beta}\left(e^{2\beta(t-s)}-1\right)$. This function is the density function of the normal distribution $N\left(\mu(s,t,y),\sigma^{2}\upsilon^{2}(s,t)\right)$.

5.3 TF of the VIR process

By the characteristics of normal distribution, the CTF of the model, for $t>s$, is

$$E\left(x(t)|x_{s}\right) = e^{\beta(t-s)}x_{s} + \frac{\alpha}{\beta}\left(e^{\beta(t-s)}-1\right). \tag{19}$$

Finally, from (19) and by considering the condition $P\left(x\left(t_{0}\right)=x_{0}\right)=1$, the TF of the VIR model is given by

$$E(x(t)) = e^{\beta(t-t_{0})}x_{0} + \frac{\alpha}{\beta}\left(e^{\beta(t-t_{0})}-1\right). \tag{20}$$

Remark 5.2. We can also analyze the asymptotic behavior in time of the TF of the VIR process, if $\beta<0$, thus obtaining

$$\lim_{t\to\infty} E(x(t)) = \frac{-\alpha}{\beta}.$$

6 Transformation the SGDP into the VIR model

6.1 The fourth result

Theorem 6.1. *If* $\{x(t); t \in [t_0; T]\}$ *is a SGDP. Then, the stochastic diffusion process* $\{\log(x(t)); t \in [t_0; T]\}$ *is a VIR model.*

Proof. The SGDP (6):

$$dx(t) = (\alpha x(t) + \beta x(t) \log(x(t)))\, dt + cx(t)dw(t); \quad x(t_0) = x_0.$$

Hence, the equation (6) becomes:

$$\frac{dx(t)}{x(t)} = (\alpha + \beta \log(x(t)))\, dt + cdw(t); \quad x(t_0) = x_0.$$

By applying the Itô's lemma to $\log(x(t))$, we have

$$d(\log(x(t))) = \left(\left(\alpha - \frac{c^2}{2}\right) + \beta \log(x(t))\right) dt + cdw(t); \quad x(t_0) = x_0. \quad (21)$$

Then, the equation (21) becomes:

$$d(y(t)) = \left(\left(\alpha - \frac{c^2}{2}\right) + \beta y(t)\right) dt + cdw(t); \quad y(t_0) = y_0.$$

with $y(t) = \log(x(t))$ and $y_0 = \log(x_0)$. The process $\{y(t), t \in [t_0; T]\}$ in the last equation is the VIR model with infinitesimal moments $a(t, y) = \left(\alpha - \frac{c^2}{2}\right) + \beta y$; and $b(t, y) = c^2$. □

6.2 The PTDF of the SGDP model using the VIR model

The PTDF of the SGDP is:

$$\begin{aligned} h(x, t|y, s) &= \frac{dH(x, t|y, s)}{dx} = \frac{dP(x(t)|x(s) = y \leq x)}{dx} \\ &= \frac{dP(\log(x(t))|\log(x(s)) = \log(y) \leq \log(x))}{dx} \\ &= \frac{dG(\log(x(t)), t|\log(y), s)}{dx} \\ &= \frac{1}{x} g(\log(x(t)), t|\log(y), s). \end{aligned}$$

Finally, the PTDF of the SGDP model $x(t)$, given $x(s)$ for $t > s$ is

$$h(x,t|y,s) = \frac{1}{x\sqrt{2\pi c^2 \upsilon^2(s,t)}} \exp\left(\frac{-\left[\log(x) - \mu(s,t,y)\right]^2}{2c^2\upsilon^2(s,t)} \right), \tag{22}$$

with $\mu(t,s,y) = e^{\beta(t-s)}\log(y) + \frac{(2\alpha - c^2)}{2\beta}\left(e^{\beta(t-s)} - 1\right)$ and
$\upsilon^2(s,t) = \frac{1}{2\beta}\left(e^{2\beta(t-s)} - 1\right)$. This function is the density function of the log-normal distribution $\Lambda_1\left(\mu(t,s,y), c^2\upsilon^2(s,t)\right)$.

6.3 TF of the SGDP

By the properties of log-normal distribution, the k^{th} CTF of the SGDP, for $t > s$, is

$$E\left(x(t)^k | x(s) = x_s\right) = \exp\left(k\mu(s,t,y) + \frac{k^2c^2\upsilon^2(s,t)}{2}\right),$$

for $k = 1$, the CTF of the SGDP, for $t > s$, is

$$\begin{aligned} E\left(x(t)|x(s) = x_s\right) = &\exp\left\{\log(y)e^{\beta(t-s)} + \frac{c^2}{4\beta}\left(e^{2\beta(t-s)} - 1\right)\right\} \\ &\times \exp\left\{\frac{2\alpha - c^2}{2\beta}\left(e^{\beta(t-s)} - 1\right)\right\}. \end{aligned} \tag{23}$$

Finally, from (23) and by considering the condition $P\left(x\left(t_0\right) = x_0\right) = 1$, the TF of the SGDP is given by

$$\begin{aligned} E(x(t)) = &\exp\left\{\log\left(x_0\right)e^{\beta(t-t_0)} + \frac{c^2}{4\beta}\left(e^{2\beta(t-t_0)} - 1\right)\right\} \\ &\times \exp\left\{\frac{2\alpha - c^2}{2\beta}\left(e^{\beta(t-t_0)} - 1\right)\right\}. \end{aligned}$$

Remark 6.2. We can also analyze the asymptotic behavior in time of the TF of the Gompertz process, if $\beta < 0$, thus obtaining

$$\lim_{t\to\infty} E(x(t)) = \exp\left(\frac{c^2 - 4\alpha}{4\beta}\right).$$

7 Conclusion

Using the elementary stochastic calculus and ItÃt''s formula, we show four results. First, we have shown that the stochastic CIR model is a SRDP and the SRDP is a stochastic CIR model. The other hand, we have shown that the VIR model is a SGDP and the SGDP is a VIR model.

Appendix

The kummer function

The Kummer Function is defined by (see [9]): $K(a,b,x) = \sum_{j=0}^{\infty} \frac{(a)_j}{(b)_j} \frac{x^j}{j!}$. It is analytic, regular at zero entire single-valued transcendental function of all a, b, x except $b = 0, -1, -2, -3, ...,$ for which it has simple poles. $K(a,b,x)$ is a notation introduced by Humbert,

$$(\lambda)_j = \lambda(\lambda+1)(\lambda+2)...(\lambda+j-1) = \frac{\Gamma(\lambda+j)}{\Gamma(\lambda)}, (\lambda)_0 = 1, (1)_j = j!$$

with λ stands for any number and j for any positive integer or zero, is the Pochhammer's symbol and Γ is the Euler gamma function.

(i) We show that $K(a,b,0) = 1$. Therefore, by the definition of the Kummer function K, we have

$$K(a,b,x) = \sum_{j=0}^{\infty} \frac{(a)_j}{(b)_j} \frac{x^j}{j!} = 1 + \sum_{j=1}^{\infty} \frac{(a)_j}{(b)_j} \frac{x^j}{j!}.$$

Finally, for $x = 0$ we deduce

$$K(a,b,0) = 1 + \sum_{j=1}^{\infty} \frac{(a)_j}{(b)_j} \frac{0^j}{j!} = 1. \tag{24}$$

(ii) We show that $K(q+2, q+1, z) = e^z + \frac{\Gamma(q+1)}{\Gamma(q+2)} z e^z$. Therefore, by the definition of the Kummer function K, we have

$$
\begin{aligned}
K(q+2,q+1,z) =& \sum_{j=0}^{\infty} \frac{\frac{\Gamma(q+2+j)}{\Gamma(q+2)}}{\frac{\Gamma(q+1+j)}{\Gamma(q+1)}} \frac{z^j}{j!} = \frac{\Gamma(q+1)}{\Gamma(q+2)} \sum_{j=0}^{\infty} \frac{\Gamma(q+2+j)}{\Gamma(q+1+j)} \frac{z^j}{j!} \\
=& \frac{\Gamma(q+1)}{\Gamma(q+2)} \sum_{j=0}^{\infty} (q+1+j) \frac{z^j}{j!} \\
=& \frac{\Gamma(q+1)}{\Gamma(q+2)} \left\{ (q+1) \sum_{j=0}^{\infty} \frac{z^j}{j!} + \sum_{j=0}^{\infty} \frac{j z^j}{j!} \right\} \\
=& \frac{\Gamma(q+1)}{\Gamma(q+2)} \left\{ (q+1)e^z + \sum_{j=1}^{\infty} \frac{j z^j}{j!} \right\} \\
=& \frac{\Gamma(q+1)}{\Gamma(q+2)} \left\{ (q+1)e^z + \sum_{j=1}^{\infty} \frac{z^j}{(j-1)!} \right\} \\
=& \frac{\Gamma(q+1)}{\Gamma(q+2)} \left\{ (q+1)e^z + \sum_{j=0}^{\infty} \frac{z^{j+1}}{(j)!} \right\} \\
=& \frac{\Gamma(q+1)}{\Gamma(q+2)} \left\{ (q+1)e^z + z \sum_{j=0}^{\infty} \frac{z^j}{(j)!} \right\} \\
=& \frac{\Gamma(q+1)}{\Gamma(q+2)} \{ (q+1)e^z + z e^z \} = e^z + \frac{\Gamma(q+1)}{\Gamma(q+2)} z e^z.
\end{aligned} \tag{25}
$$

(iii) We show that $K(a,b,z) = e^z K(b-a,b,-z)$. Therefore, based on Euler's integral representation for the Kummer function, one might expect that the Kummer function satisfies

$$
K(a,b,z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 e^{zt} t^{a-1} (1-t)^{b-a-1} dt, \quad b > a > 0.
$$

Hence,

$$
\begin{aligned}
K(a,b,z) &= \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 e^{zt} t^{a-1} (1-t)^{b-a-1} dt \\
&= \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} e^z \int_0^1 e^{z(t-1)} t^{a-1} (1-t)^{b-a-1} dt.
\end{aligned}
$$

By applying the change of variable $s = 1 - t$, we get

$$\begin{aligned} K(a,b,z) =& \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} e^z \int_0^1 e^{-zs}(1-s)^{a-1} s^{b-a-1} ds \\ =& e^z K(b-a, b, -z). \end{aligned} \tag{26}$$

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